LAGRANGIAN FORMULATION OF BUBBLE DYNAMICS*

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Abstract. The dynamical problem of two ideal fluids separated by an interface is formulated in terms of general coordinates in Lagrangian variables. The same problem is shown to be equivalent to a Hamiltonian variational principle which takes into account explicitly the surface energy of the interface between the two fluids. The formulation is applied to the motion of slightly nonspherical bubbles. It is shown that, although we start from an entirely different set of differential equations, the results of the linear stability analysis in Eulerian formulation are recovered.

1. Introduction. Most studies of bubble dynamics deal with spherical bubbles. This is quite understandable both from the theoretical and practical points of view. On the one hand, most small bubbles are essentially spherical due to the action of surface tension; on the other hand, nonspherical bubbles are extremely difficult to analyze. However, certain questions of a theoretical and practical nature compelled us to take up the analysis of nonspherical bubbles. Thus, when theoreticians were debating the asymptotic behavior of bubble collapse [1, 2], the question could be proved only academic by the linear analysis of the bubble stability from the spherical shape [3]. The stability analysis was also applied to account for the observed jet formation as a bubble collapses [4]. Recently, as more and more detailed observation on the collapse and rebound, distortion and break-up of bubbles become available [5, 6], it seems that the understanding of the dynamics of nonspherical bubbles is even more urgent. The study on the surface waves of bubbles [8] is certainly another area relating to nonspherical bubbles.

With advances in the computer technology, it is likely that many specific questions may be best answered by numerical methods. Indeed, the complete history of collapse under specific conditions has been traced numerically [9]. Still, an analytical study should help us to gain a deeper understanding of the problem and should point to directions for new approximation schemes and better numerical approaches. A variational principle was recently formulated for the study of nonspherical bubble dynamics [10, 11], and a specific application of the variation method to the collapse of bubble has also been presented [12]. These analyses are based on the Eulerian formulation. Although the Eulerian formulation in fluid mechanics is in general more convenient to use than the Lagrangian formulation, there are certain areas in which the Eulerian formulation is not so adequate. Take the process of bubble break-up. Before we reach the point of break-up, it is evident we should have already admitted the multi-valuedness of the surface elevation as a function of Eulerian variables, whereas we could

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avoid this awkward feature in terms of Lagrangian variables at least up to the breaking point.

The formulation of fluid dynamics in terms of Lagrangian coordinates and the corresponding variational principle is nothing new at all. Here we only mention the more recent work of Seliger and Whitham [15], which lists a series of previous papers, and the article by Truesdell and Toupin [14], to serve as a general background. What is new here is the adoption of this formulation to the problem of bubble dynamics. Thus, in order to deal with the essentially curved bubble surface, a general curvilinear coordinate system instead of Cartesian system has to be adopted. Since surface tension plays an essential role in the study of bubble dynamics, surface tensors are introduced in order to compute the mean curvature of the interface. The surface energy and the motion of the bubble surface are also new features to be incorporated in the variational formulation. The general formulation and the variational principle are presented in Secs. 2 and 3. In Sec. 4, we shall apply the Lagrangian formulation to the problem of slightly nonspherical bubbles. It is shown that the results of the linear stability analysis in the Eulerian formulation can be recovered. Although the result is to be expected, it is illuminating since the mathematical equations are entirely different.

2. Lagrangian formulation for two fluids separated by an interface. Let

$$F(\mathbf{r}, t) = 0 \tag{2.1}$$

be a surface that divides the whole space into two regions G and G', each occupied by an ideal, compressible fluid. Then the governing equations in G are [15]:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{2.2}$$

$$\frac{\partial}{\partial t} (\rho s) + \nabla \cdot (\rho s \mathbf{v}) = 0, \tag{2.3}$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho} \nabla p, \tag{2.4}$$

where ρ is the density, s, the entropy, p, the pressure and $\mathbf{v}$, the velocity of the fluid in G. If we introduce the internal energy function $U(\rho, s)$, then we also have the following thermodynamic relations:

$$\partial U/\partial \rho = p/\rho^2, \tag{2.5}$$

$$\partial U/\partial s = T, \tag{2.6}$$

where T is the temperature.

An identical set of equations in primed variables can be similarly written down for the fluid in the region G'.

On the interface $F(\mathbf{r}, t) = 0$, the kinematic and dynamic interfacial conditions are as follows:

$$\frac{\partial F}{\partial t} + (\mathbf{v} \cdot \nabla)F = 0, \tag{2.7}$$

$$\frac{\partial F}{\partial t} + (\mathbf{v}' \cdot \nabla)F = 0, \tag{2.8}$$

$$p = p' = \sigma\left(\frac{1}{r_1} + \frac{1}{r_2}\right), \tag{2.9}$$

where σ is the coefficient of surface tension and r_1 and r_2 are the two principal radii of curvature at the point of interest on $F = 0$. They are taken to be positive if the centers of curvature lie on the side of G, and negative if otherwise.

In tensor notation, Eqs. (2.2), (2.3) and (2.4) can be rewritten as

$$\rho_{,t} + (\rho v^i)_{,i} = 0 \tag{2.10}$$

$$(\rho s)_{,t} + (\rho s v^i)_{,i} = 0 \tag{2.11}$$

$$v_{i,t} + v^j v_{i,j} = -\frac{1}{\rho} p_{,i} \tag{2.12}$$

where, for instance, $v^i{}_{,j}$ denotes the covariant derivative of the contravariant vector v^i.

The preceding is the Eulerian formulation of the problem. To formulate the problem in terms of Lagrangian coordinates, we need to establish the generalized coordinates to describe both the present position and the initial position of the fluid particles. Let (X^1, X^2, X^3) be the generalized coordinates at the initial moment $t = 0$ of a particle, which will have generalized coordinates (x^1, x^2, x^3) at time t. Thus in general, we have

$$X^i = X^i(x^1, x^2, x^3, t), \tag{2.13}$$

and conversely

$$x^i = x^i(X^1, X^2, X^3, \tau), \tag{2.14}$$

where $\tau = t$.

The metric tensors for the initial and present general coordinates are denoted respectively g_{JK} and g_{ik}. We shall use capital letters to designate the initial coordinates and lower-case letters to designate the present coordinates. Thus, for example, the tensor $t_K{}^i$ is a mixed tensor contravariant in the present coordinates and covariant in the initial coordinates. The covariant differentiation with respect to either coordinate is also defined accordingly. A relevant summary in more mathematical detail on mixed tensors is given in Appendix I.

