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TURBULENT DISPERSION FROM A PLANE SURFACE*

By

MARK J. BERAN (*Tel-Aviv University*)

ALAN M. WHITMAN (*University of Pennsylvania, Philadelphia*)

Abstract. We consider here the turbulent dispersion of a neutral concentrant from a plane surface. Very near the surface the dispersion is by molecular diffusion since the turbulent velocity is zero at the surface. Within a layer near the surface the mode of transport changes to turbulent exchange. Our principal concern in this paper is to determine the concentration distribution near the surface and the flux of concentrant that is transported from or to the surface. We use as our governing equation an integro-differential equation form which was derived in an earlier paper. In this equation we are required to assume a form for the integral kernel which is compatible with a turbulent state. A kernel was chosen using simple arguments and the resulting concentration distribution is presented. It is shown that under suitable conditions the concentration decays at large distances as an inverse power of the distance from the surface. We also present the dependence of flux on the molecular diffusion coefficient. For some physical situations the concentration will decay at a rate inversely proportional to the square of the distance, and the flux will be proportional to $D^{2/3}$. (D is the molecular diffusion coefficient.) For two particular physical applications the validity of the above solution is discussed. We also compare the results obtained with those found using an eddy diffusivity. It is shown that in the presence of sources the two results may differ substantially.

1. Introduction. We wish to consider here the one-dimensional steady-state transport of a neutral concentrant from a plane surface. We assume that on a surface $z = 0$ the concentration is fixed at C_0, and suitable conditions are given at $z = L$. We shall be particularly interested in the case in which $L \to \infty$. We suppose that above the surface the mean velocity $\{\mathbf{v}(z)\}$ is independent of x, y, and t and has no component in the z direction. The turbulence is assumed stationary and statistically homogeneous in the x and y directions. At the surface the velocity is zero. The non-steady equation governing the concentration distribution is taken to be

$$\frac{\partial C(\mathbf{x}, t)}{\partial t} + \{v_i(z)\} \frac{\{C(\mathbf{x}, t)\}}{\partial x_i} + v_i'(\mathbf{x}, t) \frac{\partial C(\mathbf{x}, t)}{\partial x_i} = D\nabla^2 C(\mathbf{x}, t) + s(\mathbf{x}, t). \qquad (1)$$

$v_i'(\mathbf{x}, t)$ is the turbulent velocity and $s(\mathbf{x}, t)$ is a non-random source term.

We desire an equation for the mean concentration $\{C(\mathbf{x}, t)\}$ where $C(\mathbf{x}, t) = \{C(\mathbf{x}, t)\} + C'(\mathbf{x}, t)$. The brackets, $\{\ \}$, indicate an ensemble average.

In previous work [1, 2] we showed that if Eq. (1) is *formally* iterated, $\{C(\mathbf{x}, t)\}$ satisfies the equation

$$\frac{\partial \{C(\mathbf{x}, t)\}}{\partial t} + \{v_i(z)\} \frac{\partial \{C(\mathbf{x}, t)\}}{\partial x_i} = D\nabla^2 \{C(\mathbf{x}, t)\}$$

$$+ \frac{\partial}{\partial x_i} \int_V \int_{-\infty}^t E_{ij}(\mathbf{x}, t, \mathbf{x}_1, t_1) \frac{\partial}{\partial x_{1j}} \{C(\mathbf{x}_1, t_1)\}\, d\mathbf{x}_1\, dt_1 + s(\mathbf{x}, t). \qquad (2)$$

$E_{ij}(\mathbf{x}, \mathbf{x}_1, t, t_1)$ is a function of the Green's function of the three non-integral terms and all order velocity correlation functions of the form

$$N_{ij} \cdots {}_p(\mathbf{x}_1, t_1, \mathbf{x}_2, t_2, \cdots, \mathbf{x}_n, t_n) = \{v_i(\mathbf{x}_1, t_1)v_j(\mathbf{x}_2, t_2) \cdots v_p(\mathbf{x}_n, t_n)\}.$$

It is given as an infinite series, the convergence properties of which are generally unknown.

In the case where molecular diffusion dominates the turbulent diffusion $E_{ij}(\mathbf{x}, \mathbf{x}_1, t, t_1)$ is given by the first term of the series. Here

$$E_{ij}(\mathbf{x}, t, \mathbf{x}_1, t_1) = G(\mathbf{x}, t, \mathbf{x}_1, t_1)N_{ij}(\mathbf{x}, t, \mathbf{x}_1, t_1).$$

G is an appropriate Green's function of Eq. (1).

In this paper, and in most applications, we are interested in cases in which this approximation is inapplicable. We thus are faced with the problem of evaluating an infinite series to find $E_{ij}(\mathbf{x}, t, \mathbf{x}_1, t_1)$. This is clearly impractical, and we instead take the position in this paper that Eq. (2) is the correct form of the equation governing $\{C(\mathbf{x}, t)\}$, but $E_{ij}(\mathbf{x}, t, \mathbf{x}_1, t_1)$ is an *unknown* function that must be determined experimentally or formulated from other arguments.

We observe that by assuming

$$E_{ij}(\mathbf{x}, t, \mathbf{x}_1, t_1) = D_{Tij}(\mathbf{x})\delta(\mathbf{x} - \mathbf{x}_1)\delta(t - t_1) \qquad (3)$$

and evaluating the integral in Eq. (2) we can obtain the usual eddy diffusivity form of the turbulent dispersion equation

$$\frac{\partial \{C(\mathbf{x}, t)\}}{\partial t} + \{v_i(z)\} \frac{\partial \{C(\mathbf{x}, t)\}}{\partial x_i} = D\nabla^2 \{C(\mathbf{x}, t)\}$$

$$+ \frac{\partial}{\partial x_i} D_{Tij}(\mathbf{x}) \frac{\partial}{\partial x_j} \{C(\mathbf{x}, t)\} + s(\mathbf{x}, t). \qquad (4)$$

There is a large literature (see for example, [3]) in which Eq. (3) is taken as the phenomenological equation governing turbulent dispersion. Various forms are assumed for $D_{Tij}(\mathbf{x})$, depending upon the physical problem. The physical argument justifying the use of Eq. (4) is that the characteristic size of eddies responsible for the turbulent transport is small compared to the characteristic distance over which $\{C(\mathbf{x}, t)\}$ varies significantly. Near boundaries this condition is not met and in this region there is no physical basis for the assumption contained in Eq. (3).

In this paper we assume that the integral equation form given in Eq. (2) is the correct phenomenological form of the equation representing turbulent dispersion. We maintain that near boundaries $E_{ij}(\mathbf{x}, t, \mathbf{x}_1, t_1)$ is very different from that given in Eq. (3), and that a proper understanding of turbulent dispersion near boundaries cannot be obtained using Eq. (4).

