LAMINAR BOUNDARY LAYER AT A DISCONTINUITY IN WALL CURVATURE*

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Abstract. The discontinuity in pressure gradient predicted for two-dimensional inviscid subsonic or supersonic flow at a jump discontinuity in wall curvature is smoothed by means of local solutions which take into account the interaction of a laminar boundary layer with the external flow.

1. Introduction. Solutions to the equations describing fluid motion in a laminar viscous boundary layer along a solid boundary are typically obtained for a specified pressure gradient at the surface. If, however, it is postulated that a small pressure change occurs over a suitably small distance along the boundary layer, an interaction with the external flow must be taken into account. The details of the resulting local pressure distribution then cannot be specified in advance, but must be found by studying changes in the boundary layer coupled with small perturbations on the external flow. A description in terms of asymptotic expansions for large Reynolds numbers has been proposed for the initial pressure rise caused by an oblique shock wave impinging on a laminar boundary layer [1]; for the incompressible flow near the trailing edge of a flat plate [2, 3]; for a boundary layer which is deflected through a small angle at a convex corner, in either a subsonic or supersonic external flow [4]; for the interaction of a boundary layer and a weak shock wave at transonic speeds [5]; and for other related problems. In each of the examples cited, the approximate equations are nonlinear (except for the particular case studied in [4]), and an explicit solution for the pressure can be obtained only by numerical integration. An example for which a solution can be derived analytically, in the form of an integral representation, arises when a boundary layer encounters a jump discontinuity in wall curvature. At the discontinuity, inviscid-flow theory predicts a jump in the pressure gradient if the external flow is supersonic and gives a logarithmically infinite pressure gradient if the flow is subsonic. By means of appropriate local solutions, these discontinuities can be removed, and continuous expressions for the pressure gradient can be obtained which are presumed to be correct asymptotic representations as the viscosity coefficient approaches zero. In Sec. 2, the relevant orders of magnitude are discussed, a linearized boundary-layer problem is formulated, and the solution is obtained in terms of the still unknown pressure distribution along the wall. Interaction with the external flow is then taken into account; equations

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for the pressure are given, and solutions are derived in Sec. 3 for both supersonic and subsonic external flows. The results may be helpful in suggesting local approximations for use in numerical integration of the boundary-layer equations near discontinuities of this kind.

2. Perturbations in the boundary-layer flow. We will consider two-dimensional laminar steady flow, in the direction of increasing X, above a wall described locally by $Y = 0$ for $X < 0$ and $Y = -\frac{1}{2}\kappa X^2$ for $X > 0$, where $\kappa > 0$ and X, Y are rectangular coordinates made nondimensional with the distance L along the wall from its leading edge. Solutions for nonzero upstream curvature can also be found quite directly from the results to be given here. Nondimensional dependent variables will be defined by

$$u = U/U_\infty = (T_w/T_\infty)\psi_Y , \qquad v = V/U_\infty = -(T_w/T_\infty)\psi_X ,$$

and

$$p = (P - P_\infty)/\rho_\infty U_\infty^2,$$

where U, V are velocity components in the X, Y directions respectively; P, ρ, T are the pressure, density, and temperature respectively; T_w is the temperature at the wall, taken to be constant; and the reference values U_∞, P_∞, ρ_∞, T_∞ are the values of U, P, ρ, T predicted at $X = 0$, $Y = 0$ by inviscid-flow theory. Also $\beta = |1 - M_\infty^2|^{1/2}$, where M_∞ is the Mach number corresponding to U_∞, p_∞, ρ_∞. The viscosity coefficient μ is assumed given by $\mu/\mu_\infty = CT/T_\infty$, where $C = \mu_w T_\infty/(\mu_\infty T_w)$ and μ_w, μ_∞ are the wall and reference values of μ respectively. We will seek asymptotic solutions near $X = 0$, $Y = 0$ in the limit as the Reynolds number $R = \rho_\infty U_\infty L/\mu_\infty \to \infty$.

