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CONVERGENCE IN ITERATIVE DESIGN*

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Abstract. Earlier results for the monotone convergence of iterative design for a simple model of the truss problem are generalized for the case in which the objective function is a homogeneous, convex function.

Introduction. This paper deals with the question of the convergence of an iterative structural design algorithm to an optimal design. Since there is an abundance of literature available to establish the state of the art in structural design (e.g. [1]), only a brief attempt will be made here to relate the present work to the literature.

Optimal structural design as discussed here is a mathematical programming problem and has received considerable attention as such. In general, algorithmic difficulties have impeded the automation of structural design, and a rather curious situation has developed in which it is possible to deal formally only with structures of modest size while the designer deals routinely with systems which involve literally thousands of parameters. (The designer, of course, deals with these large systems heuristically.)

The procedure commonly used by designers is an analysis-redesign procedure which is referred to here as "iterative design." In this procedure the designer makes some assumption concerning the initial values of the system parameters and analyzes the system. On the basis of this analysis the system parameters are modified and the system again is analyzed. This procedure is repeated until a "satisfactory" design is achieved.

An interesting characteristic of iterative design is the fact that while the redesign rules usually deal with questions of strength (e.g. keeping the stress less than the allowable), they produce satisfactory and sometimes optimal results in terms of weight. In a sequence of papers [2, 3] the phenomenon of iterative design has been studied. Of particular interest here is [4] in which it was possible to show global convergence for a simple model of the truss problem.

In [2], the truss problem has been generalized to cases such as frames and sandwich plates which can be described by a mathematical programming problem with a convex, homogeneous objective function and linear equality constraints. In this paper, convergence of the algorithm described in [2] is discussed and shown to be monotone globally in one case and locally in general. This extends the result obtained in [4].

Iterative design. Using the notation of (2), the optimal structural design problem is to find an optimal F which will

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$$\text{minimize } t(F) \text{ subject to } \tilde{N}F = P. \tag{1}$$

Here t is a scalar, convex, homogeneous, positive function of degree one, F is an $n \times 1$ (row $\times$ column) matrix of independent variables, and $\tilde{N}F = P$ represents a given system of m ($m \leq n$) linear equality constraints which any feasible F must satisfy. In Eq. (1), N and P may be regarded to be $n \times m$ and $m \times 1$ matrices which are given and the tilde is used to indicate the matrix transpose. For the cases discussed in [2], the system (1) refers to minimizing weight (in terms of force variables) while satisfying the requirements of equilibrium.

Convex programming with linear constraints is, of course, not new. Zangwill [5], for example, has derived the Kuhn–Tucker conditions for an optimal point $\bar{F}$ as

$$\tilde{N}F = P \quad \text{and} \quad \nabla t(\bar{F}) = N\delta \tag{2}$$

from the Lagrangian function

$$L = t(F) + \tilde{\delta}(P - \tilde{N}F) \tag{3}$$

where δ is an $m \times 1$ matrix of Lagrange multipliers and the symbol $\nabla t(\bar{F})$ is used to represent the $n \times 1$ gradient matrix of the function t evaluated at the point $\bar{F}$, i.e. $(\nabla t(\bar{F}))_i = \partial t/\partial F_i|_{F=\bar{F}}$. Beyond this Zangwill is concerned with the application of gradient techniques to this problem.

In [2], Eq. (2) is replaced by

$$\tilde{N}\bar{F} = P \quad \text{and} \quad 2t\nabla t(\bar{F}) = 2tN\delta \tag{4}$$

which can be rewritten as

$$\tilde{N}\bar{F} = P \quad \text{and} \quad \nabla\varphi = 2\varphi^{1/2}N\delta \tag{5}$$

where

$$\varphi = t^2 \Rightarrow \nabla\varphi = 2t\nabla t \tag{6}$$

in order to avoid the Hessian matrix of t which is singular (since t is homogeneous of degree 1). $\nabla\varphi$ is then linearized at point F' as

$$\nabla\varphi(F) \sim \nabla\varphi(F') + H(F')(F - F') = H(F')F. \tag{7}$$

Here H is the positive definite Hessian matrix of φ. Eq. (5) is now solved iteratively as

$$\tilde{N}F^{(n)} = P \quad \text{and} \quad H^{(n-1)}F^{(n)} = 2(\phi^{(n-1)})^{1/2}N\delta^{(n)} \tag{8}$$

or

$$\tilde{N}F^{(n)} = P \quad \text{and} \quad F^{(n)} = K^{(n-1)}N\delta^{(n)} \tag{9}$$

where

$$K^{(n-1)} = 2(\phi^{(n-1)})^{1/2}(H^{(n-1)})^{-1}. \tag{10}$$

Eq. (9) then has the form of a linear structural analysis problem.