Just as workers in the past have attempted to determine the form of $D_{T ij}(\mathbf{x})$ based on simple physical arguments, we shall in this paper attempt to determine the form of $E_{ij}(\mathbf{x}, t, \mathbf{x}_1, t_1)$ based on simple physical arguments. Criticism of this approach has been that it is more desirable to find $E_{ij}(\mathbf{x}, t, \mathbf{x}_1, t_1)$ from first principles by terminating in some way the hierarchy of statistical equations governing $\{C(\mathbf{x}, t)\}$. The authors themselves are very partial to these efforts, but unfortunately found that despite a great deal of work, the proper termination is still in doubt. We therefore have decided to explore the alternate phenomenological approach in an attempt to understand the nature of the integral kernel and obtain useful solutions of Eq. (2) in the vicinity of boundaries.

By making use of the assumptions of stationarity and statistical homogeneity in x and y and taking $s(\mathbf{x}, t) = s(z)$, we have here

$$\{C(\mathbf{x}, t)\} = \{C(z)\} \equiv \bar{C}(z)$$

and Eq. (2) becomes simply

$$D \frac{d^2}{dz^2} \bar{C}(z) + \frac{d}{dz} \int_0^\infty B(z, z_1, D) \frac{d}{dz_1} \bar{C}(z_1)\, dz_1 + s(z) = 0 \tag{5}$$

where

$$B(z, z_1, D) = \frac{1}{2} \iiint\limits_{-\infty}^{\infty} E_{33}(\mathbf{x}, t, \mathbf{x}_1, t_1)\, dx_1\, dy_1\, dt_1 .$$

We now explicitly show the D dependence in the kernel B.

We thus only have to determine $B(z, z_1, D)$, a function of two variables and D. Further, Eq. (5) may be integrated once. We find

$$D \frac{d\bar{C}(z)}{dz} + \int_0^\infty B(z, z_1, D) \frac{d}{dz_1} \bar{C}(z_1)\, dz_1 = F + S(z) \tag{6}$$

where F is a constant and $S(z) = - \int_0^z s(z')\, dz'$.

One property of $B(z, z_1, D)$ may be immediately introduced if we examine the formal series representation of E_{ij}. This is the condition

$$B(z, z_1, D) = 0 \quad \text{if either} \quad z \text{ or } z_1 = 0. \tag{7}$$

This follows from the physical fact that $v_i = 0$ on the plane surface. In this case $F = DC_0'$ where C_0' is derivative of $\bar{C}$ at $z = 0$. Thus

$$D \frac{d\bar{C}(z)}{dz} + \int_0^\infty B(z, z_1, D) \frac{d}{dz_1} \bar{C}(z_1)\, dz_1 = DC_0' + S(z) \tag{8}$$

Specifically, we shall consider in this paper the case in which turbulent diffusion dominates molecular diffusion except in the very immediate vicinity of the wall. If we characterize the turbulent diffusion by the length l_c of the turbulent intensity variation and the molecular diffusion by the length $l_D = D/\bar{v}$ of molecular diffusion variation ($\bar{v}$ is some characteristic velocity), then our interest is in the case

$$l_D \ll l_c . \tag{9}$$

If it is also true that any overall length scale, L (distance between two parallel walls, etc.), is much greater than l_c , then the domain of definition of Eq. (8) is $0 \leq z \leq \infty$ and we can impose the boundary conditions

$$\bar{C} = \bar{C}_0 \quad \text{at} \quad z = 0,$$

$$\bar{C} \to \bar{C}_F \quad \text{as} \quad z \to \infty.$$

In the next section we shall choose a form for $B(z, z_1 , D)$ subject to the above restrictions and then solve Eq. (8) in the infinite half-space $0 \leq z < \infty$. Finally, we consider in Sec. 4 two physical problems to which we may apply the basic solution.

2. Form of the kernel function $B(z, z_1 , D)$.

2.1 *Measurement of $B(z, z_1 , D)$.*   In this section we shall choose a form for the kernel function $B(z, z_1 , D)$ and then give a solution to Eq. (8). Before doing this, however, it is important to note that in principle $B(z, z_1 , D)$ may be measured if a sufficient number of experiments are performed with an arbitrary concentration source $s(z)$.

If we choose $\bar{p}(z) \approx 0$ for $z > L$, Eq. (8) may be written as a finite integral in the form

$$D\bar{p}(z) + \int_0^L B(z, z_1 , D)\bar{p}(z_1) \, dz_1 = DC_0' + \int_0^z s(z') \, dz' \tag{10}$$

where $\bar{p}(z) \equiv d\bar{C}(z)/dz$.

Let us suppose that the integral may be evaluated by replacing it by

$$\sum_{n=1}^N B(z, z_n , D)\bar{p}(z_n) \, \Delta z$$

where $\Delta z = L/(N + 1)$. For the points $z = z_m$, Eq. (10) may be written as

$$\bar{p}_m + \sum_{n=1}^N B_{mn}\bar{p}_n = T_m$$

where

$$\bar{p}_m = \bar{p}(z_m), \qquad B_{mn} = B(z_m , z_n , D)(\Delta z/D), \qquad T_m = C_0' + \frac{1}{D}\int_0^{z_m} s(z') \, dz'.$$

If $\bar{p}_m$ and T_m are measured, we have m equations which the constants B_{mn} must satisfy. For example, the constants B_{1n} must satisfy the equation

$$\bar{p}_1 + \sum_{n=1}^N B_{1n}\bar{p}_n = T_1 . \tag{11}$$

To actually determine the constants B_{1n} we must make N independent measurements; that is, N experiments with the same turbulent conditions and value of D but for different values of T_m . These same experiments may be used to determine B_{2n} and in general B_{in} . Some care must be taken in following this procedure, however, since the problem set in Eq. (10) for unknown $B(z, z_1 , D)$ is properly posed only in the sense of Tykhonov [4].

If $N > 10$, the above procedure is impractical. If, however, Eq. (11) is considered as a constraint condition, it may be used very advantageously to check theoretical choices of $B(z, z_1 , D)$. We feel that it is worth considerable effort to gain as much experimental evidence about $B(z, z_1 , D)$ as possible since it contains the entire effect of turbulent steady-state transport. Once experience with this function has been gained, it will not be difficult to make reasonable theoretical choices for $B(z, z_1 , D)$.