We shall introduce now the material derivative with respect to time of a tensor $A{:::}$ by

$$\dot{A}{:::} \equiv \frac{\partial A{:::}}{\partial \tau} + A{:::}_{;\alpha}\dot{x}^\alpha \tag{2.15}$$

where $\dot{x}^\alpha \equiv \partial x^\alpha/\partial\tau$. Then the equation of motion (2.4) can be rewritten in terms of Lagrangian coordinates as

$$\rho x_{;J}{}^i \ddot{x}_i = -p_{;J}\,. \tag{2.16}$$

The continuity equation (2.2) can be rewritten as

$$\rho J = \rho_0\,, \tag{2.17}$$

where $\rho_0 = \rho(X^1, X^2, X^3, 0)$ is the density distribution at the initial instant, and

$$J = \frac{[\det g_{km}]^{1/2}}{[\det g_{KM}]^{1/2}} \frac{\partial(x^1, x^2, x^3)}{\partial(X^1, X^2, X^3)}$$

is the Jacobian in terms of the Cartesian coordinates.

The equation for the conservation of entropy (2.3) takes simply the form:

$$s = s_0(X^1, X^2, X^3). \tag{2.18}$$

The interface is expressed by

$$F(x^1, x^2, x^3, t) = 0. \tag{2.1}$$

At the initial instant, it is given by

$$F_0(X^1, X^2, X^3) = 0. \tag{2.19}$$

Thus, we have

$$F(x^1, x^2, x^3, t) = F_0(X^1(x^i, t), X^2(x^i, t), X^3(x^i, t)). \tag{2.20}$$

The kinematic interfacial conditions (2.7) and (2.8) are contained in (2.19). Now the initial interface can also be described by two independent parameters u^1 and u^2. Hence Eq. (2.19) is equivalent to:

$$X^1 = X^1(u^1, u^2), \qquad X^2 = X^2(u^1, u^2), \qquad X^3 = X^3(u^1, u^2). \tag{2.21}$$

Since $x^i = x^i(X^1, X^2, X^3, \tau)$, the present interface can also be represented by

$$x^1 = x^1(u^1, u^2, \tau), \qquad x^2 = x^2(u^1, u^2, \tau), \qquad x^3 = x^3(u^1, u^2, \tau). \tag{2.22}$$

With the introduction of the surface tensors $a_{\alpha\beta}$ and $b_{\alpha\beta}$ (Appendix II), the mean curvature H is expressible as

$$H = \tfrac{1}{2}a^{\alpha\beta}b_{\alpha\beta}\ . \tag{2.23}$$

The dynamic interfacial condition is thus given by

$$p - p' = -2\sigma H. \tag{2.24}$$

In the above expressions, the orientation is chosen in such a manner that the u^1-curve, u^2-curve and the outward unit normal vector from the region G, ξ^i, form a right-handed system.

Eqs. (2.16), (2.17), (2.18), (2.19), (2.24) and the corresponding equations in G' together with the thermodynamic relations constitute the Lagrangian formulation of the problem.

3. The variational principle. The previous set of equations is equivalent to the following variational principle: the flow field of the system and the interfacial conditions are such that the functional:

$$I = \int_{t_1}^{t_2} d\tau \int_V dX^1\, dX^2\, dX^3 [\det g_{KM}]^{1/2} [\tfrac{1}{2}\rho_0 g_{ij}\dot{x}^i\dot{x}^j - \rho_0 U] - \int_{t_1}^{t_2} d\tau \int_A \sigma a^{1/2}\, du^1\, du^2, \tag{3.1}$$

is an extremum, subject to the conditions:

$$\rho J = \rho_0\ , \tag{3.2}$$

$$s = s_0(X^1, X^2, X^3). \tag{3.3}$$

To prove the variational principle, let us compute term by term the expression of δJ due to the variation $\{\delta x^k\}$ from (3.1). Here we should state the further condition that $\delta x^k = 0$ at $\tau = t_1$, $\tau = t_2$ and on the boundary of the region.

Let us denote the variation of the first and second terms in the volume integral in (3.1) by δI_1 and δI_2 and the variation of the term in the surface integral by δI_s. Since g_{KM} and ρ_0 do not depend on x^k, thus we have

$$\delta I_1 = \int_{t_1}^{t_2} d\tau \int_V dX^1\, dX^2\, dX^3 [\det g_{KM}]^{1/2} \tfrac{1}{2}\rho_0 \left[\frac{\partial g_{ij}}{\partial x^k} \dot{x}^i \dot{x}^j\, \delta x^k + 2 g_{ki} \dot{x}^i\, \delta \dot{x}^k \right]. \tag{3.4}$$

We can rewrite:

$$\frac{\partial g_{ij}}{\partial x^k} \dot{x}^i \dot{x}^j = \left[\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ki}}{\partial x^j} \right] \dot{x}^i \dot{x}^j = 2[ki, j] \dot{x}^i \dot{x}^j = 2 g_{lj} \begin{Bmatrix} l \\ k\; i \end{Bmatrix} \dot{x}^i \dot{x}^j = 2 \begin{Bmatrix} l \\ k\; i \end{Bmatrix} \dot{x}^i \dot{x}_l$$

Thus after integration by parts of the second term in (3.4) with respect to τ, we obtain

$$\delta I_1 = \int_{t_1}^{t_2} d\tau \int_V dX^1\, dX^2\, dX^3 [\det g_{KM}]^{1/2} \rho_0 \left[\begin{Bmatrix} l \\ k\; i \end{Bmatrix} \dot{x}^i \dot{x}_l - \frac{\partial \dot{x}_k}{\partial \tau} \right] \delta x^k$$

$$= -\int_{t_1}^{t_2} d\tau \int_V dX^1\, dX^2\, dX^3 [\det g_{KM}]^{1/2} \rho_0 \ddot{x}_k\, \delta x^k. \tag{3.5}$$

Since g_{KM} and ρ_0 are functions of X^i only, we have

$$\delta I_2 = -\int_{t_1}^{t_2} d\tau \int_V dX^1\, dX^2\, dX^3 [\det g_{KM}]^{1/2} \rho_0\, \delta U. \tag{3.6}$$

Using the thermodynamic relations and (3.3), we obtain

$$\delta U = \frac{\partial U}{\partial \rho}\, \delta\rho = \frac{p}{\rho^2}\, \delta\rho. \tag{3.7}$$

