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ON THE STABILITY OF SWIRLING FLOW IN MAGNETOGASDYNAMICS*

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1. Introduction. The stability of the steady circular nondissipative flow of an incompressible fluid betwen two concentric cylinders was first studied by Rayleigh [1] who assumed the disturbances to be axisymmetric. He showed that this problem has a remarkable analogy with that of the stability of a density stratified fluid at rest under gravity. Michael [2] extended this problem to the case of a perfectly conducting liquid with an electric current distribution parallel to the axis of cylinders and found that Rayleigh's analogy holds in a slightly modified form. Using this analogy, Howard and Gupta [3] investigated the stability of nondissipative swirling flow of an incompressible fluid between two concentric cylinders with respect to axisymmetric disturbances. They found that stability is ensured if a Richardson number based on the swirl velocity and the shear in the axial flow exceeds $\frac{1}{4}$ everywhere. Recently Howard [4] using a modification of the analysis due to Chimonas [5] on compressible stratified shear flow, derived a Richardson-number theorem for the linear stability to axisymmetric perturbations of compressible nondissipative swirling flow.

The present note is an extension of Howard's [4] problem to the case of a perfectly conducting fluid permeated by an axial distribution of electric current. It is important to note that in a compressible swirling flow, the swirl velocity distribution $V(r)$ not only plays a role similar to that in incompressible flows, but also gives rise, through the centrifugal acceleration V^2/r, to a radial effective gravity which, combined with a radial density stratification, affects the perturbations significantly.

We discuss the axisymmetric stability of pure axial flow of a compressible perfectly conducting fluid between two concentric cylinders permeated by a uniform axial magnetic field. We show that the complex wave speed for any unstable wave lies in a semicircle in the upper half plane, having the same range of axial velocity as the diameter.

2. Compressible swirling flow with an axial current. Consider the steady swirling flow of an inviscid, compressible and perfectly conducting fluid between two concentric cylinders of radii a and b $(a < b)$, the flow being subjected to a volume distribution of current parallel to the axis of the cylinders. Using cylindrical coordinates (r, θ, z), we take the basic velocity and the magnetic field as $[0, V(r), W(r)]$ and $[0, H_0(r), 0]$. The radial momentum equation in the undisturbed state gives

$$\frac{\rho_0 V^2}{r} = \frac{\mu_e H_0^2}{4\pi r} + \left[p_0 + \frac{\mu_e H_0^2}{8\pi}\right]', \tag{1}$$

where $p_0(r)$ and $\rho_0(r)$ denote the basic pressure and density distribution and a prime denotes derivative with respect to r. Let a perturbed state of this flow be

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$$\mathbf{q} = (u,\ V + v,\ W + w), \qquad p^* = p_0 + p,$$
$$\mathbf{H} = (h_r\ ,\ H_0 + h_\theta\ ,\ h_z), \qquad \rho^* = \rho_0 + \rho. \tag{2}$$

Since the gas is in thermodynamic equilibrium, the relation $p^* = f(\rho^*, S)$ must hold, S being the specific entropy. We can, of course, avoid the explicit dependence on S by using the equation of entropy

$$dS/dt = 0, \tag{3}$$

all dissipative phenomena in the gas being negligible. In fact, using (3), one finds

$$dp^*/dt = (\partial p^*/\partial \rho^*)_S (d\rho^*/dt) = -a^2 \rho^* \nabla \cdot \mathbf{q}, \tag{4}$$

where $a(\rho^*, S)$ is the adiabatic sound speed and use is made of the equation of continuity

$$(d\rho^*/dt) + \rho^* \nabla \cdot \mathbf{q} = 0. \tag{5}$$

The magnetic induction equation is

$$\frac{\partial \mathbf{H}}{\partial t} + (\mathbf{q} \cdot \nabla)\mathbf{H} = (\mathbf{H} \cdot \nabla)\mathbf{q} - \mathbf{H} \nabla \cdot \mathbf{q} \tag{6}$$

along with the solenoidal condition

$$\nabla \cdot \mathbf{H} = 0. \tag{7}$$

As is usual in linear stability problems, we now look for solutions for which the $z - t$ dependence of the perturbation quantities in (2) is taken in the form $\exp[i\alpha(z - ct)]$. The three linearized components of the momentum equation now give

$$\rho_0[i\alpha(W - c)u - 2Vv/r] + \mu_e H_0 h_\theta / 2\pi r - \rho V^2/r = -[p + \mu_e h_\theta H_0/4\pi]', \tag{8}$$

$$\rho_0[i\alpha(W - c)v + (rV)'u/r] = 0, \tag{9}$$

$$\rho_0[i\alpha(W - c)w + uW'] = -i\alpha[p + \mu_e h_\theta H_0/4\pi]. \tag{10}$$

Eqs. (4) and (5) give

$$i\alpha(W - c)p + up_0' + a_0^2 \rho_0[(ru)'/r + i\alpha w] = 0, \tag{11}$$

$$i\alpha(W - c)\rho + u\rho_0' + \rho_0[(ru)'/r + i\alpha w] = 0. \tag{12}$$

Further, (6) gives

$$h_r = h_z = 0,$$
$$i\alpha(W - c)h_\theta + u(H_0' - H_0/r) + H_0[(ru)'/r + i\alpha w] = 0, \tag{13}$$

Eq. (7) being identically satisfied.