2.2 *D dependence in* $B(z, z_1, D)$. $B(z, z_1, D)$ is a function of all order velocity correlation functions and a Green's function that depends on D. In terms of the characteristic velocity $\bar{v}$ and characteristic distance l_c, we may write

$$B(z, z_1, D) = \bar{v}\mathcal{B}(\bar{z}, \bar{z}_1, \epsilon)$$

where $\bar{z} = z/l_c$, $\epsilon = D/\bar{v}l_c$, and $\mathcal{B}$ is a dimensionless function. Using these transformations we can write Eq. (8) in the following form:

$$\epsilon \frac{d\bar{C}}{d\bar{z}} + \int_0^\infty \mathcal{B}(\bar{z}, \bar{z}_1, \epsilon) \frac{d\bar{C}}{d\bar{z}_1} d\bar{z}_1 = \epsilon C_0' l_c + \frac{l_c}{\bar{v}} \int_0^{\bar{z}} s(z_1) dz_1 .$$

As we have remarked earlier, our only interest in this paper is the case $\epsilon \ll 1$. From the definition above we see that ϵ measures the ratio of molecular diffusion to turbulent dispersion and thus, as we have indicated earlier, this restriction refers to the case for which molecular diffusion is negligible compared with turbulent dispersion on the length scale l_c. In the limit $\epsilon \to 0$ we obtain from the equation above

$$\int_0^\infty \mathcal{B}(\bar{z}, \bar{z}_1, 0) \frac{d\bar{C}}{d\bar{z}_1} d\bar{z}_1 \sim \frac{l_c}{\bar{v}} \int_0^{\bar{z}} s(\bar{z}_1) d\bar{z}_1$$

This Fredholm equation yields a solution which is necessarily inconsistent in the vicinity of the wall. At the wall itself, molecular diffusion is dominant and therefore there must be a region in the vicinity of the wall in which the molecular and turbulent processes are of the same order. $\mathcal{B}$ approaches zero with $\bar{z}$ and $\bar{z}_1$, so let us suppose that under the scaling $\bar{z} = \epsilon^k \eta$ we obtain

$$\mathcal{B}(\bar{z}, \bar{z}_1, \epsilon) = \epsilon^{ik} G(\eta, \eta_1, \epsilon).$$

Here we have recognized that in general G is a different function of η, η_1, ϵ than $\mathcal{B}$ is of $\bar{z}$, $\bar{z}_1$, ϵ. Hence, even though $\mathcal{B}(\bar{z}, \bar{z}_1, 0)$ is independent of diffusion, $G(\eta, \eta_1, 0)$ contains a diffusion effect[1].

We now choose k so that the molecular diffusion and turbulent dispersion terms are the same order of magnitude. This gives $k = 1/(1 + j)$. The boundary layer equation is then, to leading order in ϵ,

$$\frac{d\bar{C}}{d\eta} + \int_0^\infty G(\eta, \eta_1, 0) \frac{d\bar{C}}{d\eta_1} d\eta_1 \sim N' + \bar{S}(\eta). \tag{12}$$

In Eq. (12) N' is a pure number such that

$$C_0' = \frac{N'}{\epsilon^{1/(j+1)} l_c}$$

and $\bar{S}(\eta)$ is a scaled source term

$$\bar{S}(\eta) = \frac{l_c \epsilon^{(1-i)/(1+i)}}{\bar{v}} \int_0^\eta \bar{s}(\eta_1) d\eta_1 .$$

We see that in terms of the scaled variable, molecular diffusion enters Eq. (12) in two ways. First, there is the direct effect which is denoted by the first term on the left.

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[1] We expect that the asymptotic series for $\mathcal{B}(\bar{z}, \bar{z}_1, \epsilon)$ will be nonuniformly valid in ϵ. For example with $\mathcal{B}(\bar{z}, \bar{z}_1, \epsilon) = \bar{z}\bar{z}_1 + \epsilon(\bar{z}\bar{z}_1 + \bar{z}^{1/2}\bar{z}_1^{1/2})$ we find under the scaling $\bar{z} = \epsilon\eta$, $\mathcal{B}(\bar{z}, \bar{z}_1, \epsilon) = \epsilon^2[\eta\eta_1 + \eta^{1/2}\eta_1^{1/2} + \epsilon\eta\eta_1] = \epsilon^2 G(\eta, \eta_1, \epsilon)$. In this case $\mathcal{B}(\bar{z}, \bar{z}_1, 0) = \bar{z}\bar{z}_1 \neq \epsilon^2 G(\eta, \eta_1, 0)$ since $G(\eta, \eta_1, 0) = \eta\eta_1 + \eta^{1/2}\eta_1^{1/2}$.

Second, there is the indirect effect contained in G in which molecular diffusion changes the particle streamline and hence the ultimate turbulent dispersion. We note that on physical grounds:

$$\lim_{\eta \to \infty} G(\eta, \eta_1, 0) = \mathcal{B}(\eta, \eta_1, 0),$$

$$\lim_{\eta_1 \to \infty} G(\eta, \eta_1, 0) = \mathcal{B}(\eta, \eta_1, 0).$$

Here $\mathcal{B}(\eta, \eta_1, 0)$ is obtained by transforming $\mathcal{B}(\bar{z}, \bar{z}_1, 0)$ using $\bar{z} = \epsilon^k \eta$ and dividing by ϵ^{ik}. The physical content of these statements is that both the direct and indirect effects of molecular diffusion must be important in exactly the same region, namely the boundary layer discussed above. In mathematical terms we may state it as follows. The non-uniformly valid asymptotic expansion (in ϵ) of $\mathcal{B}$ fails on exactly the same scale for which the integral and derivative terms balance. Then the limit statements above express the fact that there is a common domain of validity between the boundary layer and the main region.

Despite the boundary layer simplification, the problem of deducing the form of $\mathcal{B}(\bar{z}, \bar{z}_1, \epsilon)$ and consequently $G(\eta, \eta_1, 0)$ is still very difficult. We therefore consider instead the equation

$$\frac{d\bar{C}}{d\eta} + \int_0^\infty \mathcal{B}(\eta, \eta_1, 0) \frac{d\bar{C}}{d\eta_1} \, d\eta_1 \sim N + \bar{S}(\eta). \tag{13}$$

In Eq. (13) only the direct effect of molecular diffusion is included. The solutions of Eqs. (12) and (13), however, are very much alike. In the region $\eta \gg 1$, they are asymptotically identical by virtue of the limit conditions above. In addition, when $\eta \ll 1$, they are also asymptotically identical because in this region they depend principally on molecular diffusion. Therefore, only in the vicinity of $\eta = 1$ do the two solutions differ and hence, N' and N, which are obtained by integrating $d\bar{C}/d\eta$ from 0 to ∞, are not very different. It is also clear that the scaling factor k must be the same in each case since the size of $\mathcal{B}(\bar{z}, \bar{z}_1, \epsilon)$ in the boundary layer is, in any event, governed by $\mathcal{B}(\bar{z}, \bar{z}_1, 0)$. In the remainder of the paper we shall thus replace $\mathcal{B}(\bar{z}, \bar{z}_1, \epsilon)$ by $\mathcal{B}(\bar{z}, \bar{z}_1, 0)$. For convenience we shall use the abbreviated notation $\mathcal{B}(\bar{z}, \bar{z}_1) \equiv \mathcal{B}(\bar{z}, \bar{z}_1, 0)$.