To calculate $\delta\rho$, we note from equation (3.2) or Eq. (2.17) that

$$\rho = \rho_0 \frac{[\det g_{KM}]^{1/2}}{[\det g_{km}]^{1/2}} \Big/ \frac{\partial(x^1, x^2, x^3)}{\partial(X^1, X^2, X^3)}. \tag{3.8}$$

Let us denote

$$K = \frac{\partial(x^1, x^2, x^3)}{\partial(X^1, X^2, X^3)},$$

and let $M_k{}^J$ be the cofactor of $\partial x^k / \partial X^J$ in the expression of the determinant K. Thus

$$\delta_l{}^k K = \frac{\partial x^k}{\partial X^J} M_l{}^J.$$

Then the variation δK due to the variation $\{\delta x^k\}$ is given by

$$\delta K = M_k{}^J \frac{\partial}{\partial X^J}\, \delta x^k. \tag{3.9}$$

Since the cofactor of g_{ij} in the expression of $(\det g_{ij})$ is $(\det g_{ij}) g^{ij}$, we obtain

$$\delta[\det g_{ij}] = [\det g_{ij}] g^{lm} \frac{\partial g_{lm}}{\partial x^k}\, \delta x^k.$$

By using $g_{lm;k} = 0$, we obtain

$$\delta[\det g_{ij}] = 2[\det g_{ij}]\begin{Bmatrix} l \\ l\ k \end{Bmatrix} \delta x^k. \tag{3.10}$$

Using (3.9) and (3.10), we obtain from (3.8):

$$\begin{aligned} \delta\rho &= -\rho\left\{\frac{\delta K}{K} + \frac{1}{2}\frac{\delta[\det g_{ij}]}{[\det g_{ij}]}\right\} \\ &= -\frac{\rho}{K}\left\{M_k{}^J \frac{\partial}{\partial X^J}\delta x^k + K\begin{Bmatrix} l \\ l\ p \end{Bmatrix}\delta x^p\right\} \\ &= -\frac{\rho}{K} M_k{}^J(\delta x^k)_{;J}\ , \end{aligned} \tag{3.11}$$

since

$$\begin{aligned} M_k{}^J(\delta x^k)_{;J} &= M_k{}^J\left[\frac{\partial}{\partial X^J}(\delta x^k) + \begin{Bmatrix} k \\ l\ p \end{Bmatrix} x_{;J}{}^l\, \delta x^p\right] \\ &= M_k{}^J \frac{\partial}{\partial X^J}\delta x^k + \begin{Bmatrix} k \\ l\ p \end{Bmatrix} K\, \delta_k{}^l\, \delta x^p. \end{aligned}$$

From (3.11), (3.7), (3.8) and (3.6) we obtain

$$\delta I_2 = \int_{t_1}^{t_2} d\tau \int_V dX^1\, dX^2\, dX^3[\det g_{KM}]^{1/2} \frac{[\det g_{km}]^{1/2}}{[\det g_{KM}]^{1/2}}\, pM_k{}^J(\delta x^k)_{;J} \tag{3.12}$$

It may be noted that

$$dx^1dx^2dx^3 = KdX^1dX^2dX^3 = dX^1dX^2dX^3\ M_k{}^J/X_{;k}{}^J,$$

since we have $x_{;J}{}^l X_{;k}{}^J = \delta_k{}^l$. Therefore we can rewrite (3.12) as

$$\begin{aligned} \delta I_2 = &-\int_{t_1}^{t_2} d\tau \int_V dX^1\, dX^2\, dX^3[\det g_{KM}]^{1/2} p_{;J} M_k{}^J \frac{[\det g_{km}]^{1/2}}{[\det g_{KM}]^{1/2}}\, \delta x^k \\ &+ \int_{t_1}^{t_2} d\tau \int_V dx^1\, dx^2\, dx^3[\det g_{km}]^{1/2}(p\ \delta x^k)_{;k} \end{aligned} \tag{3.13}$$

The second integral can be transformed into the surface integral by the divergence theorem. Let us denote the outward normal vector to the interface from the region G by $\{\xi^i\}$. Let us also choose the orientations such that the u^1-curve, the u^2-curve and the normal at the point form a right-handed system. Thus we obtain

$$\begin{aligned} \delta I_2 = &-\int_{t_1}^{t_2} d\tau \int_V dX^1\, dX^2\, dX^3[\det g_{KM}]^{1/2} \frac{\rho_0}{\rho K}\, p_{;J} M_k{}^J\ \delta x^k \\ &+ \int_{t_1}^{t_2} d\tau \int_A (p - p')g_{ki}\xi^i\ \delta x^k a^{1/2}\, du^1\, du^2. \end{aligned} \tag{3.14}$$

To find δI_s, we may recall that

$$a_{\alpha\beta} = g_{ij}\frac{\partial x^i}{\partial u^\alpha}\frac{\partial x^j}{\partial u^\beta}.$$

Thus

$$\delta a_{\alpha\beta} = g_{ki} \frac{\partial x^i}{\partial u^\beta} \frac{\partial}{\partial u^\alpha} \delta x^k + g_{ik} \frac{\partial x^i}{\partial u^\alpha} \frac{\partial}{\partial u^\beta} \delta x^k + \frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^j}{\partial u^\beta} \frac{\partial g_{ij}}{\partial x^k} \delta x^k.$$

Making use of the relation that $g_{ij;k} = 0$, we obtain

$$\delta a_{\alpha\beta} = g_{ki} \frac{\partial x^i}{\partial u^\beta} (\delta x^k)_{;\alpha} + g_{ik} \frac{\partial x^i}{\partial u^\alpha} (\delta x^k)_{;\beta} . \tag{3.15}$$

Also, since

$$\delta a^{1/2} = \tfrac{1}{2} a^{-1/2}\, \delta a = \tfrac{1}{2} a^{1/2} a^{\alpha\beta}\, \delta a_{\alpha\beta} ,$$

then by making use of the two-dimensional divergence theorem and the facts that the interface is a closed surface, that the variables and their variations are single-valued, and that $g_{ik;\beta} = 0$ and $a_{\alpha\gamma;\beta} = 0$, we obtain:

$$\begin{aligned} \delta I_s &= -\int du^1\, du^2 \sigma\, \delta a^{1/2} \\ &= \int du^1\, du^2\, a^{1/2} \sigma g_{ki}\, a^{\alpha\beta} x^i{}_{;\alpha\beta}\, \delta x^k \\ &= \int du^1\, du^2 a^{1/2} \sigma g_{ki} \xi^i a^{\alpha\beta} b_{\alpha\beta}\, \delta x^k. \end{aligned} \tag{3.16}$$

From (3.5), (3.14) and (3.16) we obtain

$$\begin{aligned} \delta I &= \delta I_1 + \delta I_2 + \delta I_s \\ &= -\int_{t_1}^{t_2} d\tau \int_V dX^1\, dX^2\, dX^3 [\det g_{KM}]^{1/2} \left[\rho_0 \ddot{x}_k + \frac{\rho_0}{\rho K} M_k{}^J p_{;J} \right] \delta x^k \\ &\quad + \int_{t_1}^{t_2} d\tau \int_A du^1\, du^2 a^{1/2} g_{ki} \xi^i [p - p' + a^{\alpha\beta} b_{\alpha\beta}]\, \delta x^k. \end{aligned} \tag{3.17}$$

Thus $\delta I = 0$ is equivalent to the Eqs. (2.16) and (2.24) formulated in the previous section.

