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A STABILITY THEOREM FOR DIFFUSION PROBLEMS WITH SHARPLY CHANGING TEMPERATURE-DEPENDENT COEFFICIENTS*

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1. Introduction. In a previous work [1] we considered a nonlinear heat conduction problem and proved the continuous dependence of its solution on the heat capacity and thermal conductivity of the medium. The results we obtained there can be summarized as follows: the L^2 norm of the difference between temperatures corresponding to the same initial and boundary data and to different thermal coefficients tends to zero together with the L^1 norms of the differences between the respective coefficients.

However, considerable drawbacks arise in applying the foregoing estimate to some special practical cases. Indeed, the condition that the L^1 norms of the differences between thermal coefficients are small could be hardly verified when sharp variations in thermal properties occur over a narrow temperature range: as a matter of fact, the evaluation of those norms can be performed only with large inaccuracy, owing to the difficulties involved in measuring heat capacity and thermal conductivity in such temperature intervals. Moreover, the estimate given in [1] is influenced quite critically by the maximum value of the heat capacities and it becomes practically useless when high peaks in specific heat occur.

On the other hand, this is the actual experimental situation encountered, for instance, when materials dealt with undergo a change in physical structure and this change does not take place at a certain temperature but over a small temperature interval: this is the typical behavior of impure substances in their phase changes.

In order to provide a meaningful stability theorem for these special cases, a more sophisticated approach is needed, starting from a new weak formulation of the heat conduction problem considered, which will be introduced in the next section.

In Sec. 3 the main result is stated and discussed in view of applications. As a matter of fact, on the basis of our stability theorem thermal fields in impure substances undergoing change of phase can be computed using approximate thermal coefficients, evaluated by means of a very simple experimental procedure. Applications to the technique of freezing and thawing of foodstuffs, together with numerical and experimental results, were carried out in collaboration with the Laboratorio per la Tecnica del Freddo in Padova [7].

The stability theorem is then proved in Sec. 4 and 5 by means of techniques based upon the concept of weak solution (classical tools are unsuitable for our purposes, as pointed out in [1]). Some procedures, which appeared in [2] and were then developed in [3] and [4] will also be applied.

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2. A weak formulation of nonlinear heat conduction problems. Let us consider the following heat conduction problem during the time interval $(0, T)$ in a medium occupying a n-dimensional region B, assuming heat capacity c and thermal conductivity k to be known functions of temperature:

$$c(u)(\partial u/\partial t) = \text{div}\,[k(u)\,\text{grad}\,u]^1 \quad \text{in} \quad D \equiv B \times (0, T), \tag{2.1}$$

$$u(x, 0) = h(x), \qquad x \in \bar{B}, \tag{2.2}$$

$$k(u)(\partial u/\partial n) = \varphi(x, t) \qquad (x, t) \in S \equiv \partial B \times (0, T); \tag{2.3}$$

here u represents the temperature at each point $(x, t) \in D$, $h(x)$ is the initial temperature distribution prescribed on $\bar{B}$, and $\varphi(x, t)$ is the given specific heat flux entering B per unit time ($\partial/\partial n$ = derivative in the direction of the outer normal to ∂B).

Throughout the paper we shall suppose that $c(u)$, $k(u)$ are positive, bounded and locally integrable functions:

$$0 < \alpha \leq c(u) \leq \beta < +\infty, \tag{2.4}$$

$$0 < \alpha' \leq k(u) \leq \beta' < +\infty. \tag{2.5}$$

Moreover, we shall assume that h and φ are bounded measurable functions and that ∂B has local representation of the form $x_j = x_j(x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n)$ for some j of class C^2.

Let us define

$$a(u) = \int_{u_0}^{u} c(\eta)\, d\eta, \tag{2.6}$$

$$b(u) = \int_{u_0}^{u} k(\eta)\, d\eta, \tag{2.7}$$

where u_0 is an arbitrarily fixed constant.