2.3 *Theoretical form for* $\mathcal{B}(\bar{z}, \bar{z}_1)$. To determine the form of $\mathcal{B}(\bar{z}, \bar{z}_1)$, we recognize that there are two aspects of the turbulent field that must be considered. First, we must account for the variation of intensity of turbulence as a function of position, and secondly, we must account for the correlation effects associated with the turbulence. There is, however, a very natural way to exhibit these effects in $\mathcal{B}(\bar{z}, \bar{z}_1)$. The intensity effect may be represented by $\mathcal{B}(\bar{z}, \bar{z})$ and the correlation effect by the characteristic distance, $\bar{z} - \bar{z}_1$, in which $\mathcal{B}(\bar{z}, \bar{z}_1)$ is significantly reduced in magnitude.

In our subsequent formulation we shall be guided by this representation and shall consider $\mathcal{B}(\bar{z}, \bar{z}_1)$ to have the same character as the function

$$\bar{R}_{33} = \iiint\limits_{-\infty}^{\infty} R_{33}(\bar{z}, \bar{z}_1, \Delta x, \Delta y, \tau) \, d\Delta x \, d\Delta y \, d\tau,$$

where R_{33} is the two-point correlation function of velocity components in the z-direction. (Here $\Delta x = x - x_1$, $\Delta y = y - y_1$, $\tau = t - t_1$.) If we chose $\mathcal{B}$ to be proportional to $\bar{R}_{33}$,

and R_{33} was known from measurement, we could, in principle, determine $\bar{C}(\bar{z})$ exactly. At this time, however, complete measurements of R_{33} are not available, and since the identification is at best approximate, we assume here only that $\mathfrak{B}$ and $\bar{R}_{33}$ have the same overall characteristics. In choosing the functional form of $\mathfrak{B}$ in what follows, we shall be guided by what we know of the physical properties of $\bar{R}_{33}$.

If we choose to describe $\mathfrak{B}(\bar{z}, \bar{z}_1)$ over a large distance from the wall, this function will be quite complex. As we stated earlier, we focus our attention here, however, on the cases in which turbulent dispersion is the dominant transport mechanism except in the very immediate neighborhood of the wall. It is easier to consider the two effects by writing $\mathfrak{B}(\bar{z}, \bar{z}_1)$ in terms of the variables $\bar{z}\bar{z}_1$ and $\bar{z} - \bar{z}_1$, i.e. $\widetilde{\mathfrak{B}}(\bar{z}\bar{z}_1, \bar{z} - \bar{z}_1)$.[2] The variable $\bar{z}\bar{z}_1$ is chosen rather than $\bar{z} + \bar{z}_1$ so that we may easily meet the conditions $\mathfrak{B}(0, \bar{z}_1) = \mathfrak{B}(\bar{z}, 0) = 0$. We begin by assuming that $\widetilde{\mathfrak{B}}(\bar{z}\bar{z}_1, \bar{z} - \bar{z}_1)$ may be expanded in a Taylor series of $\bar{z}\bar{z}_1$ about the point $\bar{z}\bar{z}_1 = 0$. Thus, we have

$$\widetilde{\mathfrak{B}}(\bar{z}\bar{z}_1, \bar{z} - \bar{z}_1) = a_1(\bar{z} - \bar{z}_1)\bar{z}\bar{z}_1 + a_2(\bar{z} - \bar{z}_1)(\bar{z}\bar{z}_1)^2 + \cdots . \tag{14}$$

To study the intensity effect we set $\bar{z} - \bar{z}_1 = 0$ so that

$$\widetilde{\mathfrak{B}}(\bar{z}^2, 0) = a_1(0)\bar{z}^2 + a_2(0)\bar{z}^4 + \cdots \tag{15}$$

where $a_i(0)$ are now constants. Since we are concerned with effects very close to the wall, we shall keep only the first non-zero term in Eq. (15). We assume initially that $a_1 \neq 0$. For generality, however, we shall write a_n for the first non-zero term in the event that experiment shows that the first non-zero term is a_n ($n > 1$).[3]

Thus, we shall in our subsequent work consider the form

$$\mathfrak{B}(\bar{z}^2, 0) = H_n\bar{z}^{2n}$$

where H_n is a numerical factor of order unity.

The form of $a_n(\bar{z} - \bar{z}_1)$ is less easy to picture. We shall be guided by simplicity and assume that in this function the correlation distance is proportional to the distance from the wall. Physically, this means that the "eddies" responsible for the turbulent transport increase in size at a rate proportional to distance from the wall. In the context of mixing-length theory it is similar to Prandtl's idea that the mixing length is proportional to distance from the wall. While we do not feel that the turbulent diffusion coefficient is a valid concept near the wall, we do agree that the correlation ("eddy size") effects in the transport mechanism increase with distance from the wall. A linear assumption is perhaps the simplest way to incorporate this effect in our equation.

We incorporate this linear effect in a simple manner by assuming the correlation between z and z_1 is a constant or zero. The form we choose for $\mathfrak{B}(\bar{z}, \bar{z}_1)$ is finally

$$\mathfrak{B}(\bar{z}, \bar{z}_1) = H_n\bar{z}^n\bar{z}_1{}^n, \qquad \alpha\bar{z} \leq \bar{z}_1 \leq \frac{1}{\alpha}\bar{z},$$

$$\tag{16}$$

$$= 0, \qquad \bar{z}_1 < \alpha\bar{z} \quad \text{and} \quad \bar{z}_1 > \frac{1}{\alpha}\bar{z}$$

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[2] We write $\widetilde{\mathfrak{B}}$ to note that the functional form of $\mathfrak{B}(\bar{z}, \bar{z}_1)$ is not the same as $\widetilde{\mathfrak{B}}(\bar{z}\bar{z}_1, \bar{z} - \bar{z}_1)$.

[3] We note that $\bar{R}_{33}(\bar{z}, \bar{z})$ is proportional to z^4, but do not insist that $\mathfrak{B}(\bar{z}, \bar{z})$ have exactly the same functional dependence.

where we set $((1/\alpha) - \alpha)\bar{z} = \bar{z}$ ($\alpha = .618$). The constants α and $1/\alpha$ were used in the limits so that the correlation distances would be the same whether $\bar{z}_1$ was greater or less than $\bar{z}$. The condition $((1/\alpha) - \alpha) = 1$ assumes the correlation interval is equal to the distance from the wall.