4. Slightly nonspherical bubbles. To treat bubbles which are almost spherical, it is most convenient to use the spherical polar coordinates. Let us identify (r, θ, ϕ) as the present general coordinates (x^1, x^2, x^3). They are related to the present Cartesian coordinates by the relations:

$$z^1 = r \sin\theta \cos\phi, \qquad z^2 = r \sin\theta \sin\phi, \qquad z^3 = r\cos\theta.$$

Similarly, we shall identify (R, Θ, Φ) as the initial general coordinates (X^1, X^2, X^3). The metric tensors g_{km}, g_{KM} and the related Christoffel symbols are readily found; they are listed in Appendix III for reference.

Since

$$\ddot{x}_i = g_{im} \left[\frac{\partial^2 x^m}{\partial \tau^2} + \left\{ {m \atop k\ l} \right\} \dot{x}^k \dot{x}^l \right]$$

Eqs. (2.16) in spherical coordinates are

$$\frac{\partial r}{\partial X^J}(r_{\tau\tau} - r\theta_\tau^2 - r\sin^2\theta\phi_\tau^2) + \frac{\partial\theta}{\partial X^J} r^2\left(\theta_{\tau\tau} - \frac{2}{r} r_\tau\theta_\tau - \sin\theta\cos\theta\phi_\tau^2\right)$$
$$+ \frac{\partial\phi}{\partial X^J} r^2\sin^2\theta\left(\phi_{\tau\tau} + \frac{2}{r} r_\tau\phi_\tau + 2\cot\theta\theta_\tau\phi_\tau\right) = -\frac{1}{\rho}\frac{\partial p}{\partial X^J}, \qquad J = 1, 2, 3 \tag{4.1}$$

where (X^1, X^2, X^3) are (R, Θ, Φ), as defined before, and the subscript τ denotes partial differentiation with respect to τ. It is easily verified that det $g_{km} = r^4 \sin^2\theta$, det $g_{KM} = R^4\sin^2\theta$; thus the continuity equation (2.17) becomes:

$$\frac{\rho_0 R^2 \sin^2\Theta}{\rho r^2\sin^2\theta} = \frac{\partial(r, \theta, \phi)}{\partial(R, \Theta, \Phi)} = \begin{vmatrix} r_R & \theta_R & \phi_R \\ r_\Theta & \theta_\Theta & \phi_\Theta \\ r_\Phi & \theta_\Phi & \phi_\Phi \end{vmatrix}. \tag{4.2}$$

The dynamical interfacial condition is given by (2.24):

$$p - p' = -2\sigma H, \tag{4.3}$$

and we shall leave the form of the equation as it is for the time being.

Let us, for simplicity, consider the case in which the liquid outside the bubble can be treated as incompressible. The region occupied by the liquid is the region G, and the region inside the bubble is G'. We shall first consider the case that the motion is spherically symmetric. Then we have

$$\begin{aligned} r &= r(R, \tau), \\ \theta &= \Theta, \\ \phi &= \Phi. \end{aligned} \tag{4.4}$$

From Eqs. (4.1) and (4.2), since $\rho = \rho_0$ is constant, we obtain:

$$\rho r_R r_{\tau\tau} = -p_R, \tag{4.5}$$

and

$$R^2 = r^2 r_R. \tag{4.6}$$

Integration of Eq. (4.6) leads to

$$r^3 = R^3 + D^3(t) - D_0^3, \tag{4.7}$$

where $D(t)$ can be interpreted as the present bubble radius, while D_0 is the initial radius. Substituting (4.7) in (4.5), we obtain

$$\rho\frac{R^2}{r^4}\left[D^2\ddot{D} + \frac{2D\dot{D}^2}{r^3}(R^3 - D_0^3)\right] = -p_R. \tag{4.8}$$

Integrate (4.8) from $R = D_0$ to $R = \infty$, and obtain

$$\rho(D\ddot{D} + \tfrac{3}{2}\dot{D}^2) = p(D_0, \tau) - p_\infty, \tag{4.9}$$

where p_∞ is the pressure at infinity. For this case of spherical symmetry, it is found (Appendix IV) that

$$H = 1/D. \tag{4.10}$$

Thus, we finally obtain the well-known equation of motion of the spherical bubble [15]:

$$D\ddot{D} + \frac{3}{2}\dot{D}^2 = \frac{1}{\rho}\left(p' + \frac{2\sigma}{D} - p_\infty\right). \tag{4.11}$$

Now, let us deal with the case when the deviation from the spherical symmetry is small. We write:

$$\begin{aligned} r &= F(R, \tau) + f(R, \Theta, \Phi, \tau), \\ \theta &= \Theta + g(R, \Theta, \Phi, \tau), \\ \phi &= \Phi + h(R, \Theta, \Phi, \tau), \\ p &= p_0(R, \tau) + p_1(R, \Theta, \Phi, \tau), \end{aligned} \tag{4.12}$$

where $p_0(R, \tau)$ and $F(R, \tau)$ are the solutions for p and r in the spherically symmetric case, and more explicitly by (4.7) we have

$$F^3 = R^3 + D^3 - D_0^3. \tag{4.13}$$

If we substitute (4.17) into Eqs. (4.1) and (4.2), treating f, g, h, p, as small quantities and retaining only terms up to those linear in f, g, h and p_1 , we obtain, after some straightforward calculations,

$$F_R f_{\tau\tau} + f_R F_{\tau\tau} = -\frac{1}{\rho}(p_1)_R , \tag{4.14}$$

$$F_{\tau\tau} f_\Theta + [F^2 g_\tau]_\tau = -\frac{1}{\rho}(p_1)_\Theta , \tag{4.15}$$

$$F_{\tau\tau} f_\Phi - \sin^2\Theta[F^2 h_\tau]_\tau = -\frac{1}{\rho}(p_1)_\Phi , \tag{4.16}$$

and

$$\frac{2f}{F} + \frac{f_R}{F_R} + \frac{1}{\sin\Theta}(g\sin\Theta)_\Theta + h_\Phi = 0. \tag{4.17}$$