For $M_\infty > 1$, the pressure at the wall according to inviscid-flow theory for $X \to 0$ is given by

$$p \sim -(\kappa/\beta)XH(X) \tag{2.1}$$

where $H(X)$ is the unit step function: $H(X) = 0$ for $X < 0$ and $H(X) = 1$ for $X > 0$. If instead $M_\infty < 1$, the complex velocity locally is $u - iv/\beta \sim -(p + iv/\beta)$, where

$$p + iv/\beta \sim \kappa\pi^{-1}\beta^{-1}z \log z - i\kappa\beta^{-1}z \tag{2.2}$$

as $z = X + i\beta Y \to 0$; $v \sim -\kappa X$ as $\arg z \to 0$ and $v/X \to 0$ as $\arg z \to \pi$. The thin boundary layer approaching $X = 0$ has a velocity profile which is linear in the stretched coordinate $R^{1/2}Y$ as $R^{1/2}Y \to 0$. Locally the most important changes in the profile shape occur in a still thinner sublayer close to the wall [1, 2, 3, 4, 5], where the changes in the viscous, pressure, and inertia forces are all of the same order as $R \to \infty$. The remainder of the boundary layer experiences primarily a displacement effect because of the small acceleration of the fluid in the sublayer, and the resulting small decrease in flow deflection angle is nearly constant across most of the boundary layer. The appropriate linear-theory pressure-angle relation describes the corresponding perturbations in the external flow. This information is sufficient to suggest [1, 2, 3, 4] that an interaction of the boundary layer with the external flow occurs in a streamwise distance $X = O(R^{-3/8})$ and that the sublayer thickness is given by $Y = O(R^{-5/8})$. If the local changes were just large enough for separation to occur, the inertia term in the sublayer equation would be nonlinear and it would follow that locally $p = O(R^{-1/4})$. In the present case the pressure should match asymptotically with (2.1) or (2.2), and so p is either $O(R^{-3/8})$ or $O(R^{-3/8} \log R)$.

The following stretched coordinates and asymptotic expansions are introduced for the description of the sublayer:

$$x = \frac{a_1^{5/4}\beta^{3/4}R^{3/8}}{(T_w/T_\infty)^{3/2}C^{3/8}}X, \qquad y = \frac{a_1^{3/4}\beta^{1/4}R^{5/8}}{(T_w/T_\infty)^{3/2}C^{5/8}}Y \tag{2.3}$$

$$\psi \sim \frac{(T_w/T_\infty)C^{3/4}}{a_1^{1/2}\beta^{1/2}R^{3/4}}\frac{y^2}{2} + \kappa\frac{(T_w/T_\infty)^{5/2}C^{7/8}}{a_1^{9/4}\beta^{7/4}R^{7/8}} \cdot\left\{j\frac{y^3}{6\pi}\log\left(\frac{(T_w/T_\infty)^{3/2}C^{3/8}}{a_1^{5/4}\beta^{3/4}R^{3/8}}\right) + j\psi_{1l}(x, y) + \psi_1(x, y)\right\}, \tag{2.4}$$

$$p \sim \kappa\frac{(T_w/T_\infty)^{3/2}C^{3/8}}{a_1^{5/4}\beta^{7/4}R^{3/8}}\left\{j\frac{x}{\pi}\log\left(\frac{(T_w/T_\infty)^{3/2}C^{3/8}\,|x|}{a_1^{5/4}\beta^{3/4}R^{3/8}}\right) + p_1(x)\right\} \tag{2.5}$$