We now proceed to solve Eq. (13) using Eq. (16) and the boundary condition $\bar{C}(\bar{z}) \rightarrow C_F$ as $\bar{z} \rightarrow \infty$. We note that even before the solution is compared to experiment, the solution itself may serve to test the validity of Eq. (16). The solution must satisfy the constraint $\bar{C}(\bar{z}) \geq 0$. This is a meaningful constraint, and a number of other choices of $\mathfrak{B}(\bar{z}, \bar{z}_1)$ lead to negative results.

3. Solution of Eq. (13) using Eq. (16) and $\bar{S}(z) = 0$.

3.1 *Nature of the solution.*   In this section we study the solution of the integral equation, Eq. (13), with the kernel given by Eq. (16). Note that the scaling exponent, k, with this choice of $\mathfrak{B}$, is equal to $k = 1/(2n + 1)$.

We write now

$$q = \frac{d\bar{C}}{d\eta} \Big/ N_n \tag{17}$$

with

$$\eta = \bar{z}\epsilon_n^{-1/(2n+1)}. \tag{18}$$

For convenience, we have modified ϵ and N to be

$$\epsilon_n = D/H_n \tilde{v} l_c , \qquad N_n = C_0' l_c \epsilon_n^{1/(2n+1)} \tag{19}$$

We have then, substituting Eq. (16) in Eq. (13),

$$q(\eta) + \eta^n \int_{\alpha\eta}^{(1/\alpha)\eta} \eta_1^n q(\eta_1) \, d\eta_1 = 1. \tag{20}$$

Eq. (20) is a universal equation (for fixed α, n) and hence, once Eq. (20) is solved, we may find $\bar{C}(z)$ for all l_c and ϵ_n. Since we are interested only in $\alpha = .618$ and one value of n, it is almost as useful to have a numerical solution of Eq. (20) as an analytical solution. Moreover, we note that we may find from Eq. (20):

$$q(0) = 1, \qquad q(\eta) \sim \left[\frac{n}{\left(\frac{1}{\alpha}\right)^n - \alpha^n} \right] \frac{1}{\eta^{\frac{1}{2n+1}}} , \qquad \eta \rightarrow \infty \tag{21}$$

We were unable to find an analytic solution to Eq. (20). To find a numerical solution we converted Eq. (20) into a differential equation. Differentiating Eq. (2) yields

$$q'(\eta) + n\eta^{n-1} \int_{\alpha\eta}^{(1/\alpha)\eta} \eta_1^n q(\eta_1) \, d\eta_1 + \frac{\eta^{2n}}{n+1} q\left(\frac{1}{\alpha}\eta\right) - \alpha^{n+1}\eta^{2n}q(\alpha\eta) = 0. \tag{22}$$

The integral term in Eq. (22) may be eliminated by using Eq. (20). We have finally

$$q\left(\frac{1}{\alpha}\eta\right) = \alpha^{2(n+1)}q(\alpha\eta) + \frac{\alpha^{(n+1)}}{\eta^{2n}} \left[\frac{n(q(\eta) - 1)}{\eta} - q'(\eta) \right]. \tag{23}$$

Eq. (23) may be evaluated away from the point $\eta = 0$ if it is known in any finite interval $0 \leq \eta \leq \eta_F$. To find $q(\eta)$ in such an interval we expand $q(\eta)$ in a power series

near $\eta = 0$. The result is

$$q(\eta) = 1 + a_1\eta^{2n+1} + a_2\eta^{4n+2} + \cdots \tag{24}$$

where

$$a_{k+1} = -\frac{a_k}{F(k, n)}\left[\left(\frac{1}{\alpha}\right)^{F(k,n)} - \alpha^{F(k,n)}\right],$$

$$a_0 = 1, \qquad F(k, n) = (2n + 1)k + n + 1.$$

The series in Eq. (24) is asymptotic, but for small η it is adequate for use as a boundary condition for Eq. (23). Tables 1 and 2 were essentially computed using the values given

TABLE 1.

Values of $q(\eta)$ and $h(\eta)$ as a function of η for $n = 1$.

η	$q(\eta)$	$h(\eta)$
0	1.0	0
.1	.999	.10
.2	.99	.20
.3	.97	.30
.4	.93	.40
.5	.88	.49
.6	.82	.57
.7	.75	.65
.8	.67	.73
.9	.58	.79
1.0	.52	.84
1.1	.45	.89
1.2	.39	.94
1.3	.34	.98
1.4	.29	1.01
1.5	.25	1.03
1.6	.21	1.06
1.7	.18	1.08
1.8	.16	1.10
1.9	.13	1.11
2.0	.12	1.13

$$q(\eta) = \frac{1.00}{\eta^3} - \frac{.60}{\eta^6} ; \eta > 2$$

$$h(\eta) = 1.25 - \frac{1}{2\eta^2} + \frac{.12}{\eta^5} ; \eta > 2$$

by Eq. (24) for $0 \leq \eta \leq 2$. The results given in these tables satisfy Eq. (20) to within 5%. Better accuracy than this did not seem necessary at this time. For comparison we also present calculations for $n = 2$.

In some initial work we thought it useful to solve Eq. (20) by expanding $q(\eta_1)$ under the integral in a Taylor series about $\eta_1 = \eta$.

$$q(\eta_1) = \sum_{k=0}^{\infty} \frac{(\eta_1 - \eta)^k}{k!} q^{(k)}(\eta). \tag{25}$$

TABLE 2.

Values of $q(\eta)$ and $h(\eta)$ as a function of η for $n = 2$.

η	$q(\eta)$	$h(\eta)$
0	1.0	0
.1	.9999	.10
.2	.999	.20
.3	.997	.30
.4	.98	.40
.5	.95	.50
.6	.92	.59
.7	.85	.67
.8	.76	.75
.9	.66	.82
1.0	.55	.88
1.1	.46	.93
1.2	.37	.97
1.3	.27	1.00
1.4	.18	1.02
1.5	.12	1.04
1.6	.08	1.05
1.7	.06	1.06
1.8	.05	1.06
1.9	.04	1.07
2.0	.03	1.07

$$q(\eta) = \frac{.894}{\eta^5} - \frac{.216}{\eta^{10}} \; ; \eta > 2$$

$$h(\eta) = 1.08 - \frac{.224}{\eta^4} + \frac{.024}{\eta^9} \; ; \eta > 2$$

This yielded a differential equation of the form

$$\sum_{i=1}^{\infty} b_i \eta^{2n+i+1} q^{(i)}(\eta) + (1 + b_0 \eta^{2n+1}) q(\eta) = 1 \qquad (26)$$

where $b_0 = d(n + 1)$, $b_1 = d(n + 2) - d(n + 1)$, $b_2 = \frac{1}{2}[d(n + 3) - 2d(n + 2) + d(n + 1)]$, and

$$d(n + m) = \frac{1}{n + m} \left(\left(\frac{1}{\alpha}\right)^{n+m} - \alpha^{n+m} \right).$$

An approximate solution of Eq. (22) is obtained by replacing the infinite sum by a finite sum. For $\eta < 1$ this works well, and considering terms up to $j = 3$ yields a good solution. For large values of η, however, values of j of at least 4 or 5 (depending on n) are necessary to get reasonable solutions. In these cases the numerical (computer) solutions become very difficult since extraneous homogeneous solutions appear. The results obtained using Eq. (26) were, however, very useful for complementing the computations when $\eta < 1$.