It is the purpose of this paper to discuss the convergence of the procedure described in Eq. (9).

Monotone convergence. In this section it will be shown that the objective function t behaves monotonically under the algorithm described in the preceding section.

Global convergence is shown in the case in which the Hessian matrix H is constant; otherwise the convergence is local.

The convergence proof makes use of four lemmas which are presented first. These lemmas in turn use some results for convex, homogeneous functions (6) which are now listed.

Since t is a convex, homogeneous function of degree 1, for any x, $y \neq 0$

$$t(x) = \tilde{x}\nabla t(x) \geq \tilde{x}\nabla t(y) \Rightarrow \tilde{x}\epsilon \leq 0 \tag{11}$$

where $\epsilon = \nabla t(y) - \nabla t(x)$. Since t is homogeneous of degree 1, $\varphi = t^2$ is homogeneous of degree 2 and

$$2\varphi = \tilde{x}\nabla\varphi \Rightarrow \nabla\varphi = Hx. \tag{12}$$

It follows that

$$t(x) = \tilde{\nabla} t(x)K(x)\nabla t(x). \tag{13}$$

LEMMA 1. (A local result.) $\tilde{\nabla} t^{(n)}K^{(n)}\nabla t^{(n)} \geq \tilde{\nabla} t^{(n-1)}K^{(n)}\nabla t^{(n-1)}$ when $|\nabla t^{(n)} - \nabla t^{(n-1)}|$ is sufficiently small.

Proof. Let $\epsilon = \nabla t^{(n-1)} - \nabla t^{(n)}$. It follows that $\tilde{\nabla} t^{(n-1)}K^{(n)}\nabla t^{(n-1)} = \tilde{\nabla} t^{(n)}K^{(n)}\nabla t^{(n)} + 2\tilde{\epsilon}K^{(n)}\nabla t^{(n)} + \tilde{\epsilon}K^{(n)}\epsilon$. Since the last term in the above equation can be made as small as desired by requiring $|\epsilon|$ to be sufficiently small and since the next-to-last term is negative by virtue of Eq. (11), the lemma is proved.

CONJECTURE. Lemma 1 is valid for any $\nabla t^{(n)}$ and $\nabla t^{(n-1)}$.

LEMMA 2. (Virtual work.) $\tilde{F}^{(n)}\,\Delta^{(n)} = \tilde{F}^{(n-1)})\,\Delta^{(n)}$ where $\Delta \equiv N\delta \Rightarrow \Delta^{(n)} = N\delta^{(n)}$.

Proof. $\tilde{N}F^{(n)} = \tilde{N}F^{(n-1)} = P \Rightarrow \tilde{\delta}^{(n)}\tilde{N}F^{(n)} = \tilde{\delta}^{(n)}\tilde{N}F^{(n-1)}$ or $\tilde{\Delta}^{(n)}F^{(n)} = \tilde{\Delta}^{(n)}F^{(n-1)}$.

LEMMA 3. $\tilde{\Delta}^{(n)}K^{(n-1)}\Delta^{(n)} \leq t(F^{(n)}) = t^{(n)}$.

Proof.

$$\begin{aligned}
t^{(n)} &= \tilde{F}^{(n)}\,\nabla t^{(n)} \\
&\geq \tilde{F}^{(n)}\,\nabla t^{(n-1)} && \text{(using Eq. (11))} \\
&= \tilde{F}^{(n)}(K^{(n-1)})^{-1}K^{(n-1)}\,\nabla t^{(n-1)} \\
&= \tilde{\Delta}^{(n)}\,F^{(n-1)} && \text{(using Eq. (9))} \\
&= \tilde{\Delta}^{(n)}\,F^{(n)} && \text{(using Lemma 2)} \\
&= \tilde{\Delta}^{(n)}\,K^{(n-1)}\,\Delta^{(n)}.
\end{aligned}$$

LEMMA 4. $\tilde{\Delta}^{(n)}\,K^{(n-1)}\,\Delta^{(n)} \leq t^{(n-1)}$.

