THE DRAG AND SPHERICITY INDEX OF A SPINDLE*

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Introduction. Several years ago, Payne and Pell [1, 2, 3] published a series of articles pertaining to the Stokes flow of a viscous, incompressible fluid about a body in which the flow is two-dimensional or has radial symmetry. Some of the shapes of bodies that were considered were a lens, a torus, a sphere, and oblate and prolate spheroids. Using this procedure, the differential equation to be satisfied in the flow region is found to be

\[ L^{-1} \psi = 0, \]

where

\[ L^{-1} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r}. \]

If the trace of the boundary of the body is called C, then the condition of vanishing velocity on C can be stated in the form

\[ \psi = 0, \quad \frac{\partial \psi}{\partial n} = 0, \text{ on } C. \]

Here \( n \) is the unit normal to C exterior to the body.

Recently, our interest has been in the general area of the flow of diseased states of blood. The above papers have been extremely useful in a problem of current medical concern, sickle-cell anemia. The red cells in this diseased state approximate lens, sickles, hemispheres and spindles. It is the purpose of this communication to show the calculation of the flow about a spindle.

Flow about a spindle. A convenient set of coordinates to represent a spindle are the bipolar coordinates (\( \varphi, \eta \)) [4]. In terms of these coordinates

\[ x = \frac{b \sinh \eta}{\cosh \eta - \cos \varphi}, \]

\[ r = \frac{b \sin \varphi}{\cosh \eta - \cos \varphi}, \]

\[ x^2 + r^2 - 2br \cot \varphi = b^2, \]

with \( \varphi > \pi/2; r \geq 0 \). The calculation of the drag, \( P \), can be made by using

\[ P = 8\pi \mu \lim_{r \to 0} (\rho \psi_1/r^2). \]

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In this equation \( p = (x^2 + r^2)^{1/2} \), \( \mu \) is the coefficient of viscosity of the suspending fluid and \( \psi_1 \) is a solution to

\[ L_{-1}^2 \psi_1 = 0. \quad (1) \]

Payne and Pell [3] suggest that the drag can best be evaluated from

\[ P = 8\pi \mu U \int_0^{\infty} \frac{F(\alpha)}{\cosh \alpha \pi} d\alpha, \quad (8) \]

where

\[ F(\alpha) = \int_{t_0}^{1} K_a(-\tau)K_a^{(2)}(\tau) \, d\tau \int_{t_0}^{1} K_a(\tau)K_a^{(2)}(\tau) \, d\tau. \quad (9) \]

In Eq. (9) \( K_a(\tau) \) is known as the conal function (5) and defined as

\[ K_a(\tau) = P_i-a_{-1/2}(\tau) \quad (10) \]

and

\[ K_a^{(n)}(\tau) = d^n K_a(\tau)/d\tau^n, \quad (11) \]

where \( P(\tau) \) is the Legendre function, and \( \tau = \cos \phi \).

To date, the drag for a spindle has not been determined, although Payne and Pell suggest that certain tables should facilitate the computation [6, 7]. However, in examining these tables, it was found that they were not adequate with regard to the choice of the angle \( \phi \) or with ease of determining the derivatives of the conal functions. To overcome these difficulties the approach that follows was finally used.

The functions \( B(\alpha) \) and \( A(\alpha) \) are evaluated from the boundary conditions of the problem and can be written as

\[ A(\alpha) = \frac{2^{1/2}}{\Omega \cosh \alpha \pi} \left[ K_a(-t_0)K_a^{(1)}(t_0) - K_a^{(1)}(-t_0)K_a(t_0) \right], \quad (13) \]

\[ B(\alpha) = \frac{2^{1/2}}{\Omega \cosh \alpha \pi} \left[ t_0K_a^{(1)}(t_0)K_a^{(1)}(-t_0) - K_a(t_0) \frac{d}{dt_0} (t_0K_a^{(1)}(t_0)) \right] \quad (14) \]

where \( t_0 = \tau_0 = \cos \phi_0 \) and

\[ \Omega = t_0[K_a^{(1)}(t_0)]^2 - K_a(t_0) \frac{d}{dt_0} (t_0K_a^{(1)}(t_0)) \quad (15) \]

The value of \( t_0 \) defines the shape of the spindle.

The calculation was facilitated by using the following series representation for \( K_a(\tau) \) [8]:

\[ K_a(\tau) = 1 + \frac{1 + 4\alpha^2}{4(1)!^2} \left( \frac{1 - \tau}{2} \right) \]

\[ + \frac{(1 + 4\alpha^2)(3^2 + 4\alpha^2)}{4^2(2)!^2} \left( \frac{1 - \tau}{2} \right)^2 \]

\[ + \frac{(1 + 4\alpha^2)(3^2 + 4\alpha^2)(5^2 + 4\alpha^2)}{4^3(3)!^2} \left( \frac{1 - \tau}{2} \right)^3 + \cdots \quad (16) \]
for $|r - 1| < 2$, where appropriate recursive relationships allowed rapid calculation of the series terms for $K_w(r)$ and its derivatives.

In biological cellular flow systems, especially blood, an arbitrary parameter that is used as a reference to changing shapes is the sphericity index (S.I.) [9]. It is defined as

$$\text{S.I.} = 4.84(V^{2/3}/S),$$

where $V$ is the volume and $S$ is the surface area of the particle respectively. If $b$ is taken as unity in Eq. (6), the volume and surface area of a spindle are found to be

$$V = 2\pi\left[a^2 + 2/3 + a(a^2 + 1)\sin^{-1}\left(\frac{1}{a^2 + 1}\right)^{1/2}\right],$$

and

$$S = 4\pi[(\cot \varphi + 1)(a^2 + 1)^{1/2}]$$

where $a = \cot \varphi$.

These values were used in Eq. (17) to calculate the sphericity index as a function of the changing shape of the spindle.

In Table I, the drag coefficient and sphericity index are listed for several different spindles.

A recent paper by Gluckman, Weinbaum and Pfeffer [10] also presents a solution to the problem of axisymmetric slow viscous flow past a convex body of revolution. Although these authors' presentation is thorough and interesting, the unique use of peripolar or bipolar coordinates as suggested by Pell and Payne [13] seems to encompass all of the above authors' bodies as well as several not included in their publication.

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References