Finally, we note that the solutions of Eq. (20) that approach zero at infinity are not unique. However, there is only one solution finite at $\eta = 0$. An inverse power series expansion yields a convergent series, but it is of unbounded variation in the neighborhood

of $\eta = 0$. The solution we presented in Tables 1 and 2 goes asymptotically to this solution as $\eta \to \infty$.

Since we desire the distribution of $\bar{C}(z)$, we also tabulated in Tables 1 and 2 the function

$$h(\eta) = \int_0^\eta q(\eta_1) \, d\eta_1 \tag{27}$$

for $n = 1, 2$ respectively.

In dimensional terms the solution for $\bar{C}(z)$ is

$$\bar{C}(z) = C_0 - \frac{(C_0 - C_F)}{h_n(\infty)} h[(z/l_c)(\epsilon_n^{-1/2n+1})], \tag{28}$$

$$F = D \frac{d\bar{C}(z)}{dz}\bigg|_{z=0} = -\frac{(C_0 - C_F)}{h_n(\infty)} \left(\frac{D}{l_c}\right)\epsilon_n^{-1/2n+1}. \tag{29}$$

As $z \to \infty$

$$\bar{C}(z) \sim C_F + \frac{(C_0 - C_F)}{2h_n(\infty)} \frac{(l_c^{2n})\epsilon_n^{2n/2n+1}}{\left[\left(\frac{1}{\alpha}\right)^n - \alpha^n\right]z^{2n}}. \tag{30}$$

Thus for $n = 1$

$$\bar{C}(z) \sim C_F + A_1/z^2; \qquad z \to \infty$$

and for $n = 2$

$$\bar{C}(z) \sim C_F + A_2/z^4; \qquad z \to \infty.$$

For $n = 1$ the flux is

$$F = -\frac{(C_0 - C_F)}{h_1(\infty)} H_1^{1/3} D^{2/3} \tilde{v}^{1/3} l_c^{-2/3}. \tag{31}$$

For $n = 2$ the flux is

$$F = -\frac{(C_0 - C_F)}{h_2(\infty)} H_2^{1/5} D^{4/5} \tilde{v}^{1/5} l_c^{-4/5}. \tag{32}$$

Note that

$$N_n = -\frac{(C_0 - C_F)}{h_n(\infty)} = \frac{F}{\tilde{v} H_n \epsilon_n^{2n/2n+1}}.$$

3.2 *Comparison of Eq. (8) and an eddy diffusivity equation.* It is perhaps worthwhile to conclude this section by asking in what sense Eq. (6) is superior to the corresponding eddy-diffusivity-type equation

$$D \frac{d}{dz} \bar{C}(z) + D_T(z) \frac{d\bar{C}(z)}{dz} = DC_0' + S(z). \tag{33}$$

This is a relevant question since by proper choice of $D_T(z)$ the essential aspects of the solution given in Eq. (28) may be reproduced.

The answer to this question lies in consideration of non-zero sources. The equation governing $\bar{C}(z)$ must hold for all choices of $s(z)$. Even if the solutions of Eq. (33) and

Eq. (6) agree for $s = 0$, there is no guarantee that the solutions will agree for $s(z) \neq 0$. In fact, simple choices of $s(z)$ may be given to show that they do not if $D_T(z)$ is assumed independent of $s(z)$. (If $D_T(z)$ is considered to be a functional of $s(z)$, the solutions may, of course, always be made to agree.)

To illustrate the differences that may arise, let us compare solutions of Eq. (10) and Eq. (33), with a source included, in the turbulence-dominated region. That is, we compare solutions of the equations

$$\bar{v}\bar{z} \int_{\alpha\bar{z}}^{(1/\alpha)\bar{z}} \bar{z}_1 \bar{p}(\bar{z}_1) \, d\bar{z}_1 = DC_0' + S(\bar{z}), \tag{34}$$

$$\frac{D_T(\bar{z})}{l_c} \bar{p}_T(\bar{z}) = DC_0' + S(\bar{z}). \tag{35}$$

Here $S(\bar{z}) = l_c \int_0^{\bar{z}} s(\bar{z}) d\bar{z}$. We assume $n = 1$, and for simplicity we have taken $H_1 = 1$ in Eq. (34).

When $S(\bar{z}) = 0$, Eqs. (34) and (35) yield the same solution:

$$\bar{p}(\bar{z}) = DC_0'/\bar{v}\bar{z}^3 \tag{36}$$

if $D_T(\bar{z}) = \bar{v}l_c\bar{z}^3$.

If $S(\bar{z}) = A_m\bar{z}^m$ $(m > 1)$, the particular solution is easily obtained as a power. (For $m = 1$ the solution of Eq. (34) is more complex.)

The solution of Eq. (34) is

$$\bar{p}(\bar{z}) = \frac{DC_0'}{\bar{v}\bar{z}^3} + \frac{A_m l_c \gamma_m}{\bar{v}} \bar{z}^{m-3} \tag{37}$$

where

$$\gamma_m = \frac{m-1}{\left[\left(\frac{1}{\alpha}\right)^{m-1} - \alpha^{m-1}\right]} \qquad (m > 1).$$

The solution of Eq. (35) is

$$\bar{p}_T(\bar{z}) = \frac{DC_0'}{\bar{v}\bar{z}^3} + \frac{A_m l_c \bar{z}^{m-3}}{\bar{v}}. \tag{38}$$

The two solutions differ by the factor γ_m. Values of γ_m are given in Table 3.

TABLE 3.

Values of γ_m vs. m $(m > 1)$.

m	γ_m
2	1
3	.89
4	.75
5	.60
6	.45
8	.24
10	.12

We thus see that if the source varies relatively slowly, the two solutions are almost the same. If $s(\bar{z})$ is linear ($m = 2$), the solutions are identical, while a parabolic variation ($m = 3$) gives an 11% difference. If, however, $s(\bar{z})$ varies as $\bar{z}^{+5}$, the solutions differ by a factor of over 2, and for $\bar{z}^{+9}$ the factor is about 8.