Since $b'(u) = k(u) \geq \alpha' > 0$, the function $b(u)$ has an inverse

$$u = z(b). \tag{2.8}$$

Thus, we can define

$$A(b) = a(z(b)). \tag{2.9}$$

Consequently, the problem (2.1)–(2.3) can be rewritten as follows:

$$\partial A(b)/\partial t = \nabla^2 b \quad \text{in} \quad D, \tag{2.10}$$

$$b[u(x, 0)] = b[h(x)] \quad \text{in} \quad \bar{B}, \tag{2.11}$$

$$\partial b/\partial n = \varphi(x, t) \quad \text{on} \quad S. \tag{2.12}$$

In order to introduce a weak formulation of this problem, let us define as *test functions* for problem (2.10)–(2.12) any smooth[2] $F(x, t)$ in $\bar{D}$, such that:

$$\partial F/\partial n = 0 \quad \text{on} \quad S, \tag{2.13}$$

$$F(x, T) = 0 \quad \text{for} \quad x \in \bar{B}. \tag{2.14}$$

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[1] Throughout the operators div, grad, $\nabla^2(\nabla^2 = \text{div grad})$ are meant to act on space variables only.

[2] More precisely suppose F_t, grad F, $\nabla^2 F$ exist and are continuous in $\bar{D}$.

Multiplying both sides of (2.10) by F, integrating over D and performing elementary calculations gives

$$\iint_D \left(b\nabla^2 F + A(b)\frac{\partial F}{\partial t} \right) dx\, dt = -\int_B a(h(x))F(x, 0)\, dx - \int_S F\varphi\, dS, \qquad (2.15)$$

where conditions (2.11), (2.12), (2.13), (2.14) were taken into account.

We shall define a weak solution of (2.1)–(2.3) as a bounded measurable function $u(x, t)$ such that $b(u(x, t))$ satisfies (2.15) for any test function F.

In the spirit of the physical problems sketched in Sec. 1, we shall suppose that c has peaks (and, correspondingly, k presents high rates of variation) in given temperature intervals (U_l^*, U_l^{**}), $l = 1, 2, \cdots, m$; without loss of generality we shall suppose, from now on, $m = 1$ and drop the subscript l; thus, besides the constants α, β, α', β' defined in (2.4), (2.5), we introduce bounds α_0, β_0, α_0', β_0' for c and k when the temperature does not belong to (U^*, U^{**}):

$$0 < \alpha_0 \le c(\eta) \le \beta_0 < +\infty \qquad (2.16)$$

$$\eta \notin (U^*, U^{**}).$$

$$0 < \alpha_0' \le k(\eta) \le \beta_0' < +\infty \qquad (2.16')$$

In order to get a priori bounds on weak solutions to problem (2.10)–(2.12), consider the following problem:

$$\partial A(\tilde{b})/\partial t \equiv \frac{c}{k}(\partial \tilde{b}/\partial t) = \nabla^2 \tilde{b} \quad \text{in} \quad D, \qquad (2.17)$$

$$\tilde{b}(x, 0) = \int_{u_0}^{h_0} k(\eta)\, d\eta \qquad \text{in} \quad \bar{B}, \qquad (2.18)$$

$$\partial \tilde{b}/\partial n = \Phi \qquad \text{on} \quad S, \qquad (2.19)$$

where

$$h_0 = \max\{U^{**}, \text{ess sup } h(x)\}, \qquad (2.20)$$

$$\Phi = \text{ess sup } \varphi(x, t), \qquad (2.21)$$

and the coefficient c/k in (2.17) must be evaluated for the temperature $v = z(\tilde{b})$.

Let us introduce sequences $\{c_m\}$, $\{k_m\}$ of C^∞ functions satisfying (2.16), (2.16′) and such that the corresponding sequences $\{a_m\}$, $\{b_m\}$ are uniformly convergent on bounded sets to a and b respectively; then consider the solutions $\{\tilde{b}_m\}$ of the corresponding family of problems of type (2.17)–(2.19). From the assumptions made on coefficients, boundary and data, it follows (see [9], p. 491) that $b_m(x, t)$ exist and are unique. Moreover,

$$\tilde{b}_m > \int_{u_0}^{U^{**}} k_m(\eta)\, d\eta,$$

and therefore $v_m > U^{**}$ and c_m, k_m are evaluated outside (U^*, U^{**}); thus, from (2.16)–(2.16′),

$$\alpha_0'/\beta_0 \le c_m/k_m \le \beta_0'/\alpha_0 .$$

At this point, when we recall also the assumption on ∂B, an a-priori estimate for linear equations with linear initial-boundary conditions of the form (2.18), (2.19) can be deduced from the maximum principle (see, e.g., Theorem 2.3 of [5], p. 16) yielding

$$v_m(x, t) \le V'',$$

V'' being a computable constant depending on α_0, β_0, α_0', β_0', h_0, Φ and ∂B but *not* on (α and) β.

