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A HOMOGENEOUS SOLUTION FOR VISCOUS FLOW AROUND A HALF-PLANE*

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1. Introduction. Many questions remain unanswered about the classical problem of the two-dimensional steady flow of a viscous fluid in the presence of a half-plane. In the context of the Navier–Stokes equations, neither existence nor uniqueness of a solution has been rigorously established. Various approximate theories as well as numerical analysis do provide convincing evidence of the existence and general character of the expected solution. The uniqueness question has been less well explored. For the Oseen equations Olmstead and Hector [7] have demonstrated the nonuniqueness of the half-plane problem. By application of the Wiener–Hopf technique, they obtained a rather complicated integral expression for the homogeneous solution. The purpose of this note is to present a very simple and concise form of that homogeneous solution, and to examine its behavior. The possible connection with the Navier–Stokes problem is also discussed.

Consider the steady two-dimensional flow exterior to the half-plane ($y = 0$, $x \geq 0$) where the velocity components $u(x, y)$ and $v(x, y)$ together with the pressure $p(x, y)$ are required to satisfy the Oseen equations.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \qquad m\frac{\partial u}{\partial x} = -\frac{\partial p}{\partial x} + \nabla^2 u, \qquad m\frac{\partial v}{\partial x} = -\frac{\partial p}{\partial y} + \nabla^2 v. \tag{1.1}$$

Here $m > 0$ is some parameter of the problem, although any such parameter could be eliminated from the half-plane problem by a redefinition of the space coordinates. However, it will be convenient to have this parameter available for later discussion.

In the homogeneous problem, the boundary conditions take the form

$$\begin{aligned} &u = v = 0 \quad \text{on} \quad y = 0, \qquad x \geq 0; \\ &u \to 0, \quad v \to 0, \quad p \to 0 \quad \text{as} \quad r \to \infty \end{aligned} \tag{1.2}$$

where $r = (x^2 + y^2)^{1/2}$.

A nontrivial solution of (1.1, 2) is provided by the expressions

$$\begin{aligned} u(x, y) &= \frac{C_0 y}{r(r - x)^{1/2}}\,[1 - \exp[-\tfrac{1}{2}m(r - x)]], \\ v(x, y) &= \frac{C_0(r - x)^{1/2}}{r}\,[1 - \exp[-\tfrac{1}{2}m(r - x)]], \\ p(x, y) &= -\frac{C_0 m y}{r(r - x)^{1/2}}, \end{aligned} \tag{1.3}$$

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for any choice of the multiplicative constant C_0. These velocity components correspond to the stream function

$$\psi(x, y) = C_0\{2(r - x)^{1/2} - (2\pi/m)^{1/2} \operatorname{erf} [(m/2)^{1/2}(r - x)^{1/2}]\} \tag{1.4}$$

where the half-plane ($y = 0$, $x \geq 0$) coincides with the streamline $\psi = 0$.

Verification that (1.3) is in fact a solution of (1.1) is tedious but straightforward. A shortcut to verification comes from observing that the stream function can be expressed as $\psi = \psi_0 + \psi_1$ where $\psi_0 = 2C_0(r - x)^{1/2}$ and $\psi_1 = -(2\pi/m)^{1/2}C_0 \operatorname{erf} [(m/2)^{1/2}(r - x)^{1/2}]$. Then it is relatively easy to verify that $\nabla^2\psi_0 = 0$ and $\nabla^2\psi_1 = m\partial\psi_1/\partial x$. Olmstead [5] has shown that these conditions are sufficient to yield a solution of (1.1) with $p = -m\,\partial\psi_0/\partial y$. Verification of the boundary conditions easily follows.

The details by which the concise expressions of (1.3) were obtained from the integral expressions of [7] are omitted here. It is simply a matter of recognizing that the desired integrals can be put in terms of some others which have been evaluated by Gautesen [1].

The homogeneous solution (1.3) is interesting not only in its own right, but also for the implication of nonuniqueness in certain inhomogeneous half-plane problems. For example, in the case of uniform flow past the half-plane, the boundary conditions attached to (1.1) are

$$\begin{aligned} &u = v = 0 \quad \text{on} \quad y = 0, \qquad x \geq 0; \\ &u \to 1, \quad v \to 0, \quad p \to 0 \quad \text{as} \quad r \to \infty, \qquad 0 < \theta < 2\pi. \end{aligned} \tag{1.5}$$

Here $\theta = \tan^{-1}(y/x)$.