where a_1 is the initial value of $(T_w/T_\infty)(R/C)^{-1/2}u_Y$ at $Y = 0$ as $x \to -\infty$, equal to 0.332 for a flat plate. For $M_\infty > 1$, $j = 0$ and (2.5) matches asymptotically with (2.1) as $R \to \infty$ with $|X| \to 0$ and $|x| \to \infty$, provided that $p_1 \sim -xH(x)$ as $|x| \to \infty$. For $M\infty < 1$, $j = 1$ and (2.5) matches with (2.2), for $\arg z \to 0$ or π, if $p_1 \to 0$ as $|x| \to \infty$. We will choose ψ_{1l} as the portion of ψ associated with the term $O(R^{-3/8}x \log |x|)$ in p, and consequently $\psi_1 \to 0$ as $x \to -\infty$ for $M_\infty < 1$ as well as for $M_\infty > 1$. Suitable factors have been introduced in (2.3), (2.4), and (2.5) so that ψ_{1l}, ψ_1, and p_1 will be independent of the parameters. The transformation (2.3) remains the same as in [1] and [4], but (2.4) and (2.5) are modified since the requirement of nonlinearity in the sublayer equations has been replaced by the matching conditions for p.

We first consider the solution for $\psi_1(x, y)$. In the limit as $R \to \infty$ with x, y fixed, the largest terms in the Navier–Stokes equations lead to a linearized boundary-layer momentum equation

$$y\psi_{1xy} - \psi_{1x} = -p_1' + \psi_{1yyy} \tag{2.6}$$

Differentiating with respect to y, and defining $\tau_1 = \psi_{1yy}$,

$$\tau_{1yy} - y\tau_{1x} = 0. \tag{2.7}$$

Since $\psi_1 = \psi_{1y} = 0$ at the wall, and the displacement of the wall from $Y = 0$ is $o(R^{-5/8})$ for $X = O(R^{-3/8})$, one boundary condition is $\tau_{1y}(x, 0) = p_1'(x)$. We also require $\tau_1 \to 0$ as $y \to \infty$ and as $x \to -\infty$.

If we specify instead that $\tau_1(0, y) = 0$ for $y > 0$, $\tau_{1y}(x, 0) = 0$ for $x > 0$, and $\tau_1 \to 0$ as $y \to \infty$ for $x > 0$, a solution to (2.7) can be obtained which is analogous to the one-dimensional temperature distribution $T(x, t)$ due to a unit heat source at $x = t = 0$. When the right-hand side of (2.7) is replaced by $\delta(x)\ \delta(y)$, the resulting differential equation is invariant under a transformation $x \to k^3x$, $y \to ky$, $\tau_1 \to k^{-2}\tau_1$. Hence a solution for x, $y > 0$ has the form $\tau_1 = x^{-2/3}f(\eta)$, where $\eta = y/x^{1/3}$, and it is found that $f(\eta) = c \exp(-\eta^3/9)$. Integration of the modified (2.7) gives the added condition that the integral of $-y\tau_1$ over y from 0 to ∞ is equal to one, and it follows that $1/c = -3^{1/3}\Gamma(2/3)$. By analogy with the one-dimensional heat-conduction problem, the solution of (2.7) satisfying $\tau_{1y}(x, 0) = p_1'(x)$ is then obtained by distributing "sources" along $y = 0$, with "source strength" $p_1'(x)$:

$$\tau_1(x, y) = -\frac{1}{3^{1/3}\Gamma(2/3)}\int_{-\infty}^{x}\frac{p_1'(\xi)}{(x - \xi)^{2/3}}\exp\left[-(1/9)y^3/(x - \xi)\right] d\xi. \tag{2.8}$$

It can be verified by direct substitition that (2.8) does in fact satisfy the required wall boundary condition. The solution (2.8) could instead have been derived from the results of [6], and has also been used in [5]. Since $\tau_1 = \psi_{1yy}$, the corresponding solution for ψ_1 is obtained by repeated integration. For $y \to \infty$, one finds

$$\psi_1 \sim -Ay \int_{-\infty}^{x} \frac{p_1'(\xi)}{(x-\xi)^{1/3}}\, d\xi + p_1'(x) \tag{2.9}$$

where $A = 3^{-2/3}\Gamma(1/3)/\Gamma(2/3)$.