Following the techniques of [4], it is possible to prove that problem (2.17)–(2.19) has a weak solution $\tilde{b}$ and that the function $v = z(\tilde{b})$ is the pointwise limit of a subsequence $\{v_{m'}\}$ of $\{v_m\}$; consequently

$$v \le V''. \tag{2.22}$$

Let us now compare b and $\tilde{b}$; from (2.15) we have immediately

$$\iint_D \left\{(\tilde{b} - b)\nabla^2 F + [A(\tilde{b}) - A(b)]\frac{\partial F}{\partial t}\right\} dx\, dt$$

$$= -\int_B [a(h_0) - a(h)]F(x, 0)\, dx - \int_S (\Phi - \varphi)F(x, t)\, dS. \tag{2.23}$$

Select now a sequence $\{\rho_j\}$ of functions from C^∞ converging in the L^2 norm to the function

$$\rho(x, t) = \frac{\tilde{b} - b}{A(\tilde{b}) - A(b)}, \qquad \tilde{b} \ne b, \tag{2.24}$$

$$= \alpha'/\beta, \qquad \tilde{b} = b,$$

and possessing the same bounds as $\rho(x, t)$, i.e. $\alpha'/\beta \le \rho \le \beta'/\alpha$; next, choose test functions such that, besides (2.13), (2.14)

$$F_t^{(j)} + \rho_j \nabla^2 F^{(j)} = Q(x, t) \ge 0 \quad \text{in} \quad D, \tag{2.25}$$

where Q is an arbitrary Hölder-continuous function.

The maximum principle yields

$$F^{(j)}(x, t) \le 0 \quad \text{in} \quad D. \tag{2.26}$$

Indeed, the function $V^{(j)}(x, t) = F^{(j)}(x, T - t)$ cannot possess any maximum in D since $\rho_j \nabla^2 V^{(j)} - (\partial V^{(j)}/\partial t) \ge 0$, nor on S because of (2.13) and the Vyborny–Friedman theorem [6]. Therefore, the maximum of $V^{(j)}$ is assumed for $t = 0$: then (2.26) follows immediately from (2.14).

Thus, the right-hand side of (2.23) is positive; i.e.:

$$\iint_D [A(\tilde{b}) - A(b)]\left[\frac{\partial F^{(j)}}{\partial t} + \rho_j \nabla^2 F^{(j)}\right] dx\, dt$$

$$- \iint_D [A(\tilde{b}) - A(b)](\rho_j - \rho)\nabla^2 F^{(j)}\, dx\, dt \ge 0.$$

It is possible to show that the second term tends to zero as j tends to infinity by applying the technique of Lemma 3 below and recalling that $A(\tilde{b})$ and $A(b)$ are bounded because of the definition of weak solution. Therefore

$$\iint_D [A(\tilde{b}) - A(b)]Q(x, t)\, dx\, dt \ge 0.$$

But since $Q(x, t)$ is an arbitrary positive function of (x, t) in D, it follows that

$$A(\bar{b}) \geq A(b), \qquad \text{a.e. in } D;$$

that is,

$$v(x, t) \geq u(x, t), \qquad \text{a.e. in } D. \tag{2.27}$$

Finally, recalling (2.22), we have

$$u(x, t) \leq V'', \qquad \text{a.e. in } D. \tag{2.28}$$

Then, applying the same technique, a lower bound V' for u can be found; setting

$$V = \max \{|V'|, |V''|\} \tag{2.29}$$

we conclude that

$$|u(x, t)| \leq V \qquad \text{a.e. in } D. \tag{2.30}$$

For sake of simplicity, we shall choose the constant u_0 of definitions (2.6) and (2.7) equal to V'.