Using Eqs. (4.15) and (4.16), we can rewrite Eq. (4.17) as

$$\begin{aligned} &\left[F^2\left(\frac{2f}{F} + \frac{f_R}{F_R}\right)_\tau\right]_\tau - F_{\tau\tau}\left[\frac{1}{\sin\Theta}(\sin\Theta f_\Theta)_\Theta + \frac{1}{\sin^2\Theta} f_{\Phi\Phi}\right] \\ &\qquad - \frac{1}{\rho}\left[\frac{1}{\sin\Theta}(\sin\Theta p_{1\Theta})_\Theta + \frac{1}{\sin^2\Theta}(p_1)_{\Phi\Phi}\right] = 0. \end{aligned} \tag{4.18}$$

Now let us express f and p_1 in terms of spherical harmonics, i.e.

$$f(R, \Theta, \Phi, \tau) = \sum_{l,m} f_{lm}(R, \tau) Y_{lm}(\Theta, \Phi),$$

$$p_1(R, \Theta, \Phi, \tau) = \sum_{l,m} p_{lm}(R, \tau) Y_{lm}(\Theta, \Phi).$$

Then Eq. (4.8) can be rewritten as:

$$\left[F^2\left(\frac{2f_{lm}}{F} + \frac{(f_{lm})_R}{F_R}\right)_\tau\right]_\tau + l(l+1)F_{\tau\tau}f_{lm} + \frac{1}{\rho}\, l(l+1)p_{lm} = 0. \tag{4.19}$$

Eqs. (4.14) and (4.19), for each l and m, form a coupled set of linear partial differential equations for f_{lm} and p_{lm}. We can easily eliminate p_{lm} and obtain a single third-order equation for f_{lm}. It is not immediately clear how we can solve this partial differential equation in a general manner. However, with the solutions for the corresponding Eulerian problem as a guide, it is possible to find a solution of the following form. Let us look for solutions such that

$$(F^2 f_{lm})_\tau = -\frac{(l+1)}{F^l}\,\psi_{lm}(\tau), \tag{4.20}$$

and we shall express $\psi_{lm}(\tau)$ as:

$$\psi_{lm} = -\frac{D^l}{l+1}\frac{d}{d\tau}\,[D^2 a_{lm}(\tau)]. \tag{4.21}$$

Since ψ_{lm} is a function of τ only, thus $(\partial/\partial R)\psi_{lm} = 0$, and we obtain

$$(F^2 f_{lm})_{\tau R} = -\frac{l}{F}\,R_R(F^2 f_{lm})_\tau\,.$$

Also, from (4.13), it is easily seen that

$$F_R = \frac{R^2}{F^2}, \qquad F_\tau = \frac{D^2\dot{D}}{F^2}, \qquad F_{\tau R} = -\frac{2F_R F_\tau}{F},$$

$$F_{\tau\tau} = \frac{1}{F^2}\left[D^2\ddot{D} + 2D\dot{D}^2 - \frac{2D^4\dot{D}^2}{F^3}\right].$$

With the aid of these relations, it is straightforward to show from direct differentiation of (4.19) with respect to R that Eq. (4.14) is recovered. Moreover, Eq. (4.19) may be rewritten as

$$\frac{l}{F}\,(F^2 f_{lm})_{\tau\tau} - \frac{lF_\tau}{F^2}\,(F^2 f_{lm})_\tau - \frac{l(l+1)}{F^2}\left(D^2\ddot{D} + 2D\dot{D}^2 - \frac{2D^4\dot{D}^2}{F^3}\right)f_{lm} = \frac{l(l+1)}{\rho}\,p_{lm}\,. \tag{4.22}$$

At the bubble surface, we have $R = D_0$ or $F = D$; thus we obtain:

$$D\ddot{a}_{lm} + 3\dot{D}\dot{a}_{lm} - (l-1)\ddot{D}a_{lm} = \frac{(l+1)}{\rho}\,p_{lm}(D,\tau). \tag{4.23}$$

To find the pressure at the bubble surface, we need to know the value of the mean curvature H, which is computed explicitly in Appendix IV for the linear case. In terms of the spherical harmonic expansions, we get

$$H = \frac{1}{D} + \frac{1}{D^2}\sum_{l,m}\tfrac{1}{2}(l+2)(l-1)a_{lm}\,. \tag{4.24}$$

If the internal pressure p' remains unchanged with the perturbation, then from (4.3) we obtain

$$p_{lm} = -(2\sigma)\frac{1}{2D^2}(l+2)(l-1)\,a_{lm}\,. \tag{4.25}$$

Hence Eq. (4.23) becomes

$$\ddot{a}_{lm} + \frac{3\dot{D}}{D}\dot{a}_{lm} - \left[\frac{(l-1)\ddot{D}}{D} - (l-1)(l+1)(l+2)\frac{\sigma}{\rho D^3}\right]a_{lm} = 0, \tag{4.26}$$

which is exactly the same equation as that derived from the Eulerian formulation [3].

With a_{lm} solved from Eq. (4.26), we can then compute f_{lm} from Eqs. (4.20) and (4.21), p_{lm} from Eq. (4.19), and then calculate g and h from Eqs. (4.15) and (4.16) respectively to obtain the complete flow field of the system.

5. Discussion. The bubble surface is characterized by the equation $R = D_0$. When we have solved the complete flow problem and obtained $r(R, \Theta, \Phi, \tau)$, $\theta(R, \Theta, \Phi, \tau)$ and $\phi(R, \Theta, \Phi, \tau)$, the elimination of Θ and Φ from the expressions of $r(D_0, \Theta, \Phi, \tau)$, $\theta(D_0, \Theta, \Phi, \tau)$ and $\phi(D_0, \Theta, \Phi, \tau)$ will yield the relation

$$r = r_s(\theta, \phi, \tau; D_0). \tag{5.1}$$

The last equation characterizes the bubble surface in the present coordinates. When the deviation from sphericity is indeed small enough, this result will agree with that obtained from the Eulerian formulations. Now the linearized solutions are often extrapolated into nonlinear region, sometimes yielding fairly satisfactory results [9]. It is evident that the extrapolation from (5.1) will in general be different from the corresponding Eulerian extrapolation. This aspect, when explored in more detail, could lead to very interesting results.