Elimination of p, h_θ , v and ρ from (8), (9), (10), (12) and (13) gives

$$\begin{aligned}
&(W - c)\frac{d}{dr}[\rho_0\{i\alpha(W - c)w + uW'\}] \\
&\quad = \rho_0\left[\frac{2V(V' + V/r)}{r} - \alpha^2(W - c)^2\right]u + \frac{\mu_e H_0}{2\pi r}[H_0 \rho_0' u/\rho_0 + u(H_0/r - H_0')] \\
&\qquad - \left[\frac{\mu_e H_0^2}{2\pi r \rho_0} - \frac{V^2}{r}\right][u\rho_0' + \rho_0(ru)'/r + \rho_0 i\alpha w].
\end{aligned} \tag{14}$$

Again, elimination of p and h_θ from (10), (11) and (13) yields

$$i\alpha w\left[\frac{\mu_e H_0^2}{4\pi} + \rho_0 a_0^2 - \rho_0(W - c)^2\right]$$
$$= \left[\frac{\mu_e H_0^2}{2\pi r} + \rho_0(W - c)W' - \rho_0 V^2/r\right]u - \left[\rho_0 a_0^2 + \frac{\mu_e H_0^2}{4\pi}\right]\frac{(ru)'}{r}. \tag{15}$$

We now eliminate w from (14) and (15) and set

$$u = F(r)\cdot Q(r)\cdot (W - c) \tag{16}$$

in the resulting equation where

$$\frac{dQ}{dr} = Q\left(\frac{2V_A^2}{r} - \frac{V^2}{r}\right)\Big/ (a_0^2 + V_A^2) \tag{17}$$

with $V_A = (\mu_e H_0^2/4\pi\rho_0)^{1/2}$ representing Alfvén velocity along the circular magnetic lines of force. The equation for F now becomes

$$\frac{d}{dr}\left[\frac{\rho_0 Q^2(a_0^2 + V_A^2)(W - c)^2}{(W - c)^2 - (a_0^2 + V_A^2)}(rF)'/r\right] + [\alpha^2(W - c)^2 - \psi - N_H^2]\rho_0 Q^2 F = 0, \tag{18}$$

where

$$\psi = \phi - \frac{\mu_e r}{4\pi\rho_0}\left(\frac{H_0^2}{r^2}\right)', \qquad \phi = \frac{1}{r^3}(r^2V^2)',$$
$$N_H^2 = \frac{V^2}{r}\frac{\rho_0'}{\rho_0} - \frac{(2V_A^2 - V^2)^2}{r^2(a_0^2 + V_A^2)}. \tag{19}$$

Since u vanishes on the walls, the boundary conditions for F are

$$F = 0 \quad \text{at} \quad r = a \quad \text{and} \quad r = b. \tag{20}$$

Eq. (18) with the boundary conditions (20) is similar in structure to the corresponding equations for F derived by Howard [4] for the non-magnetic case, the only difference being that a_0^2 , ϕ and N^2 in Eq. (21) of [4] are replaced by $a_0^2 + V_A^2$, ψ and N_H^2 respectively, where N^2 is the square of the adiabatic Brunt-Väisälä frequency. The expression for ψ is Michael's discriminant [2], while N_H^2 may be recognized as the square of the adiabatic Brunt-Väisälä frequency modified by the presence of the circular magnetic field.

Following Howard [4], we therefore conclude that if $\psi + N_H^2 \geq 0$, then the complex wave speed of any unstable mode must lie in a semicircle in the upper half-plane which has the range of $W(r)$ as diameter. Further, we may define a local Richardson number $\mathrm{Ri} = (\psi + N_H^2)/W'^2$ such that the flow will be stable with respect to axisymmetric perturbations if $\mathrm{Ri} \geq \frac{1}{4}$ everywhere in the flow. It may be seen from (19) that, unlike the stability of the corresponding incompressible MHD swirling flow studied by Howard and Gupta [3], the stability characteristics in the present problem are affected by a circular magnetic field even when $H_0(r)$ is proportional to r.