3. The stability theorem. Let us denote by $u_1(x, t)$, $u_2(x, t)$ the temperature distributions corresponding to initial and boundary conditions (2.2)–(2.3) and to temperature-dependent thermal coefficients c_1, k_1 and c_2, k_2 respectively, each of them satisfying (2.4), (2.5), (2.16). We define functions a_i, b_i, z_i, A_i, $i = 1, 2$, according to formulae (2.6)–(2.9). Starting from (2.15), it is easily found that

$$\iint_D \left\{[A_1(b_1) - A_2(b_2)] \frac{\partial F}{\partial t} + (b_1 - b_2)\nabla^2 F\right\} dx\, dt = -\int_B \{a_1[h(x)] - a_2[h(x)]\}F(x, 0)\, dx. \tag{3.1}$$

Subtracting $\iint_D A_1(b_2)(\partial F/\partial t)\, dx\, dt$ on both sides and setting

$$\begin{aligned} s(x, t) &= \frac{b_1 - b_2}{A_1(b_1) - A_1(b_2)} \quad \text{for} \quad b_1 \neq b_2 \\ &= \alpha'/\beta \qquad\qquad\qquad \text{for} \quad b_1 = b_2, \end{aligned} \tag{3.2}$$

we get

$$\iint_D [A_1(b_1) - A_1(b_2)]\left(\frac{\partial F}{\partial t} + s\nabla^2 F\right) dx\, dt = -\iint_D [A_1(b_2) - A_2(b_2)] \frac{\partial F}{\partial t}\, dx\, dt - \int_B [a_1(h) - a_2(h)]F(x, 0)\, dx. \tag{3.3}$$

Remark that

$$\alpha'/\beta \leq s(x, t) \leq \beta'/\alpha \quad \text{in} \quad D. \tag{3.4}$$

In order to state the stability theorem let us introduce some notation. First, define the following subsets of D and B.[3]

[3] Obviously these subsets are defined apart from sets of zero measure.

$$\begin{aligned} D^{(1)} &\equiv \{(x, t) : u_2(x, t) < U^*\}, \\ D^{(2)} &\equiv \{(x, t) : U^* \leq u_2(x, t) \leq U^{**}\}, \\ D^{(3)} &\equiv \{(x, t) : U^{**} < u_2(x, t)\}; \end{aligned} \tag{3.5}$$

$$\begin{aligned} B^{(1)} &\equiv \{x : h(x) < U^*\}, \\ B^{(2)} &\equiv \{x : U^* \leq h(x) \leq U^{**}\}, \\ B^{(3)} &\equiv \{x : U^{**} < h(x)\}; \end{aligned} \tag{3.6}$$

and denote by $\mu^{(l)}$, $\mu_0{}^{(l)}$ the Lebesgue measures of $D^{(l)}$, $B^{(l)}$ in R^{n+1} and R^n respectively ($l = 1, 2, 3$) and set

$$\mu = \sum_{l=1}^{3} \mu^{(l)}, \qquad \mu_0 = \sum_{l=1}^{3} \mu_0{}^{(l)}.$$

Next, set

$$\Delta' c = \int_{V'}^{U^*} |c_1(\eta) - c_2(\eta)| \, d\eta, \tag{3.7}$$

$$\Delta'' c = \int_{U^{**}}^{V''} |c_1(\eta) - c_2(\eta)| \, d\eta, \tag{3.8}$$

$$\Delta k = \int_{V'}^{V''} |k_1(\eta) - k_2(\eta)| \, d\eta, \tag{3.9}$$

$$\lambda_i = \int_{U^*}^{U^{**}} c_i(\eta) \, d\eta, \qquad i = 1, 2, \tag{3.10}$$

$$\Lambda = \max (\lambda_1 , \lambda_2), \tag{3.10$'$}$$

$$\Delta\lambda = |\lambda_1 - \lambda_2|. \tag{3.11}$$

We shall prove the following

THEOREM. *If $u_1(x, t)$, $u_2(x, t)$ are weak solutions of problem* (2.1)–(2.3) *with respective thermal coefficients c_1, k_1 and c_2, k_2 satisfying* (2.4), (2.5), (2.16), *then*

$$||u_1 - u_2||_{L^2(D)} \leq \epsilon(\Delta k, \Delta' c, \Delta'' c, \Delta\lambda, \mu^{(2)}, \mu_0{}^{(2)}; \alpha, \alpha', \beta', \alpha_0 , \beta_0 , \Lambda, V, \mu_0 , T), \tag{3.12}$$

where ϵ is a known function of its arguments, tending to zero when Δk, $\Delta' c$, $\Delta'' c$, $\Delta\lambda$, $\mu^{(2)}$, $\mu_0{}^{(2)}$ all tend to zero.