The variational formulation, as we know, not only offers an alternate formulation of the problem, but also opens up new avenues of useful approximations to treat nonlinear problems. The success of the approximation scheme depends heavily on the choice of the trial solutions we choose to apply. The linearized solution presented in the last section should give us a good starting point in search of reasonable trial solutions.

Finally, we should point out that although our motivation is the study of bubble dynamics, the previous analyses can apply to the dynamics of a liquid drop or general water waves as well.

Appendix I: mixed tensors. A brief relevant summary of mixed tensors follows (see [14]). Let the initial (Lagrangian) configuration be described by the Cartesian coordinates Z or the general coordinates X. Let the present (Eulerian) configuration be described by the Cartesian coordinates z or the general coordinates x. The indices in capital letters refer to Lagrangian coordinates and the indices in lower-case letters refer to Eulerian coordinates. The metric tensors are defined accordingly by:

$$g_{KM} = \frac{\partial Z^P}{\partial X^K}\frac{\partial Z^P}{\partial X^M}, \qquad g_{km} = \frac{\partial z^p}{\partial x^k}\frac{\partial z^p}{\partial x^m}. \tag{AI.1}$$

A mixed tensor $T_K{}^k$ transforms contravariantly in Eulerian coordinates and covariantly in Lagrangian coordinates. The partial covariant derivatives are defined in a manner as follows:

$$T_{K,M}{}^k = \frac{\partial T_K{}^k}{\partial X^M} - \left\{\begin{matrix} P \\ KM \end{matrix}\right\}T_P{}^k, \qquad T_{K,m}{}^k = \frac{\partial T_K{}^k}{\partial x^m} + \left\{\begin{matrix} k \\ mp \end{matrix}\right\}T_K{}^p. \tag{AI.2}$$

The total covariant derivatives are defined as follows:

$$T::::_{;K} = T::::_{,K} + T::::_{,k}x_{;K}{}^{k} \qquad \text{(AI.3)}$$

$$T::::_{;k} = T::::_{,k} + T::::_{,K}X_{;k}{}^{K}$$

where

$$x_{;K}{}^{k} \equiv x_{;K}{}^{k} \equiv \frac{\partial x^{k}}{\partial X^{K}}, \qquad X_{;k}{}^{K} \equiv X_{,k}{}^{K} \equiv \frac{\partial X^{K}}{\partial x^{k}}, \qquad \text{(AI.4)}$$

and

$$x_{;K}{}^{l}X_{;h}{}^{K} = \delta_{k}{}^{l}, \qquad X_{;k}{}^{L}x_{;K}{}^{k} = \delta_{K}{}^{L}. \qquad \text{(AI.5)}$$

Also we have

$$T::::_{;k} = T::::_{;K}X_{;k}{}^{K}, \qquad T::::_{;K} = T::::_{;k}x_{K}{}^{k}. \qquad \text{(AI.6)}$$

The volume elements are given by

$$dV = [\det g_{KM}]^{1/2}dX^{1}dX^{2}dX^{3},\ dv = [\det g_{km}]^{1/2}dx^{1}dx^{2}dx^{3}. \qquad \text{(AI.7)}$$

Thus the Jacobian is given by

$$J = \frac{\partial(z^{1}, z^{2}, z^{3})}{\partial(Z^{1}, Z^{2}, Z^{3})} = \frac{[\det g_{km}]^{1/2}}{[\det g_{KM}]^{1/2}} \frac{\partial(x^{1}, x^{2}, x^{3})}{\partial(X^{1}, X^{2}, X^{3})}. \qquad \text{(AI.8)}$$

It is also readily verified that:

$$g_{km;l} = g_{km;L} = g_{KM;l} = g_{KM;L} = 0. \qquad \text{(AI.9)}$$

Appendix II: surface tensors (see [16]). Let the surface be defined by

$$x^{i} = x^{i}(u^{1}, u^{2}), \qquad i = 1, 2, 3, \qquad \text{(AII.1)}$$

where u^1 and u^2 are two independent parameters. The line element ds is given by

$$(ds)^{2} = a_{\alpha\beta}du^{\alpha}du^{\beta} \qquad \text{(AII.2)}$$

where

$$a_{\alpha\beta} = g_{ij}\frac{\partial x^{i}}{\partial u^{\alpha}}\frac{\partial x^{j}}{\partial u^{\beta}} \qquad \text{(AII.3)}$$

is the surface metric tensor. We shall use the Greek letters to designate the surface tensorial indices. The contravariant metric tensor $a^{\alpha\gamma}$ is defined by

$$a_{\alpha\beta}a^{\alpha\gamma} = \delta_{\beta}{}^{\gamma} \qquad \text{(AII.4)}$$

Set

$$a = \det a_{\alpha\beta} \qquad \text{(AII.5)}$$

$$e_{11} = e_{22} = 0, \qquad e_{12} = 1, \qquad e_{21} = -1;$$

then

$$\epsilon_{\alpha\beta} = a^{1/2}e_{\alpha\beta} \quad \text{and} \quad \epsilon^{\alpha\beta} = a^{-1/2}e_{\alpha\beta} \qquad \text{(AII.6)}$$

are surface tensors. The quantity

$$x_{;\alpha}{}^{i} \equiv \frac{\partial x^i}{\partial u^\alpha} \qquad (i = 1, 2, 3; \quad \alpha = 1, 2) \tag{AII.7}$$

is a mixed tensor contravariant to space and covariant to the surface. The covariant derivative of such a mixed tensor $A_\alpha{}^i$ with respect to u^β is given by

$$A_{\alpha;\beta}{}^{i} \equiv \frac{\partial A_\alpha{}^i}{\partial u^\beta} + \left\{\begin{matrix} i \\ j\ k \end{matrix}\right\} A_\alpha{}^j x_{;\beta}{}^k - \left\{\begin{matrix} \epsilon \\ \alpha\ \beta \end{matrix}\right\} A_\epsilon{}^i. \tag{AII.8}$$

Thus, for instance:

$$x_{;\alpha\beta}{}^{i} = \frac{\partial^2 x^i}{\partial u^\alpha\, \partial u^\beta} + \left\{\begin{matrix} i \\ j\ k \end{matrix}\right\} x_{;\alpha}{}^j x_{;\beta}{}^k - \left\{\begin{matrix} \epsilon \\ \alpha\ \beta \end{matrix}\right\} x_{;\epsilon}{}^i. \tag{AII.9}$$