A symmetric solution of (1.1, 5) has been derived by Lewis and Carrier [4] and Kaplun [2]. It can be expressed as

$$\begin{aligned} u(x, y) &= \operatorname{erf} [(m/2)^{1/2}(r - x)^{1/2}] - \frac{(r - x)^{1/2}}{(2\pi m)^{1/2} r} [1 - \exp(-\tfrac{1}{2}m(r - x))], \\ v(x, y) &= \frac{y}{(2\pi m)^{1/2}(r - x)^{1/2} r} [1 - \exp(-\tfrac{1}{2}m(r - x))], \\ p(x, y) &= (m/2\pi)^{1/2} \frac{(r - x)^{1/2}}{r}. \end{aligned} \tag{1.6}$$

These velocity components correspond to the stream function

$$\psi(x, y) = y \operatorname{erf} \left[\left(\frac{m}{2}\right)^{1/2}(r - x)^{1/2}\right] - \left(\frac{2}{m\pi}\right)^{1/2} \frac{y}{(r - x)^{1/2}} [1 - \exp(-\tfrac{1}{2}m(r - x))] \tag{1.7}$$

where the centerline ($y = 0$, $-\infty < x < \infty$) coincides with the streamline $\psi = 0$.

From a physical viewpoint it is perplexing to find that (1.3) can be added to (1.6) and still have (1.1, 5) satisfied. As pointed out in [7], other conditions must be imposed before (1.6) becomes the unique solution of (1.1, 5). In this example, it seems natural to impose a symmetry condition and thereby preclude (1.3). However, in problems where symmetry is not a logical specification, then the rôle of (1.3) is unclear. Such a situation arises, for example, in the problem of a vertical point force in front of the half-plane as treated by Olmstead and Byrne [6].

2. Properties of the homogeneous solution. The fluid motion described by the homogeneous solution (1.3) is that of circulation around the half-plane. In particular the stream function relation (1.4) implies that $\psi = F[(r - x)^{1/2}]$ where

$$F(z) = C_0\{2z - (2m/\pi)^{1/2} \operatorname{erf}[(m/2)^{1/2}z]\}$$

is monotone for $z \geq 0$ and infinitely differentiable. Inversion of the stream function relation yields $(r - x)^{1/2} = F^{-1}(\psi)$, or equivalently

$$y^2 = [F^{-1}(\psi)]^2\{[F^{-1}(\psi)]^2 + 2x\}. \tag{2.1}$$

Thus the streamlines form a family of parabolas with focus at the origin, and $\psi = 0$ corresponds to the half-plane $(y = 0, x \geq 0)$.

The means for generating this circulatory motion around the half-plane can be explained from the asymptotic behavior of the velocity and pressure as $r \to \infty$. Substitution of polar coordinates into (1.3) gives

$$\begin{aligned} u &= \frac{C_0 \cos(\theta/2)}{(2r)^{1/2}}[1 - \exp(-\tfrac{1}{2}mr(1 - \cos\theta))] \\ v &= \frac{C_0 \sin(\theta/2)}{(2r)^{1/2}}[1 - \exp(-\tfrac{1}{2}mr(1 - \cos\theta))] \\ p &= \frac{-mC_0 \cos(\theta/2)}{(2r)^{1/2}}. \end{aligned} \tag{2.2}$$

Hence it is seen that u, v, p are each $O(r^{-1/2})$ uniformly in θ as $r \to \infty$. This behavior implies that the work done over a circle of infinite radius is nonzero even though the velocity and pressure both tend to zero. That is, as $r \to \infty$,

$$\int_0^{2\pi} p(u^2 + v^2)^{1/2} r \cos(\theta/2)\, d\theta \not\to 0.$$

To determine the forces exerted on the half-plane by the flow, the normal stress $\tau_{yy} = -p + \partial v/\partial y$ and the shear stress $\tau_{xy} = \partial u/\partial y + \partial v/\partial x$ must be evaluated on each side of the half-plane. It is found that

$$\begin{gathered} \tau_{yy}(x, 0+) - \tau_{yy}(x, 0-) = p(x, 0+) - p(x, 0-) = -2^{3/2}C_0 m x^{-1/2}, \\ \tau_{xy}(x, 0+) = \tau_{xy}(x, 0-) = 0, \qquad x > 0. \end{gathered} \tag{2.3}$$

Thus this flow produces no drag on the half-plane, but the jump in pressure does provide a net lift. These results were obtained in [7] by means of the Wiener–Hopf technique.

The vorticity $\omega = \partial v/\partial x - \partial u/\partial y$ is found to be

$$\omega(x, y) = -\frac{m\pi^{1/2}(r - x)^{1/2}}{2r} \exp(-\tfrac{1}{2}m(r - x)). \tag{2.4}$$

Along any streamline, the vorticity depends only on the distance from the leading edge of the half-plane so that $\omega = c(\psi)r^{-1}$. Also, the vorticity vanishes on the half-plane for $x > 0$.