Bearing in mind the practical applications we have outlined in Sec. 1, by c_1 and k_1 we shall mean the actual thermal coefficients which can be measured only with large innacuracy in the interval (U^*, U^{**}), whereas c_2 and k_2 will denote the functions used for the computation of the approximate thermal field $u_2(x, t)$. Inequality (3.12) furnishes an estimate of the error brought about by this procedure.

The main features of this estimate are summarized in the following remarks.

Remark 1. The function ϵ in (3.12) depends only on quantities which are easily evaluated by means of either experimental measurements or computation. On the contrary, it is essential to note that the constant β and the shape of the curves c_1, k_1 vs. temperature in the interval (U^*, U^{**}) do not enter into ϵ: indeed, their appearance in (3.12) would make it meaningless because of the large value expected for β and the difficulties arising in doing local measurements of c_1, k_1 in the quoted temperature interval.

Remark 2. The estimate (3.12) provides a very useful rule for selecting functions c_2, k_2 : it is sufficient for c_2, k_2 to approximate closely c_1, k_1 where they are easily measurable, i.e. outside (U^*, U^{**}), while in this interval the conductivity can be approximated with a linear interpolation and the heat capacity in a rather arbitrary way (e.g. a bell-shaped, or even a simpler curve) provided that the integral $\lambda_2 = \int_{U_*}^{U^{**}} c_2(\eta)\, d\eta$ is nearly equal to the actual total change in enthalpy (the "latent heat" for the phase change) across the interval (U^*, U^{**}); this quantity is easily measurable with a single calorimetric experiment.

Remark 3. While all other quantities in (3.12) are known or estimated a priori, $\mu^{(2)}$ must be evaluated a posteriori. Nevertheless, this is not a significant drawback: once the approximate thermal field $u_2(x, t)$ has been computed, the measure of $D^{(2)}$ can be calculated directly.

Moreover, if an a-priori lower bound on grad $u_2(x, t)$ in $D^{(2)}$ can be established (e.g. by means of the maximum principle), $\mu^{(2)}$ is also estimated a-priori.

In this connection it should be emphasized that in practical processes (e.g. freezing of foodstuffs) high values of the thermal gradient are required in the phase-change zone, which causes a significant narrowing of the region $D^{(2)}$.

4. Preliminary lemmas. Let us introduce the following sequences of functions belonging to $C^\infty(\bar{D})$:

$$\{s_m\}, \quad 0 < \alpha'/\beta \leq s_m \leq \beta'/\alpha, \quad \lim \|s_m - s\| = 0^4 \tag{4.1}$$

$$\{\psi_m\}, \quad \|\psi_m\| \leq \|u_1 - u_2\|, \quad \lim \|\psi_m - (u_1 - u_2)\| = 0 \tag{4.2}$$

and consider the solutions of the following parabolic problems ($m = 1, 2, \cdots$):

$$(\partial F_m/\partial t) + s_m(x, t)\, \nabla^2 F_m = \psi_m(x, t)\cdot s_m(x, t);\ (x, t) \in D, \tag{4.3}$$

$$F_m(x, T) = 0 \qquad x \in B, \tag{4.4}$$

$$(\partial/\partial n)F_m(x, t) = 0, \qquad (x, t) \in S. \tag{4.5}$$

Since the F_m satisfy (2.13), (2.14), they can be used as test functions in (3.3), which can be rewritten as follows:

$$\iint_D [A_1(b_1) - A_1(b_2)]s_m\psi_m\, dx\, dt + \iint_D [A_1(b_1) - A_1(b_2)]\cdot(s - s_m)\nabla^2 F_m\, dx\, dt$$
$$= -\iint_D [A_1(b_2) - A_2(b_2)] \frac{\partial F_m}{\partial t}\, dx\, dt - \int_B [a_1(h) - a_2(h)]F_m(x, 0)\, dx. \tag{4.6}$$

Let us prove some lemmas in order to estimate $\|F_m(x, 0)\|_{L^2(B)}$, $\|\partial F_m/\partial t\|$, and the second term of the right-hand side of (4.6).

LEMMA 1. *There exists a constant N (given by (4.13)) such that:*

$$\|\partial F_m/\partial t\| \leq N\, \|u_1 - u_2\|, \tag{4.7}$$

for each m.