Let ξ^i be the unit normal vector with the orientation so that the u^1-curve, u^2-curve and the normal form a right-handed system. Then

$$\xi_i = \tfrac{1}{2}\epsilon^{\alpha\beta}\epsilon_{ijk}x_{;\alpha}{}^j x_{;\beta}{}^k \tag{AII.10}$$

where $\epsilon_{ijk} = [\det g_{lm}]^{1/2} e_{ijk}$ and $e_{ijk} = 0$, if $i \neq j \neq k \neq i$, $e_{123} = e_{231} = e_{312} = 1$, $e_{132} = e_{213} = e_{321} = -1$. Let

$$b_{\alpha\beta} = \xi_i x_{;\alpha\beta}{}^i\ ; \tag{AII.11}$$

then the mean curvature H is defined as

$$H = \tfrac{1}{2}a^{\alpha\beta}b_{\alpha\beta}\ . \tag{AII.12}$$

Now

$$a_{\alpha\beta;\gamma} = \frac{\partial a_{\alpha\beta}}{\partial u^\gamma} - \left\{\begin{matrix} \epsilon \\ \gamma\ \alpha \end{matrix}\right\} a_{\epsilon\beta} - \left\{\begin{matrix} \epsilon \\ \gamma\ \beta \end{matrix}\right\} a_{\epsilon\alpha} = 0. \tag{AII.13}$$

Thus

$$\left\{\begin{matrix} \alpha \\ \gamma\ \alpha \end{matrix}\right\} = \frac{1}{2}\, a^{\alpha\beta}\, \frac{\partial a_{\alpha\beta}}{\partial u^\gamma} = \frac{1}{2a}\, \frac{\partial a}{\partial u^\gamma}.$$

Since

$$C_{;\alpha}{}^{\alpha} = \frac{\partial C^\alpha}{\partial u^\alpha} + \left\{\begin{matrix} \alpha \\ \gamma\ \alpha \end{matrix}\right\} C^\gamma = \frac{\partial C^\alpha}{\partial u^\alpha} + \frac{1}{2a}\,\frac{\partial a}{\partial u^\alpha}\, C^\alpha,$$

we have

$$\int_A C_{;\alpha}{}^{\alpha} a^{1/2}\, du^1\, du^2 = \int_A \frac{\partial}{\partial u^\alpha}\, [a^{1/2}C^\alpha]\, du^1\, du^2. \tag{AII.14}$$

The right-hand side can be rewritten as a line integral if the surface A is bounded by a curve, or vanishes if A is a closed surface. This is the two-dimensional verison of the divergence theorem.

Appendix III: spherical coordinates. Let (x^1, x^2, x^3) be identified as the spherical polar coordinates (r, θ, ϕ); then the components of the matric tensor are given by:

$$g_{rr} = 1, \quad g_{\theta\theta} = r^2, \quad g_{\phi\phi} = r^2 \sin^2 \theta, \quad g_{r\theta} = g_{r\phi} = g_{\theta\phi} = 0. \tag{AIII.1}$$

The only non-zero Christoffel symbols of the second kind are

$$\begin{aligned} &\begin{Bmatrix} r \\ \theta\ \theta \end{Bmatrix} = -r, \quad \begin{Bmatrix} r \\ \phi\ \phi \end{Bmatrix} = r \sin^2 \theta, \quad \begin{Bmatrix} \theta \\ r\ \theta \end{Bmatrix} = \begin{Bmatrix} \theta \\ \theta\ r \end{Bmatrix} = \frac{1}{r}, \\ &\begin{Bmatrix} \theta \\ \phi\ \phi \end{Bmatrix} = -\sin \theta \cos \theta, \quad \begin{Bmatrix} \phi \\ r\ \phi \end{Bmatrix} = \begin{Bmatrix} \phi \\ \phi\ r \end{Bmatrix} = \frac{1}{r}, \\ &\begin{Bmatrix} \phi \\ \theta\ \phi \end{Bmatrix} = \begin{Bmatrix} \phi \\ \phi\ \theta \end{Bmatrix} = \cot \theta. \end{aligned} \tag{AIII.2}$$

Similar expressions hold for g_{KM} and $\{{}_K{}^J{}_L\}$.

Appendix IV: the mean curvature. Let the initial shape of the bubble be a sphere of radius D_0; then a convenient set of the surface parameters are $u^1 = \Theta$, $u^2 = \Phi$. The present surface of the bubble is described by the following set of equations:

$$\begin{aligned} r &= r_0(\Theta, \Phi, \tau) \equiv r(D_0, \Theta, \Phi, \tau), \\ \theta &= \theta_0(\Theta, \Phi, \tau) \equiv \theta(D_0, \Theta, \Phi, \tau), \\ \phi &= \phi_0(\Theta, \Phi, \tau) \equiv \phi(D_0, \Theta, \Phi, \tau). \end{aligned} \tag{AIV.1}$$

The surface metric tensors are given by

$$\begin{aligned} a_{\Theta\Theta} &= \left(\frac{\partial r_0}{\partial \Theta}\right)^2 + r_0^2\left(\frac{\partial \theta_0}{\partial \Theta}\right)^2 + r_0^2 \sin^2 \theta_0\left(\frac{\partial \phi_0}{\partial \Theta}\right)^2, \\ a_{\Phi\Phi} &= \left(\frac{\partial r_0}{\partial \Phi}\right)^2 + r_0^2\left(\frac{\partial \theta_0}{\partial \Phi}\right)^2 + r_0^2 \sin^2 \theta_0\left(\frac{\partial \phi_0}{\partial \Phi}\right)^2, \\ a_{\Theta\Phi} &= a_{\Phi\Theta} = \frac{\partial r_0}{\partial \Theta}\frac{\partial r_0}{\partial \Phi} + r_0^2 \frac{\partial \theta_0}{\partial \Theta}\frac{\partial \theta_0}{\partial \Phi} + r_0^2 \sin^2 \theta_0 \frac{\partial \phi_0}{\partial \Theta}\frac{\partial \phi_0}{\partial \Phi}. \end{aligned} \tag{AIV.2}$$