Proof. Since $F^{(n)}$ minimizes the strain energy given $K^{(n-1)}$, it follows that

$$\tilde{F}^{(n)}(K^{(n-1)})^{-1}F^{(n)} \leq \tilde{F}^{(n-1)}(K^{(n-1)})^{-1}F^{(n-1)},$$

$$\tilde{\Delta}^{(n)}\,F^{(n)} \leq \tilde{\nabla} t^{(n-1)}\,K^{(n-1)}(K^{(n-1)})^{-1}K^{(n-1)}\,\nabla t^{(n-1)},$$

$$\tilde{\Delta}^{(n)}\,K^{(n-1)}\,\Delta^{(n)} \leq \tilde{\nabla} t^{(n-1)}\,K^{(n-1)}\,\nabla t^{(n-1)} = t^{(n-1)}.$$

THEOREM. $t^{(n)} \leq t^{(n-1)}$ when either 1) $|\nabla t^{(n)} - \nabla t^{(n-1)}|$ is sufficiently small or 2) H is constant.

Proof.

$$
\begin{aligned}
t^{(n)} &= \tilde{F}^{(n)} \nabla t^{(n)} = \tilde{\Delta}^{(n)} K^{(n-1)} \nabla t^{(n)} \\
(t^{(n)})^2 &= (\tilde{\Delta}^{(n)} K^{(n-1)} \nabla t^{(n)})^2 \\
&\leq (\tilde{\Delta}^{(n)} K^{(n-1)} \Delta^{(n)})(\tilde{\nabla} t^{(n)} K^{(n-1)} \nabla t^{(n)}) \quad \text{(using Schwarz's inequality)} \\
&\leq t^{(n)}(\tilde{\nabla} t^{(n)} K^{(n-1)} \nabla t^{(n)}) \quad \text{(using Lemma 3)} \\
&\leq t^{(n)}(\tilde{\nabla} t^{(n-1)} K^{(n-1)} \nabla t^{(n-1)}) \quad \text{(using Lemma 1)} \\
&\leq t^{(n)} t^{(n-1)},
\end{aligned}
$$

or

$$t^{(n)} \leq t^{(n-1)}.$$

This is the direct generalization of the proof given in (4) and uses the local result of Lemma 1. For the case in which H is constant, i.e.

$$H^{(n)} = H^{(n-1)} \Rightarrow K^{(n-1)} = K^{(n)} t^{(n-1)}/t^{(n)}$$

it is possible to proceed using Lemma 4 as follows:

$$
\begin{aligned}
(t^{(n)})^2 &\leq (\tilde{\Delta}^{(n)} K^{(n-1)} \nabla t^{(n)})^2 \\
&\leq (\tilde{\Delta}^{(n)} K^{(n-1)} \Delta^{(n)})(\nabla t^{(n)} K^{(n-1)} \nabla t^{(n)}) \\
&\leq t^{(n-1)}(\nabla t^{(n)} K^{(n)} \nabla t^{(n)}) t^{(n-1)}/t^{(n)} \quad \text{(using Lemma 4)} \\
&\leq (t^{(n-1)})^2.
\end{aligned}
$$

It may be noted that the requirement of H being constant does not imply that t is linear (see, e.g. [6]).

Some practical considerations. The algorithm discussed here is a Newtonian scheme and suffers the usual difficulties associated with such schemes. In the two cases in which the authors have applied this algorithm the objective function has been separable, i.e. $t(F)$ has the form

$$t(F) = \sum_i t_i(F_i), \tag{14}$$

and in both cases the dimension of F_i has been small enough to allow the required Hessian matrices to be computed explicitly by hand so that recourse to numerical differentiation was not necessary. When this is true, at each step in the calculation it is only necessary to solve a sparse system of linear algebraic equations. It is here that the effectiveness of the method lies since highly efficient techniques are available to deal with, sparse systems.

It should be noted that in the separable case it is sometimes convenient to replace $2t \nabla t(\bar{F}) = 2tN\delta$ with $2t_i \nabla t_i(\bar{F}_i) = 2t_i(N\delta)_i$ $(i = 1, \cdots)$ in Eq. (4).

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