Proof. From (4.3) we have

$$\|\partial F_m/\partial t\| \leq \|s_m\, \nabla^2 F_m\| + \|\psi_m s_m\|, \tag{4.8}$$

[4] $\|\cdot\| \equiv \|\cdot\|_{L^2(D)}$.

where, by virtue of (3.1), (3.2):

$$||\psi_m s_m|| \leq (\beta'/\alpha)\ ||u_1 - u_2|| \,. \tag{4.9}$$

Concerning the first term in the right-hand side of (4.8), one can write

$$||s_m \nabla^2 F_m|| \leq (\beta'/\alpha)^{1/2} ||\ s_m^{1/2} \nabla^2 F_m||\ , \tag{4.10}$$

and estimate $||s_m^{1/2} \nabla^2 F_m||$ by multiplying both sides of (4.3) by $\nabla^2 F_m$ and integrating over D:

$$||s_m^{1/2}\nabla^2 F_m||^2 = \iint_D \psi_m s_m \nabla^2 F_m\, dx\, dt - \iint_D \frac{\partial F_m}{\partial t} \nabla^2 F_m\, dx\, dt. \tag{4.11}$$

It is easily shown that

$$\iint_D \frac{\partial F_m}{\partial t} \nabla^2 F_m\, dx\, dt = \int_S \frac{\partial F_m}{\partial n} \frac{\partial F_m}{\partial t}\, dx\, dt - \iint_D \operatorname{grad} F_m \operatorname{grad} \frac{\partial F_m}{\partial t}\, dx\, dt$$

$$= -\frac{1}{2} \iint_D (\partial/\partial t)\ (\operatorname{grad} F_m)^2\, dx\, dt = \frac{1}{2} \int_B [\operatorname{grad} F_m(x, 0)]^2\, dx.$$

Thus, returning to (4.11) and applying the Schwarz inequality to the first integral, one has:

$$||s_m^{1/2}\nabla^2 F_m||^2 \leq ||\psi_m s_m^{1/2}||\ ||s_m^{1/2}\nabla^2 F_m|| - \frac{1}{2} \int_B [\operatorname{grad} F_m(x, 0)]^2\, dx.$$

Therefore,

$$||s_m^{1/2} \nabla^2 F_m|| \leq (\beta'/\alpha)^{1/2}\ ||u_1 - u_2|| \,. \tag{4.12}$$

Summing up (4.8), (4.9), (4.10), (4.12), we achieve the proof of (4.7) with

$$N = 2\,\frac{\beta'}{\alpha}. \tag{4.13}$$

LEMMA 2. The L^2 norms of the functions $F_m(x, 0)$ satisfy the following inequality:

$$||F_m(x, 0)||_{L^2(B)} \leq \sqrt{T}\, N\ ||u_1 - u_2|| \,. \tag{4.14}$$

Proof. Inequality (4.14) follows immediately from Lemma 1 and the identity

$$F_m(x, 0) = -\int_0^T \frac{\partial F_m(x, \tau)}{\partial \tau}\, d\tau.$$

LEMMA 3. *The second term on the left-hand side of* (4.6) *tends to zero when* m *tends to infinity, that is:*

$$\lim_{m\to\infty} \iint_D [A_1(b_1) - A_1(b_2)](s - s_m)\nabla^2 F_m\, dx\, dt = 0 \tag{4.15}$$

Proof. The term in square brackets is bounded in the maximum norm:

$$[A_1(b_1) - A_1(b_2)] \leq 4\beta V. \tag{4.16}$$

Therefore,

$$\left|\iint_D [A_1(b_1) - A_1(b_2)](s - s_m)\nabla^2 F_m\, dx\, dt\right| \leq 4\beta V \iint_D \frac{|s - s_m|}{s_m^{1/2}} \cdot |s_m^{1/2}\nabla^2 F_m|\, dx\, dt$$

Since $s_m^{-1/2} \leq (\beta'/\alpha)^{1/2}$, taking into account (4.12) the Schwartz inequality yields:

$$\left| \iint_D [A_1(b_1) - A_1(b_2)](s - s_m)\nabla^2 F_m \, dx \, dt \right| \leq 8V^2\beta^{3/2}\left(\frac{\mu\beta'}{\alpha\alpha'}\right)^{1/2} ||s - s_m||.$$

Because of (4.1), the proof of Lemma 3 is completed.

Now let m tend to infinity in (4.6). Recall definitions (4.1), (4.2) and apply the results of Lemmas 1, 2 and 3 to obtain:

$$\iint_D (b_1 - b_2)(u_1 - u_2) \, dx \, dt$$

$$\leq \{N\sqrt{T}\, ||a_1(h) - a_2(h)|| + N\, ||A_1(b_2) - A_2(b_2)||\} \cdot ||u_1 - u_2||. \quad (4.17)$$

The following section contains the analysis of (4.17) necessary to prove the theorem.