When the deviation from the spherical surface is small, we have

$$\begin{aligned} r_0 &= D(\tau) + f_0(\Theta, \Phi, \tau), \\ \theta_0 &= \Theta + g_0(\Theta, \Phi, \tau), \\ \phi_0 &= \Phi + h_0(\Theta, \Phi, \tau), \end{aligned} \tag{AIV.3}$$

where $f_0 = f(D_0, \Theta\ \Phi, \tau)$, etc. Thus, if we are interested only in terms up to the first order of small quantities f_0, g_0, and h_0, we have

$$\begin{aligned} a_{\Theta\Theta} &\cong D^2\left(1 + \frac{2f_0}{D} + 2\frac{\partial g_0}{\partial \Theta}\right), \\ a_{\Phi\Phi} &\cong D^2 \sin^2 \Theta\left(1 + \frac{2f_0}{D} + 2g_0 \cot \Theta + 2\frac{\partial h_0}{\partial \Phi}\right), \end{aligned} \tag{AIV.4}$$

$$a_{\Theta\Phi} \cong D^2\left(\frac{\partial g_0}{\partial \Phi} + \sin^2\Theta \frac{\partial h_0}{\partial \Theta}\right),$$

and

$$a \cong D^4 \sin^2\Theta\left(1 + \frac{4f_0}{D} + 2g_0 \cot\Theta + 2\frac{\partial g_0}{\partial\Theta} + 2\frac{\partial h_0}{\partial \Phi_0}\right) \tag{AIV.5}$$

Hence

$$a^{\Theta\Theta} \cong \frac{1}{D^2}\left(1 - \frac{2f_0}{D} - 2\frac{\partial g_0}{\partial\Theta}\right)$$

$$a^{\Phi\Phi} \cong \frac{1}{D^2\sin^2\Theta}\left(1 - \frac{2f_0}{D} - 2g_0\cot\Theta - 2\frac{\partial h_0}{\partial\Phi}\right), \tag{AIV.6}$$

$$a^{\Theta\Phi} \cong -\frac{1}{D^2\sin^2\Theta}\left(\frac{\partial g_0}{\partial\Phi} + \sin^2\Theta\frac{\partial h_0}{\partial\Theta}\right).$$

With the identification $(x^1, x^2, x^3) = (r_0, \theta_0, \phi_0)$, we also have

$$\det g_{lm} \cong D^4\sin^2\Theta\left(1 + \frac{4f_0}{D} + 2g_0\cot\Theta\right). \tag{AIV.7}$$

From (AII.10), we then obtain

$$\xi_i = \left[\frac{\det g_{lm}}{a}\right]^{1/2} e_{ijk}x_{;\Theta}{}^{j}x_{;\Phi}{}^{k}$$

$$\cong \left[1 - \frac{\partial g_0}{\partial\Theta} - \frac{\partial h_0}{\partial\Phi}\right]e_{ijk}x_{;\Theta}{}^{j}x_{;\Phi}{}^{k}. \tag{AIV.8}$$

Thus

$$\xi_1 \cong 1, \qquad \xi_2 \cong -\partial f_0/\partial\Theta, \qquad \xi_3 \cong -\partial f_0/\partial\Phi. \tag{AIV.9}$$

Now, to obtain $b_{\alpha\beta}$ or $x_{;\alpha\beta}{}^{i}$, we need to compute the various Christoffel symbols. To get the value of the mean curvature H up to $O(\epsilon)$, where ϵ is a small parameter to characterize the smallness of f_0, g_0 and h_0, it is sufficient to note after some calculation that

$$\begin{Bmatrix}\Theta \\ \Theta\ \Theta\end{Bmatrix} \cong \begin{Bmatrix}\Phi \\ \Theta\ \Theta\end{Bmatrix} \cong \begin{Bmatrix}\Phi \\ \Phi\ \Phi\end{Bmatrix} \cong \begin{Bmatrix}\Theta \\ \Theta\ \Phi\end{Bmatrix} = O(\epsilon) \tag{AIV.10}$$

$$\begin{Bmatrix}\Theta \\ \Phi\ \Phi\end{Bmatrix} = -\sin\Theta\cos\Theta + O(\epsilon), \qquad \begin{Bmatrix}\Phi \\ \Theta\ \Phi\end{Bmatrix} = \cot\Theta + O(\epsilon).$$

Then with the aid of (AIII.2) and (AIII.10), we obtain ((AII.9)) that

$$x_{;\Theta\Theta}{}^{1} = -D\left(1 + \frac{f_0}{D} + 2\frac{\partial g_0}{\partial\Theta} - \frac{1}{D}\frac{\partial^2 f_0}{\partial\Theta^2}\right) + O(\epsilon^2),$$

$$x_{;\Phi\Phi}{}^{1} = -D\sin^2\Theta\left(1 + \frac{f_0}{D} - \frac{\cot\Theta}{D}\frac{\partial f_0}{\partial\Theta} + 2g_0\cot\Theta + 2\frac{\partial h_0}{\partial\Phi} - \frac{1}{D\sin^2\Theta}\frac{\partial^2 f_0}{\partial\Phi^2}\right),$$

$$x_{;\Theta\Phi}{}^{1} \cong x_{;\Theta\Theta}{}^{2} \cong x_{;\Phi\Phi}{}^{2} \cong x_{;\Theta\Phi}{}^{2} \cong x_{;\Theta\Theta}{}^{3} \cong x_{;\Phi\Phi}{}^{3} \cong x_{;\Theta\Phi}{}^{3} = O(\epsilon). \tag{AIV.11}$$

In view of (AIV.9, 11), we obtain from (AII.11) that

$$b_{\alpha\beta} = x_{;\alpha\beta}{}^1 + O(\epsilon^2). \tag{AIV.12}$$

Using (AIV.6, 9, 12), we obtain from (AII.12) that

$$\begin{aligned} H &= \tfrac{1}{2}[a^{\Theta\Theta}x_{;\Theta\Theta}{}^1 + a^{\Phi\Phi}x_{;\Phi\Phi}{}^1] + O(\epsilon^2) \\ &= \frac{1}{D}\left\{1 - \frac{1}{2D}\left[2f_0 + \frac{\partial^2 f_0}{\partial\Theta^2} + \cot\Theta\frac{\partial f_0}{\partial\Theta} + \frac{1}{\sin^2\Theta}\frac{\partial^2 f_0}{\partial\Phi^2}\right]\right\}. \end{aligned} \tag{AIV.13}$$

If $f_0(\Theta, \Phi, \tau)$ can be expressed as

$$f_0(\Theta, \Phi, \tau) = \sum_{l,m} a_{lm}(\tau) Y_{lm}(\Theta, \Phi), \tag{AIV.14}$$

then we obtain:

$$H = \frac{1}{D} + \frac{1}{D^2}\sum_{l,m}\tfrac{1}{2}(l+2)(l-1)a_{lm}\,. \qquad \text{AIV.15)}$$

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