For subsonic flow, an analogous result for ψ_{1l} is obtained if $p_1(x)$ is replaced by $\pi^{-1}x \log |x|$ and τ_1 by $\tau_{1l} = \psi_{1lyy}$. Convergent integrals for τ_{1ly} and for τ_{1lx} can be derived after differentiating (2.8) and (2.7) with respect to y and x respectively and then making these replacements. We find

$$\tau_{1ly} = \frac{y^2}{3^{4/3}\pi\Gamma(2/3)} \int_{-\infty}^{x} \frac{1 + \log |\xi|}{(x-\xi)^{5/3}} \exp\left[-(1/9)y^3/(x-\xi)\right] d\xi, \tag{2.10}$$

$$\tau_{1lx} = \frac{-1}{3^{1/3}\pi\Gamma(2/3)} \int_{-\infty}^{x} \frac{1}{\xi(x-\xi)^{2/3}} \exp\left[-(1/9)y^3/(x-\xi)\right] d\xi, \tag{2.11}$$

where the solution for τ_{1lx} has been obtained by the procedure of the preceding paragraph. By setting $t = (1/9)y^3/(x - \xi)$ in (2.10), expanding the logarithm and integrating, it is found that the largest terms in ψ_{1l} as $y \to \infty$ are $O(y^3 \log y)$ and $O(y^3)$. Additional terms can perhaps be derived more easily using the same substitution in (2.11) and then interchanging the order of integration over t and y. Finally, for $y \to \infty$,

$$\psi_{1l} \sim \frac{y^3}{6\pi} \log \frac{y^3}{9} - \frac{1}{6\pi}\left(\frac{9}{2} + \Psi(2/3)\right)y^3 + \sigma A x^{2/3} y - \frac{x}{\pi} \log \frac{y^3}{9\,|x|} + \frac{1}{\pi}(-4 + \Psi(2/3))x + \cdots, \tag{2.12}$$

where $\Psi(q) \equiv (d/dq) \log \Gamma(q)$; $\sigma = 3^{1/2}/2$ for $x > 0$ and $\sigma = -3^{1/2}$ for $x < 0$.

To complete the description of the changes in the boundary layer, solutions are also required for $R^{1/2}Y = O(1)$. The differential equations and matching conditions suggest a solution of the form

$$\frac{R^{1/2}}{C^{1/2}}\psi \sim \psi_0^*(y^*) + \frac{C^{1/4}}{R^{1/4}}\psi_1^*(x, y^*) + \frac{C^{3/8}}{R^{3/8}} \log\left(\frac{R^{3/8}}{C^{3/8}}\right) x\psi_{2l}^*(y^*) + \frac{C^{3/8}}{R^{3/8}}\psi_2^*(x, y^*) + \cdots, \tag{2.13}$$

$$y^* = \frac{R^{1/2}}{C^{1/2}}\frac{T_\infty}{T_w} Y = \frac{C^{1/8}}{R^{1/8}} \frac{(T_w/T_\infty)^{1/2}}{a_1^{3/4}\beta^{1/4}}\, y \tag{2.14}$$

where ψ is now defined by $\rho u = \rho_\infty \psi_Y$, $\rho v = -\rho_\infty \psi_X$; the previous definition is recovered as $y^* \to 0$. As $y^* \to \infty$, $\psi_0^* \sim (T_w/T_\infty)y^*$; and as $y^* \to 0$

$$\psi_0^* \sim \frac{1}{2} a_1 y^{*2} + \frac{j}{6\pi}\frac{\kappa}{\beta}\frac{T_w}{T_\infty}\left\{\log \frac{a_1 y^{*3}}{9} - \frac{9}{2} - \Psi(2/3)\right\} y^{*3} + \cdots \tag{2.15}$$

in agreement with the largest terms of (2.4) as $y \to \infty$; as before, $j = 0$ for $M_\infty > 1$ and $j = 1$ for $M_\infty < 1$. It is found that the largest perturbations in u, ρ/ρ_∞, and T/T_∞ at a fixed point follow from a small displacement of streamlines toward the wall, and not from changes along streamlines. It follows that ψ_1^* satisfies $(\psi_{1x}^*/\psi_0^{*\prime})_{y^*} = 0$, so that the largest term in the streamline slope is a function of x but not of y^*[1, 2, 3, 4, 5], and can be expressed in terms of the pressure perturbation $p_1(x)$ as

