AN INITIAL-VALUE PROBLEM IN SHOCK STABILITY

BY G. W. SWAN (*Washington State University*)

A study of the stability of normal shock waves in fluids with viscosity and heat conduction is presented by Morduchow and Paullay [1]. In dealing with the structure for a continuous weak shock the following equation, their (41), is obtained:

$$\frac{\partial \bar{u}}{\partial t} + \left(u_s - \frac{1+\alpha}{2}\right)\Gamma \frac{\partial \bar{u}}{\partial x} + \Gamma u_s' \bar{u} = \frac{1}{2}\,\delta\,\frac{\partial^2 \bar{u}}{\partial x^2}\,, \tag{1}$$

$(-\infty < x < \infty, t > 0)$, where $\bar{u}(x, t)$ is a small perturbation on the steady-state velocity:

$$u_s(x) = \tfrac{1}{2}[1 + \alpha - (1 - \alpha) \tanh \tfrac{1}{2}\delta^{-1}\Gamma(1 - \alpha)x], \tag{2}$$

and α, δ (> 0) and Γ are constants. The boundary conditions are

$$x \to -\infty, \quad \bar{u} \to 0, \quad u_s \to 1, \quad x \to +\infty, \quad \bar{u} \to 0, \quad u_s \to \alpha. \tag{3}$$

In [1] only the nature of the continuous eigenvalue spectrum is investigated. The complete formulation of the above problem requires that the initial form of the perturbation $\bar{u}(x, 0)$ be specified:

$$\bar{u}(x, 0) = A(x), \tag{4}$$

say.

The purpose of this note is to illustrate how one can obtain an explicit solution to the initial- and boundary-value problem posed by (1)–(4).

The coefficients of $\bar{u}_x$ and $\bar{u}$ in (1) are complicated hyperbolic functions. By introduction of a change of variables it is possible to arrange for these coefficients to be algebraic in nature. This may be achieved as follows. Introduce X, t as the new independent variables, with $X = (1 - u_s)/(1 - \alpha)$. With $\Omega(X, t)$ denoting $\bar{u}(x(u_s(X)), t)$ the problem (1)–(4) is now formulated as

$$(X - X^2)^2(\partial^2\Omega/\partial X^2) + 2(X - X^2)\Omega = k(\partial\Omega/\partial t), \tag{5}$$

$$\Omega(0, t) = \Omega(1, t) = 0, \quad \Omega(X, 0) = F(X), \tag{6}$$

where $k = 2\delta/(1 - \alpha)^2\Gamma^2 > 0$, and, for convenience, the initial form of the perturbation $A(x(u_s(X)))$ is replaced by $F(X)$.

Eq. (5) is linear and this suggests the use of integral transform techniques. Introduce the Laplace transform of $\Omega(X, t)$:

$$\Phi(X, p) = \int_0^\infty \Omega(X, t) \exp(-pt)\, dt. \tag{7}$$

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Direct application of this transform to (5) gives

$$\Phi'' + [2(X - X^2)^{-1} + \lambda(X - X^2)^{-2}]\Phi = f(X), \tag{8}$$

where

$$\lambda = -kp,\ \Phi = \Phi(X, -\lambda k^{-1}), \qquad f(X) = -k\Omega(X, 0)(X - X^2)^{-2}, \tag{9}$$

and $\Omega(X, 0)$ represents the initial form of the perturbation. Also, since $\Omega(0, t) = \Omega(1, t) = 0$,

$$\Phi(0, -\lambda k^{-1}) = \Phi(1, -\lambda k^{-1}) = 0. \tag{10}$$

Mathematically, here, we have a singular eigenfunction expansion problem. The determination of Φ and the spectrum of eigenvalues is not trivial.

Let $\varphi(X, \lambda)$, $\psi(X, \lambda)$ be two solutions of the homogeneous equation (namely (8) with $f \equiv 0$) such that their Wronskian $W(\varphi, \psi) = 1$; then it is straightforward, by differentiation, to show that

$$\Phi(X, -\lambda k^{-1}) = \psi(X, \lambda) \int_0^X \varphi(X, \lambda) f(X)\, dX + \varphi(X, \lambda) \int_X^1 \psi(X, \lambda) f(X)\, dX \tag{11}$$

is the solution of (8). To find φ and ψ proceed as follows. Introduce

$$U = X^\tau (1 - X)^m (n - X), \tag{12}$$

where τ, m and n are as yet undetermined quantities. Consider the homogenoeus equation

$$L\Phi = 0,\ L \equiv d^2/dX^2 + 2(X - X^2)^{-1} + \lambda(X - X^2)^{-2}. \tag{13}$$

Now

$$LU = X^{\tau-2}(1 - X)^{m-2}(P + QX + RX^2 + SX^3), \tag{14}$$

where

$$P = n(\tau^2 - \tau + \lambda), \qquad Q = -2n\tau(m - 1 + \tau) - \tau^2 - \tau + n + \lambda,$$
$$R = 2(\tau^2 + \tau m - n) + (m - 1 + \tau)(n\tau + mn + 2),$$
$$S = (m + 2 + \tau)(m - 1 + \tau).$$