5. Proof of the theorem. Put

$$I_1 \equiv \iint_D (u_1 - u_2) \int_{u_2}^{u_1} k_1(\eta) \, d\eta \, dx \, dt, \quad (5.1)$$

$$I_2 \equiv \iint_D (u_1 - u_2) \int_{u_0}^{u_2} [k_1(\eta) - k_2(\eta)] \, d\eta \, dx \, dt, \quad (5.2)$$

$$J_1 \equiv ||a_1(h) - a_2(h)||, \quad (5.3)$$

$$J_2 \equiv \left|\left| \int_{u_0}^{u_2} [c_1(\eta) - c_2(\eta)] \, d\eta \right|\right|, \quad (5.4)$$

$$J_3 \equiv \left|\left| \int_{s_2(b_2)}^{s_1(b_2)} c_1(\eta) \, d\eta \right|\right|. \quad (5.5)$$

From (4.17) we get:

$$I_1 + I_2 \leq N(\sqrt{T}\, J_1 + J_2 + J_3)\, ||u_1 - u_2||\,. \quad (5.6)$$

A. *Analysis of* I_1. Since $k_1 \geq \alpha'$ one has

$$I_1 \geq \alpha' \iint_D (u_1 - u_2)^2 \, dx \, dt = \alpha' \, ||u_1 - u_2||^2. \quad (5.7)$$

B. *Analysis of* I_2. One has immediately from the Schwartz inequality and definition (3.9)

$$|I_2| \leq \Delta k \iint_D |u_1 - u_2| \, dx \, dt \leq \Delta k(\mu_0 T)^{1/2} \, ||u_1 - u_2|| \quad (5.8)$$

C. *Analysis of* J_1. From definitions (3.6) it follows that

$$J_1^2 \leq \int_{B(1)} \left| \int_{u_0}^{h} (c_1 - c_2) \, d\eta \right|^2 dx$$

$$+ \int_{B(2)} \left\{ \left| \int_{u_0}^{U^*} (c_1 - c_2) \, d\eta \right| + \left| \int_{U^*}^{h} (c_1 - c_2) \, d\eta \right| \right\}^2 dx$$

$$+ \int_{B(3)} \left\{ \left| \int_{u_0}^{U^*} (c_1 - c_2) \, d\eta \right| + \left| \int_{U^*}^{U^{**}} (c_1 - c_2) \, d\eta \right| + \left| \int_{U^{**}}^{h} (c_1 - c_2) \, d\eta \right| \right\}^2 dx. \quad (5.9)$$

Recalling (3.7), (3.8), (3.10); (3.11), we obtain:

$$J_1^2 \leq \mu_0^{(1)}[\Delta' c]^2 + \mu_0^{(2)}[\Delta' c + 2\Lambda]^2 + \mu_0^{(3)}[\Delta' c + \Delta\lambda + \Delta'' c]^2.$$

Thus we can write

$$J_1 \leq \epsilon_1(\Delta' c,\ \Delta'' c,\ \Delta\lambda,\ \mu_0^{(2)};\ \Lambda,\ \mu_0) \tag{5.10}$$

where ϵ_1 is a known function tending to zero when $\Delta' c$, $\Delta'' c$, $\Delta\lambda$, $\mu_0^{(2)}$ all tend to zero.

D. *Analysis of* J_2. From definitions (3.5) we get

$$\begin{aligned} J_2^2 = &\iint_{D^{(1)}} \left| \int_{u_0}^{u_2} (c_1 - c_2)\, d\eta \right|^2 dx\, dt \\ &+ \iint_{D^{(2)}} \left\{ \int_{u_0}^{U^*} (c_1 - c_2)\, d\eta + \int_{U^*}^{u_2} (c_1 - c_2)\, d\eta \right\}^2 dx\, dt \\ &+ \iint_{D^{(3)}} \left\{ \int_{u_0}^{U^*} (c_1 - c_2)\, d\eta + \int_{U^*}^{U^{**}} (c_1 - c_2)\, d\eta + \int_{U^{**}}^{u_2} (c_1 - c_2)\, d\eta \right\}^2 dx\, dt. \end{aligned}$$

The same technique which led us from (5.9) to (5.10) now yields

$$J_2 \leq \epsilon_2(\Delta' c,\ \Delta'' c,\ \mu^{(2)},\ \Delta\lambda;\ \Lambda,\ \mu_0 T) \tag{5.11}$$

where ϵ_2 is a known function tending to zero when $\Delta' c$, $\Delta'' c$, $\mu^{(2)}$, $\Delta\lambda$ all tend to zero.