The velocity components vanish on the half-plane and at infinity; consequently they attain extremum values in the interior of the flow field. The critical points are easily determined from the gradients of u and v. It is found that each critical point depends on the number $N^* = 1.256 \cdots$ which satisfies the transcendental equation

$$\exp(N^*) = 2N^* + 1. \tag{2.5}$$

If $C_0 > 0$, then $u \operatorname{sgn} y \geq 0$ and $v \geq 0$ with

$$\begin{aligned} u_{\max} &= u\left(0, \frac{2}{m} N^*\right) = \frac{C_0(2mN^*)^{1/2}}{2N^* + 1} = -u\left(0, -\frac{2}{m} N^*\right) = -u_{\min}, \\ v_{\max} &= v\left(-\frac{1}{m} N^*, 0\right) = \frac{2C_0(2mN^*)^{1/2}}{2N^* + 1}. \end{aligned} \tag{2.6}$$

Moreover, all of the critical points

$$\left(0, \frac{2}{m} N^*\right), \left(0, -\frac{2}{m} N^*\right), \left(-\frac{1}{m} N^*, 0\right)$$

lie on the same streamline.

3. The Navier–Stokes problem. For Navier–Stokes flow the nonlinear equations

$$\begin{aligned} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \\ m\left[u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y}\right] &= -\frac{\partial p}{\partial x} + \nabla^2 u \\ m\left[u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y}\right] &= -\frac{\partial p}{\partial y} + \nabla^2 v \end{aligned} \tag{3.1}$$

are considered instead of the Oseen equations. These homogeneous equations together with the homogeneous boundary conditions (1.2) pose an interesting problem, but it should be noted that the implications are not quite the same as for the linear problem. The existence of a nontrivial solution to the nonlinear problem would not immediately imply nonuniqueness as it does in the linear case. The reason, of course, is that the homogeneous solution could not simply be added to an inhomogeneous solution and still satisfy (3.1). Nevertheless, a nontrivial solution of the homogeneous problem (3.1, 1.2) would certainly challenge whether a related inhomogeneous problem was well posed.

Some tentative arguments can be made about a possible solution of the homogeneous problem (3.1, 1.2) in the asymptotic sense as $m \to 0$. In particular, a solution with the same general character as (1.3) is sought.

With $m \to 0$ and (x, y) confined to some bounded region near the half-plane $(y = 0, x \geq 0)$, an inner problem is suggested by (3.1) with $m = 0$. A solution to these Stokes equations which satisfies the homogeneous conditions on the half-plane is provided by

$$\begin{aligned} \bar{u}(x, y) &= \frac{Cm^{3/2}y(r - x)^{1/2}}{2r}, \\ \bar{v}(x, y) &= \frac{Cm^{3/2}(r - x)^{3/2}}{2r}, \\ \bar{p}(x, y) &= -\frac{Cm^{3/2}y}{r(r - x)^{1/2}}. \end{aligned} \tag{3.2}$$

The velocity components are derivable from the stream function

$$\bar{\psi}(x, y) = (Cm^{3/2}/3)(r - x)^{3/2}. \tag{3.3}$$

As might be expected, u and v diverge as $r \to \infty$, $0 < \theta < 2\pi$, and so fail to satisfy the desired conditions at infinity.

For a flow with the same general character as (1.3), it is natural to ask that the behavior at infinity correspond to the potential flow solution for circulation around a half-plane. This well-known result (e.g., see Lamb [3]) can be expressed as

$$\begin{aligned} \hat{u}(x, y) &= Cm^{1/2}y/r(r - x)^{1/2}, \\ \hat{v}(x, y) &= Cm^{1/2}(r - x)^{1/2}/r, \\ \hat{p}(x, y) &= -C^2m^2/r \end{aligned} \tag{3.4}$$

with the stream function

$$\hat{\psi}(x, y) = 2Cm^{1/2}(r - x)^{1/2}. \tag{3.5}$$

The velocity components and pressure given by (3.4) satisfy the full nonlinear equations (3.1) and they have the required circulation behavior at infinity. Of course, the homogeneous conditions on the half-plane are not satisfied.

Both of these stream functions, $\bar{\psi}$ for the inner problem and $\hat{\psi}$ for the outer problem, are derivable as asymptotic limits from the single expression (1.4). Let $C_0 = m^{1/2}C$, $X = m^2x/2$, $Y = m^2y/2$, $R = m^2r/2$. Then (1.4) can be expressed as

$$\begin{aligned} \psi(x, y) &= Cm^{1/2}\left\{2(r - x)^{1/2} - \left(\frac{2\pi}{m}\right)^{1/2} \operatorname{erf}\left[\left(\frac{m}{2}\right)^{1/2}(r - x)^{1/2}\right]\right\} \\ &= 2^{1/2}C\{2m^{-1/2}(R - X)^{1/2} - \pi^{1/2} \operatorname{erf}[m^{-1/2}(R - X)^{1/2}]\}. \end{aligned} \tag{3.6}$$

Now for $m \to 0$ with fixed x, y, it is easily found that $\psi \to \bar{\psi}$ in the inner limit. Moreover, for $m \to 0$ with the stretched variables X, Y fixed, then $\psi \to \hat{\psi}$ in this outer limit.

These arguments do not prove the existence of a small m solution of (3.1, 1.2), but do suggest that possibility.

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