$$\frac{v}{u} \sim -\frac{a^{5/4}\beta^{3/4}C^{3/8}}{(T_w/T_\infty)^{1/2}R^{3/8}}\,\frac{\psi_{1x}^*}{\psi_0^{*\prime}} = \frac{\kappa(T_w/T_\infty)^{3/2}C^{3/8}}{a_1^{5/4}\beta^{3/4}R^{3/8}}\left\{-j\,\frac{2\sigma A}{3x^{1/3}} + A\int_{-\infty}^{x}\frac{p_1''(\xi)\,d\xi}{(x-\xi)^{1/3}}\right\} \tag{2.16}$$

where the right-hand side is found by matching with the sublayer solution. The functions ψ_{2l}^* and ψ_2^* satisfy

$$\frac{a_1^{5/4}\beta^{3/4}}{(T_w/T_\infty)^{3/2}}\left(\frac{\psi_{2l}^*}{\psi_0^{*\prime}}\right)_{y^*} = -\frac{j\kappa}{\pi\beta}\left(M_\infty^2 - \frac{1}{\rho_0 u_0^2}\right), \tag{2.17}$$

$$\frac{a_1^{5/4}\beta^{3/4}}{(T_w/T_\infty)^{3/2}}\left(\frac{\psi_{2x}^*}{\psi_0^{*\prime}}\right)_{y^*}$$

$$= \left(M_\infty^2 - \frac{1}{\rho_0 u_0^2}\right)\left\{\frac{j\kappa}{\pi\beta} + \frac{j\kappa}{\pi\beta}\log\frac{(T_w/T_\infty)^{3/2}\,|x|}{a_1^{5/4}\beta^{3/4}} + \frac{\kappa}{\beta}\,p_1'\right\} + \left(\frac{T_\infty}{T_w}\right)^2\frac{u_0'''}{\rho_0^2 u_0^2}, \tag{2.18}$$

where we have set $\rho/\rho_\infty = \rho_0(y^*) + O(R^{-1/4})$ and $\rho_0(y^*)u_0(y^*) = (T_w/T_\infty)\psi_0^{*\prime}(y^*)$. As $y^* \to 0$ the largest terms from (2.17) and (2.18) can be shown to match correctly with corresponding terms found from (2.9) and (2.12) as $y \to \infty$.

If $M_\infty < 1$, upstream solutions corresponding to (2.4) and (2.14) are also of some interest, although not really needed here. As $x \to -\infty$ and $X \to 0$, solutions obtained for y^* fixed and for $y^* \to 0$ with $\eta = y^*/X^{1/3}$ fixed are, respectively, of the forms

$$\left(\frac{R}{C}\right)^{1/2}\psi \sim \psi_0^*(y^*) + X^{2/3}\phi_1(y^*) + X\log X\phi_{2l}(y^*) + X\phi_2(y^*) + \cdots \tag{2.19}$$

$$\sim \tfrac{1}{2}a_1 y^{*2} + X\log Xf_1(\eta) + Xf_2(\eta) + \cdots. \tag{2.20}$$

The functions f_1, f_2, $\cdots$ satisfy the wall boundary conditions, and as $\eta \to \infty$ the largest terms match with corresponding terms in ψ_0^*, ϕ_1, ϕ_{2l}, ϕ_2, $\cdots$ as $y^* \to 0$. In particular, $f_1 \equiv$ (const.) η^3 and $f_2 = O(\eta^3 \log \eta)$ as $\eta \to \infty$, for matching with the term $O(y^{*3}\log y^*)$ in $\psi_0^*(y^*)$. That is, even for $x \to -\infty$ an inner solution (2.20) is necessary to correct the behavior of ψ_0^* as $y^* \to 0$. Additional details of these solutions are easily worked out.