3. Compressible axial flow with an axial magnetic field. Consider nondissipative axial flow of a compressible conducting fluid between two concentric cylinders in the presence of a uniform axial magnetic field. Thus, in cylindrical coordinates, the basic velocity and magnetic field are given by $[0, 0, W(r)]$ and $[0, 0, H_0]$ respectively. We take the perturbed state as

$$\mathbf{q} = (u, v, W + w), \qquad p^* = p_0 + p,$$
$$\mathbf{H} = (h_r\,, h_\theta\,, H_0 + h_z), \qquad \rho^* = \rho_0 + \rho, \tag{21}$$

where the subscript zero refers to the unperturbed state. The governing equations are (4)–(7) along with the equations of momentum. All perturbation quantities are assumed to be axisymmetric. If we proceed as in Sec. 2, the linearized equations of momentum are given by

$$\rho_0 i\alpha(W - c)u - \mu_e H_0 i\alpha h_r/4\pi = -(p + \mu_e H_0 h_z/4\pi)', \tag{22}$$
$$\rho_0(W - c)v - \mu_e H_0 h_\theta/4\pi = 0, \tag{23}$$
$$\rho_0 i\alpha(W - c)w + \rho_0 uW' - \mu_e H_0 i\alpha h_z/4\pi = -i\alpha(p + \mu_e H_0 h_z/4\pi). \tag{24}$$

The components of (6) give

$$(W - c)h_r - H_0 u = 0, \tag{25}$$
$$(W - c)h_\theta - H_0 v = 0, \tag{26}$$
$$i\alpha(W - c)h_z - h_r W' - i\alpha H_0 w + H_0[(ru)'/r + i\alpha w] = 0. \tag{27}$$

Further, (4) and (5) yield

$$i\alpha(W - c)p + a_0^2\rho_0[(ru)'/r + i\alpha w] = 0, \tag{28}$$
$$i\alpha(W - c)\rho + u\rho_0' + \rho_0[(ru)'/r + i\alpha w] = 0. \tag{29}$$

Again $\nabla\cdot\mathbf{H} = 0$ leads to

$$(rh_r)'/r + i\alpha h_z = 0. \tag{30}$$

Eqs. (23) and (26) give

$$v = h_\theta \equiv 0. \tag{31}$$

Eliminating all the variables except u from (22), (24), (25), (27), (28) and (30) and putting $F(r) = u/(W - c)$ in the resulting equation, we obtain

$$\frac{d}{dr}\left[\rho_0\left\{\frac{a_0^2(W - c)^2}{(W - c)^2 - a_0^2} + V_A^2\right\}(rF)'/r\right] + \rho_0\alpha^2[(W - c)^2 - V_A^2]F = 0, \tag{32}$$

where

$$V_A = (\mu_e H_0^2/4\pi\rho_0)^{1/2} \tag{33}$$

represents the Alfvén velocity.

For an incompressible homogeneous fluid, $\rho_0 =$ constant and $a_0 \to \infty$ so that (32) reduces to

$$\frac{d}{dr}\,[\{(W - c)^2 - V_A^2\}(rF)'/r] - \alpha^2[(W - c)^2 - V_A^2]F = 0, \tag{34}$$

as deduced by Howard and Gupta [3]. Multiplying (32) by $r\bar{F}$, integrating between a and b, and using $F = 0$ at $r = a$ and b, we obtain

$$\int_a^b (W - c)^2 Y(r)\, dr = T, \tag{35}$$

where

$$Y(r) = \frac{\rho_0 a_0^{\,4} r\, |D_* F|^2}{|a_0^{\,2} - (W - c)^2|^2} + \alpha^2 \rho_0 r\, |F|^2,$$

$$T = \int_a^b \rho_0 \left[\frac{a_0^{\,2}\, |W - c|^4}{|a_0^{\,2} - (W - c)^2|^2} + V_A^{\,2}\right] r\, |D_* F|^2\, dr + \alpha^2 \int_a^b \rho_0 r V_A^{\,2}\, |F|^2\, dr,$$

$$D_* \equiv \frac{d}{dr} + \frac{1}{r}. \tag{36}$$

Eq. (35) is of the same form as the one deduced by Howard [6] since both $Y(r)$ and T are non-negative. We therefore conclude that the complex wave speed c for any unstable mode must lie in the semicircle in the upper half plane which has the range of $W(r)$ for diameter. It may be noticed that Eq. (29) involving the density perturbation ρ is not needed for finding the stability characteristics. This is due to the fact that there is no basic swirl velocity and therefore the mechanism of the centrifugal acceleration playing the role of a radial effective gravity is absent in this stability analysis.

References

[1] J. W. S. Rayleigh, Proc. Roy. Soc. A, **93**, 148 (1916)
[2] D. H. Michael, Mathematika **1**, 45 (1954)
[3] L. N. Howard and A. S. Gupta, J. Fluid Mech. **14**, 463 (1962)
[4] L. N. Howard, Studies in Appl. Math. **LII**, 39, (1973)
[5] G. Chimonas, J. Fluid Mech. **43**, 833 (1970)
[6] L. N. Howard, J. Fluid Mech. **10**, 509 (1961)