The difference of the solutions of Eqs. (34) and (35) is not simply of academic interest. Many problems involve distributed concentration sources and some chemical processes can be described in this way. Before these problems can be effectively considered, Eq. (10), rather than an eddy diffusivity equation, must become the basis of our analysis. This paper is an attempt to show how this equation may be effectively utilized.

Eqs. (28)–(32) must now be compared to experiment in order to assess the validity of the theory. We now turn our attention to two physical problems where the theory may be applicable.

4. Experimental investigations. The theory presented above treats a semi-infinite one-dimensional model. In addition, it requires that the transition from molecular diffusion to turbulent dispersion takes place on a scale small compared to the scale of turbulent intensity variation. We consider in this section two real problems for which this model may be applicable.

(A) Material (and perhaps heat) transport in channels when $vL/\nu \gg 1$ and $D/\nu \ll 1$.

(B) Material transport at a liquid-gas interface when $D/\nu \ll 1$.

4.1 *Problem (A)*. In this problem we consider a channel of width $2L$, where $\bar{C}(0) = C_0$ and $\bar{C}(2L) = 0$. The concentration distribution is anti-symmetric about the center line where $C(L) = C_0/2$. We have now added another length scale, L, to the two we have considered previously, and hence in this problem it is clear that the B function ought to be written as $B(z, z_1, D) = \bar{v}\mathfrak{B}(z, z_1, \epsilon, \delta)$ where here $\delta = l_c/L$.

In order to use our previous results, we require $\epsilon \ll 1$. We must also further restrict ourselves here to the condition $\delta \ll 1$. We may write δ in the following manner:

$$\delta = \frac{l_c}{L} = \left(\frac{\bar{v}l_c}{\nu}\right)\left(\frac{\nu}{\bar{v}L}\right)$$

When the Reynolds number $\bar{v}L/\nu$ satisfies the condition $\bar{v}L/\nu \gg 1$, we find experimentally that $\bar{v}l_c/\nu = 0(1)$. Thus, the condition $\delta \ll 1$ is satisfied in channel flows at high Reynolds numbers. In addition the condition

$$\epsilon_n = \frac{D}{H_n\bar{v}l_c} = \left(\frac{D}{\nu}\right)\left(\frac{\nu}{H_n\bar{v}l_c}\right) \ll 1$$

here implies that $D/\nu \ll 1$. ($H_n = O(1)$.)

The function $\mathfrak{B}(z, z_1, 0, 0)$ varies on the length scale L. We have, however, previously identified the function $\mathfrak{B}(z, z, 0, 0)$ with the turbulent intensity, and since the turbulent intensity varies little in the range $z > l_c$, we write

$$\mathfrak{B}(\bar{z}, \bar{z}_1, 0, 0) = H_0 \tag{39}$$

where H_0 is a constant. Mathematically, this is the same as choosing $n = 0$ in Eq. (20). The solution, to lowest order for large η, can therefore be obtained from Eq. (21). Using L'Hospital's rule, we find

$$\frac{d\bar{C}}{d\eta} \sim -\frac{F}{2H_0\bar{v}\log\alpha}\frac{1}{\eta}. \tag{40}$$

Integrating and applying the boundary condition at $z = L$ gives

$$\bar{C} \sim \frac{C_0}{2} - \frac{F}{2H_0\bar{v}\log\alpha}\log\bar{z}\delta. \tag{41}$$

We note here that the logarithmic profile is observed in channel flows and can be obtained independently of the arguments given above by simple dimensional reasoning. Further, we have used only the asymptotic form ($\eta \to \infty$) in obtaining Eq. (41) since the $\delta \to 0$ limit must be bounded away from the wall by the diffusion boundary layer. This is, of course, the layer within a distance $\epsilon^{1/2n+1}$ of the wall. For convenience, we repeat here the solution, Eq. (28), valid in this latter region:

$$\bar{C}(\bar{z}) = C_0 - \frac{F\epsilon_n^{-2n/2n+1}}{\bar{v}H_n}h(\bar{z}/\epsilon_n^{1/2n+1}).$$

This equation has the asymptotic form for large η.

$$\bar{C}(z) \sim C_F - \frac{F}{2\left[\left(\frac{1}{\alpha}\right)^n - \alpha^n\right]\bar{v}H_n\bar{z}^{2n}} \tag{42}$$

One could match the central region, Eq. (41), with the boundary layer, Eq. (42) if $\mathcal{B}(\bar{z}, \bar{z}_1, \epsilon, \delta)$ was known in the transition region. Since the exact form of the solution is not crucial in this region, we shall be satisfied merely to patch the two solutions together (at a place where neither is really valid) in order to determine F.

Since

$$C_F = C_0 + h_n(\infty)N_n = C_0 + \frac{h_n(\infty)F}{\bar{v}H_n\epsilon_n^{2n/2n+1}}$$

we obtain, by patching Eq. (41) and Eq. (42) together at $\bar{z} = 1$,

$$\frac{C_0}{2} - \frac{F}{2H_0\bar{v}\log\alpha}\log\delta = C_0 + \frac{F}{\bar{v}H_n}\left\{\frac{h_n}{\epsilon_n^{2n/2n+1}} - \frac{1}{2\left[\left(\frac{1}{\alpha}\right)^n - \alpha^n\right]}\right\}.$$

Making use of the condition $\epsilon \ll 1$ then gives for F

$$F = -\frac{C_0\bar{v}\epsilon_n^{2n/2n+1}}{2\left\{\frac{h_n(\infty)}{H_n} + \frac{\epsilon_n^{2n/2n+1}\log\delta}{2H_0\log\alpha}\right\}}. \tag{43}$$

Two limits are of interest here. In the case $|\log\delta|\,\epsilon_n^{2n/2n+1} \ll 1$ we find

$$F = -\frac{C_0H_n\bar{v}\epsilon_n^{2n/2n+1}}{2h_n(\infty)}, \tag{44}$$

while in the limit $|\log\delta|\,\epsilon_n^{2n/2n+1} \gg 1$ we find

$$F = -\frac{C_0\bar{v}H_0\log\alpha}{\log\delta}. \tag{45}$$

We are most interested here in the result given in Eq. (44). The expression $\log\delta$ is of the same order as the logarithm of the Reynolds number, and in most experiments for which $D/\nu \ll 1$, the term $|\log\delta|\,\epsilon_n^{2n/2n+1} \ll 1$. It is of interest to note, however, that

for exceptionally large Reynolds numbers this term may be much greater than unity and then the flux would be given by Eq. (45). In Eq. (45) F is independent of D, and it represents a case in which the transport of flux is limited by the turbulent dispersion rather than molecular diffusion, which is the usual result.