3. Solutions for the pressure distribution. As $y^* \to \infty$, the largest terms in (2.13) should match with solutions for small perturbations in the external flow, evaluated for $R^{3/8}Y \to 0$ with $R^{3/8}X$ fixed [1, 2, 4]. For supersonic flow, since $p \sim \beta^{-1}v/u$ in this limit, it follows from (2.16) that $p_1(x)$ satisfies

$$p_1(x) + xH(x) = A\int_{-\infty}^{x}\frac{p_1''(\xi)}{(x-\xi)^{1/3}}\,d\xi \tag{3.1}$$

where $p_1(x) \to 0$ as $x \to -\infty$. For subsonic flow, p and $\beta^{-1}v/u$ are related as $y^* \to \infty$ by a solution to Laplace's equation in suitably stretched coordinates for the external flow. From (2.16) it then follows that

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{p_1(\xi)}{x-\xi}\,d\xi = -\frac{2\sigma A}{3x^{1/3}} + A\int_{-\infty}^{x}\frac{p_1''(\xi)}{(x-\xi)^{1/3}}\,d\xi \tag{3.2}$$

where $p_1''(x) \sim -1/(\pi x)$ as $x \to 0$ and the Cauchy principal value is to be taken wherever necessary.

Since (3.1) is in the form of a convolution integral, Fourier transformation gives

$$F\{p_1(x) + xH(x)\} = AF\{p_1''(x)\}F\{x^{-1/3}H(x)\} \tag{3.3}$$

where

$$F\{f(x)\} \equiv \int_{-\infty}^{\infty} \exp(-i\omega x)f(x)\,dx. \tag{3.4}$$

The left-hand side of (3.3) is found by repeated integration by parts. The integral defining $F\{x^{-1/3}H(x)\}$ can be regarded as an integral along the positive real axis in a complex z-plane, with $-\pi < \arg z \leq \pi$. For $\omega > 0$ or for $\omega < 0$, the integration can be carried out instead along the negative or positive imaginary axis respectively. Then (3.3) becomes

$$-\zeta^{-2} - \zeta^{-2}F\{p_1''(x)\} = A\zeta^{-2/3}\exp(-\pi i/3)\Gamma(2/3)F\{p_1''(x)\} \tag{3.5}$$

where ζ is complex, such that $|\zeta| = \omega$, and we will take $-3\pi/2 < \arg \zeta \leq \pi/2$.

Thus (3.3) gives an analytic function for $F\{p_1''(x)\}$. Inversion leads to an integral along the real axis

$$p_1''(x) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\exp(ix\zeta)}{1 + \exp(-\pi i/3)(\zeta/B)^{4/3}}\,d\zeta \tag{3.6}$$

where $B^{-4/3} = 3^{-2/3}\Gamma(1/3) = A\Gamma(2/3)$. For $-3\pi/2 < \arg \zeta \leq \pi/2$ the only singularity of the integrand is a simple pole at $\zeta = \exp(-\pi i/2)B$. For $x < 0$,

$$Bp_1(x) = -\tfrac{3}{4}\exp(Bx). \tag{3.7}$$

For $x > 0$, a convenient alternative form is obtained if (3.6) is replaced by twice the real part of the integral along the imaginary axis $\arg \zeta = \pi/2$. We find

$$p_1(x) = -x - \frac{3^{1/2}}{2\pi B}\int_0^{\infty} \frac{\exp(-Bxt)\,dt}{t^{2/3}(1 + t^{4/3} + t^{8/3})} \tag{3.8}$$

where $p_1(x)$ and $p_1'(x)$ have been required to be continuous at $x = 0$. The result given by Stewartson [4] for a wall with convex corner defined by $y = -\alpha xH(x)$, $\alpha \ll R^{-1/4}$, is recovered by differentiation with respect to x. A series for small x can be obtained by expanding (3.6) for $0 \leq \omega < \omega_0$ and (3.8) for $\omega_0 < \omega < \infty$, where $\omega_0 \gg 1$ and $\omega_0 x \ll 1$. The result is, for $x > 0$,

$$Bp_1(x) \sim -\frac{3}{4}\exp(Bx) + \frac{27}{56\pi}3^{1/2}\Gamma(2/3)(Bx)^{7/3} + \frac{81}{1760\pi}3^{1/2}\Gamma(1/3)(Bx)^{11/3} + \cdots. \tag{3.9}$$