The quantity S can be chosen to be zero if

$$m = 1 - \tau, \tag{15}$$

and for this value of m, $R = 2(\tau - n)$, which can be made zero for

$$n = \tau. \tag{16}$$

Also, on using (15) and (16), $Q = -(\tau^2 - \tau + \lambda)$ and if τ is chosen to satisfy

$$\tau^2 - \tau + \lambda = 0, \tag{17}$$

P and Q are now zero and $LU = 0$ with

$$U = X^\tau (1 - X)^{1-\tau}(\tau - X). \tag{18}$$

However, the coefficient S can also be chosen to be zero for $m = -2 - \tau$ and it is readily verified that this choice does not give consistency when the quantities P, Q and R are set to zero. Consequently this value of m is dismissed. The solution of (17) is

$$\tau = \tfrac{1}{2} - \tfrac{1}{2}i(4\lambda - 1)^{1/2}, \quad \tau_1 = \tfrac{1}{2} + \tfrac{1}{2}i(4\lambda - 1)^{1/2}. \tag{19}$$

Two linearly independent solutions of $L\Phi = 0$ are now (18) and

$$V = X^{\tau_1}(1 - X)^{1-\tau_1}(\tau_1 - X)$$

and, since $\tau_1 = 1 - \tau$,

$$V = X^{1-\tau}(1 - X)^{\tau}(1 - \tau - X),$$

with τ being given by the first equation of (19). Furthermore

$$W(U, V) = -(1 - 2\tau)(\tau^2 - \tau) = i\lambda(4\lambda - 1)^{1/2},$$

and hence

$$\varphi(X, \lambda) = [i\lambda(4\lambda - 1)^{1/2}]^{-1}X^{\tau}(1 - X)^{1-\tau}(\tau - X), \tag{20}$$

$$\psi(X, \lambda) = X^{1-\tau}(1 - X)^{\tau}(1 - \tau - X), \tag{21}$$

with $2\tau = 1 - i(4\lambda - 1)^{1/2}$, are two linearly independent solutions of $L\Phi = 0$ such that $W(\varphi, \psi) = 1$.

Finally, substitution of the forms (20), (21) for φ and ψ into (11) gives the solution of (8) with boundary conditions (10). Inversion of (7) gives

$$\Omega(X, t) = -(2\pi ik)^{-1} \int_{-kc+i\infty}^{-kc-i\infty} \Phi(X, -\lambda k^{-1}) \exp(-\lambda k^{-1}t)\, d\lambda, \tag{22}$$

where c is a positive constant. Since k is positive, kc is positive. There is a pole of Φ at $\lambda = 0$ and a branch-point singularity at $\lambda = \frac{1}{4}$. The evaluation of (22) is (formally) accomplished by closing the contour in the right-hand half-plane. Let C_1 be the arc of the quarter circle from $-kc - i\infty$ to ∞, C_2 be the lower branch from ∞ to $\frac{1}{4} + \delta$, C_3 be the arc of a small circle, radius δ, surrounding $\lambda = \frac{1}{4}$, C_4 be the upper branch from $\frac{1}{4} + \delta$ to ∞ and C_5 be the arc of the quarter circle from ∞ to $kc + i\infty$. On C_1 and C_5, $\lambda = \text{R}e^{i\theta}$, say, and as $R \to \infty$ the presence of the decaying exponential in the integrand in (22) assures that there are no contributions from C_1 and C_5. The residue at $\lambda = 0$ is given by

$$2\pi ik(X - X^2) \int_0^1 (X - X^2)^{-1}\Omega(X, 0)\, dX.$$

On the branch C_2, $\lambda = \frac{1}{4} + \text{r}e^{2\pi i}$ and on the branch C_4, $\lambda = \frac{1}{4} + r$. Finally, the perturbation (in the limit as $\delta \to 0$)

$$\Omega(X, t) = (X - X^2) \int_0^1 (X - X^2)^{-1}\Omega(X, 0)\, dX$$
$$+ (2\pi ik)^{-1}\left[-\int_{1/4}^{\infty} F(X, r)\, dr + \int_{1/4}^{\infty} G(X, r)\, dr\right],$$

where $F(X, r)$, $G(X, r)$ are the contributions from C_2 and C_4, respectively. After a little algebraic manipulation this expression can be cast in the form

$$\Omega(X, t) = (X - X^2) \int_0^1 (X - X^2)^{-1} \Omega(X, 0)\, dX$$

$$+ (4\pi)^{-1} \int_{1/4}^{\infty} [a(X, r) a_1(r) + b(X, r) b_1(r)] \exp(-rk^{-1} t)\, dr, \qquad (23)$$

where

$$a(X, r) = X^{1/2 + is} (1 - X)^{1/2 - is} (\tfrac{1}{2} + is - X) s^{-1} (\tfrac{1}{4} + r)^{-1}, \qquad (24)$$

$$a_1(r) = \int_0^1 X^{-3/2 - is} (1 - X)^{-3/2 + is} (\tfrac{1}{2} - is - X) \Omega(X, 0)\, dX, \qquad (25)$$

and $b(X, r)$, $b_1(r)$, respectively, are the same as $a(X, r)$, $a_1(r)$, respectively, but with i replaced by $-i$; also $s = r^{1/2}$.

The first expression on the right of (23) is interpreted as being the neutrally stable mode. It represents a translation of the weak shock structure and does not damp out with time.

Reference

[1] M. Morduchow and A. J. Paullay, *Stability of normal shock waves with viscosity and heat conduction*, Phys. Fluids **14**, 323–331 (1971)