E. *Analysis of* J_3. First, remark that

$$\sup_{|\xi| \leq 2\beta' V} |z_1(\xi) - z_2(\xi)| \leq \frac{1}{\alpha'} \sup_{|u_2| \leq V} |b_1(u_2) - b_2(u_2)| \leq \frac{1}{\alpha'} \Delta k. \tag{5.12}$$

Consequently,

$$\left| \int_{z_2(b_2)}^{z_1(b_2)} c_1(\eta)\, d\eta \right| \leq \text{ma} \left\{ \int_{u_2}^{u_2 + \Delta k/\alpha'} c_1(\eta)\, d\eta,\ \int_{u_2 - \Delta k/\alpha'}^{u_2} c_1(\eta)\, d\eta \right\}. \tag{5.13}$$

Let us look for a bound on

$$\left\| \int_{u_2}^{u_2 + \Delta k/\alpha'} c_1(\eta)\, d\eta \right\|.$$

Define the following subsets of D:

$$\begin{aligned} \mathfrak{D}^{(1)} &\equiv \{(x, t) : u_2(x, t) < U^* - \Delta k/\alpha'\}, \\ \mathfrak{D}^{(2)} &\equiv \{(x, t) : U^* - \Delta k/\alpha' \leq u_2(x, t) \leq U^{**}\}, \\ \mathfrak{D}^{(3)} &\equiv \{(x, t) : u_2(x, t) > U^{**}\}, \end{aligned}$$

whose respective measures will be indicated by $\bar{\mu}^{(1)}$, $\bar{\mu}^{(2)}$, $\bar{\mu}^{(3)}$. Then, recalling that in $\mathfrak{D}^{(1)}$ and $\mathfrak{D}^{(3)}$ $\alpha_0 \leq c_1(\eta) \leq \beta_0$, by virtue of (3.5)

$$\begin{aligned} &\left\| \int_{u_2}^{u_2 + \Delta k/\alpha'} c_1(\eta)\, d\eta \right\|^2 \\ &\leq \bar{\mu}^{(1)} \beta_0^2 \left(\frac{\Delta k}{\alpha'}\right)^2 + \iint_{\mathfrak{D}^{(2)}} \left\{ \int_{u_2}^{U^*} c_1(\eta)\, d\eta + \int_{U^*}^{u_2 + \Delta k/\alpha'} c_1(\eta)\, d\eta \right\}^2 dx\, dt + \bar{\mu}^{(3)} \beta_0^2 \left(\frac{\Delta k}{\alpha'}\right)^2 \\ &\leq 2\mu_0 T \beta_0^2 \left(\frac{\Delta k}{\alpha'}\right)^2 + \bar{\mu}^{(2)} \left[\frac{2\Delta k}{\alpha'} \beta_0 + \lambda_1\right]^2. \end{aligned} \tag{5.14}$$

Since a similar result holds for

$$\left|\left|\int_{u_2-\Delta k/\alpha'}^{u_2} c_1(\eta)\, d\eta\right|\right|$$

and the difference $\bar{\mu}^{(2)} - \mu^{(2)}$ tends to zero with Δk, we can conclude that

$$J_3 \leq \epsilon_3(\Delta k, \mu^{(2)}; \alpha', \beta_0, \Lambda, \mu_0 T) \tag{5.15}$$

where ϵ_3 is a computable function tending to zero when Δk and $\mu^{(2)}$ both tend to zero. Summing up (5.7), (5.8), (5.10), (5.11) and (5.15) and recalling the definition (4.13) of N, (5.6) furnishes the following inequality:

$$\alpha' \,||u_1 - u_2|| \leq \Delta k(\mu_0 T)^{1/2} + 2\frac{\beta'}{\alpha}(\sqrt{T}\,\epsilon_1 + \epsilon_2 + \epsilon_3), \tag{5.16}$$

from which the final estimate (3.12) follows immediately.

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