Expansion of (3.8) for $x \to \infty$ gives

$$Bp_1(x) \sim -Bx - \frac{3^{1/2}\Gamma(1/3)}{2\pi(Bx)^{1/3}} + \frac{3^{1/2}\Gamma(5/3)}{2\pi(Bx)^{5/3}} + \cdots. \tag{3.10}$$

For subsonic flow, Fourier transformation of (3.2) gives

$$(1/\pi)F\{p_1(x)\}F\{1/x\} = -(2/3)AF\{\sigma x^{-1/3}\} + AF\{x^{-1/3}H(x)\}F\{p_1''(x)\} \tag{3.11}$$

where the Cauchy principal value is understood at $x = 0$. Also $F\{p_1''(x)\} = -\omega^2 F\{p_1(x)\}$

and $F\{1/x\} = \pm\pi i$, where the upper sign is to be used for $\omega < 0$ and the lower sign for $\omega > 0$. It follows that

$$F\{p_1''(x)\} = \mp\frac{i \exp (\pm i\pi/3)(\omega/B)^{4/3}}{\exp (\pm i\pi/3)(\omega/B)^{4/3} \pm i} \tag{3.12}$$

where the same sign convention applies. Inversion gives

$$p_1''(x) = -\frac{1}{\pi x} - \frac{1}{\pi} \operatorname{Re} \int_0^\infty \frac{\exp (i\omega x)\, d\omega}{\exp (-\pi i/3)(\omega/B)^{4/3} - i} \tag{3.13}$$

where the real part is to be taken. The integral can be rewritten by setting $\omega = B \exp (-\pi i/2)t$ and $\omega = B \exp (\pi i/2)t$ for $x < 0$ and for $x > 0$ respectively. The results are

$$p_1''(x) = -\frac{1}{\pi x} - \frac{B}{\pi} \int_0^\infty \frac{\exp (Bxt)}{1 + t^{8/3}}\, dt \qquad (x < 0), \tag{3.14}$$

$$p_1''(x) = -\frac{1}{\pi x} + \frac{B}{\pi} \int_0^\infty \frac{1 - (3^{1/2}/2)t^{4/3}}{t^{8/3} - 3^{1/2}t^{4/3} + 1} \exp (-Bxt)\, dt \qquad (x > 0). \tag{3.15}$$

These expressions are the derivatives with respect to x of the results given by Stewartson [4] for subsonic flow past a convex corner. It follows from (3.14) that $p_1'(x) \sim \pi^{-1} \log |Bx| - \gamma/\pi$ as $x \to 0$, where $\gamma = 0.577 \cdots$, and that $Bp_1(0) = (3/8)2^{3/4}(1 + 2^{1/2})^{-1/2}$. For $x \to -\infty$ and $x \to +\infty$, (3.14) and (3.15) give, respectively,

$$\pi B p_1(x) \sim \Gamma(5/3)(Bx)^{-5/3} \qquad (x \to -\infty), \tag{3.16}$$

$$\pi B p_1(x) \sim \tfrac{1}{2}3^{1/2}\Gamma(1/3)(Bx)^{-1/3} + \tfrac{1}{2}\Gamma(5/3)(Bx)^{-5/3} \qquad (x \to +\infty), \tag{3.17}$$

The terms of order $x^{-1/3}$ in p_1 given above and in ψ_{11x} obtained from (2.12) are consistent, for $x \to \pm\infty$, with a term in the complex velocity (2.2) which is proportional to $(-1 + 3^{-1/2}i)Az^{-1/3}$ as $z \to 0$.

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