As Monin and Yaglom [3] point out, there are virtually no data on the concentration distribution for this problem, and we must use indirect means to verify the theory. In particular, the dependence of flux on D is something that has been experimentally studied. We find from Eq. (44) the result

$$F \propto D^{2n/2n+1}. \tag{46}$$

Thus, when $n = 1$

$$F \propto D^{2/3}. \tag{47}$$

Experimental data indicates a variation from $D^{2/3}$ to $D^{5/6}$, but the $D^{2/3}$ dependence seems to give the most satisfactory agreement. The choice $n = 1$ thus gives agreement with experiment. This means, in terms of the theory that we have presented, that, as we assumed, the first term of the Taylor series expansion given in Eq. (14) is non-zero.

4.2 *Problem (B)*.   In considering the interface of two fluids we suppose that one of the two fluids is in a turbulent state. This means that, instead of a plane surface, the fluid interface has some random shape. In some cases the turbulence is sufficiently intense to cause a breakup of the surface. Clearly, for these states the theory presented above is inapplicable. There are, however, fluid states where the integrity of the fluid surface remains intact. From a continuum point of view, this means that fluid particles on the surface remain for all time on the surface. The only mode of transfer between fluids is by molecular diffusion. If, in addition to the surface remaining intact, the surface curvature is large compared to the distance l_c we defined at the conclusion of Sec. 1, we expect the theory presented to be roughly applicable.

As in the channel flow problem, there are few data on the concentration distribution, and we must rely on the dependence of F on D to check the theory. Again, we shall consider only the cases in which $D/\nu \ll 1$.

Unlike the channel flow problem, little is known about the nature of the turbulence in the interface problem. We assume here that Eq. (16) provides an appropriate description of $\mathfrak{B}(z, z_1)$ with $n = 1$. In this case, choosing $C_F = 0$ for convenience, we have

$$F = -C_0 A \tilde{v}^{1/3} D^{2/3} l_c^{-2/3} \tag{48}$$

where A is a constant of order unity.

Numerous articles have been written about the dependence of F on D (see Scriven [5] for a review of work on this problem). Fortescue and Pearson [6] point out that results have been obtained of the form

$$F \propto D^m \tag{49}$$

where m varies between zero and one. They state, however, that it has usually not been possible to vary D over a wide enough range to determine a unique value of m. In their work they find $m = 1/2$ to be a suitable value. We could obtain the value $m = 1/2$ by choosing $n = 1/2$ in Eq. (16), but we have no reason for assuming that $\mathfrak{B}(\tilde{z}, \tilde{z})$ cannot be expanded in Taylor series about $z = 0$. In our study of the literature we were unable to find sufficient experimental data to determine the coefficient m and do not feel that the validity of the value 2/3 that we obtained can be presently evaluated.

It is interesting to note the connection between F and a characteristic decay distance of $\bar{C}(z)$. A characteristic decay distance z_c is found by taking $\eta = 1$ in Eq. (18). We find then

$$z_c = A' D^{1/3} l_c^{2/3} / \bar{v}^{1/3} \qquad (50)$$

where A' is of order unity.

Eliminating $\bar{v}$ from Eqs. (48) and (50) yields

$$|F|/C_0 = A'' D/z_c \qquad (51)$$

where A'' is of order unity. We may identify z_c with the film thickness used in models that assume there is a thin film where molecular diffusion is dominant, and that then almost discontinuously the turbulent transport becomes dominant. In our model z_c is the point where the two effects are of the same order of magnitude. Finally, for a known value of D, $\bar{v}$ may be found from Eq. (50) once z_c has been determined.

If we assume, as in the channel flow, that $\overline{w'^2}$ increases until some characteristic distance and then is a constant, we may identify $\bar{v}_c$ with $(\overline{w'^2})^{1/2}$ and l_c with this distance. Note that z_c and l_c are quite different quantities. As an example, consider the study by Broecker and Peng [7] on the vertical distribution of Radon in the North Atlantic. They found $z_c = 60 \times 10^{-4}$ cm. They chose $D \approx 1.1 \times 10^{-5}$ cm^2/sec. If $(\overline{w'^2})^{1/2}$ is measured, then l_c may be determined. Broecker and Peng estimate that the eddy diffusivity, D_E, from 0 to 20 m is > 160 cm^2/sec. Assuming $D_E = 0(\overline{w'^2})^{1/2} L_E$, where L_E is a characteristic eddy size, and choosing $L_E = 1$ m, $D_E = 160$ cm^2/sec, yields $(\overline{w'^2})^{1/2} = 0$ (1.6 cm/sec.). For this value of $(\overline{w'^2})^{1/2}$, $l_c = 0(.1$ cm).

6. Summary. In this paper we have considered the one-dimensional steady-state turbulent dispersion of a neutral concentrant from a plane surface. At the surface the turbulent velocity is zero and molecular diffusion is the dominant effect. We considered here those cases in which turbulence becomes the dominant effect very close to the surface. The governing equation that we used was Eq. (5) (or Eq. (8)). It was shown that the integral kernel could in principle be measured, but in this paper we chose the form given in Eq. (16).

Eq. (20), the non-dimensionalized form of the basic equation, was solved by using the differential form given in Eq. (23). This equation was solved numerically for $n = 1$ and $n = 2$. These values are tabulated in Tables 1 and 2 respectively. The integral of $q(\eta)$, $h(\eta)$ (see Eq. (27)) is also tabulated in these tables. $\bar{C}(z)$ is given by Eq. (28). The flux is given by Eq. (29). For $n = 1$ we found $F \propto D^{2/3}$ and

$$\bar{C}(z) - C_F \simeq \frac{A_1}{z^2} \quad \text{as} \quad z \to \infty.$$

In Sec. 3.2 we presented a comparison of Eq. (8) and an eddy diffusivity equation. It was shown that if the diffusivity is independent of the source, significant differences between the two equations occur when the source varies rapidly in space.

In Sec. 4 the theory was first applied to mass transport in channel flow when $D/\nu \ll 1$. Agreement with experiment was obtained for the flux dependence on D. Measurements do not presently exist which allow one to check the distribution $\bar{C}(z)$ close to the wall. It was, however, shown that by proper choice of $\mathfrak{B}(z, z_1)$ the logarithmic profile (valid

some distance from the wall) may be obtained and joined to the boundary layer solution. As a second application, mass transport at a fluid interface was considered. A sufficient amount of experimental data does not now exist to evaluate the theory in this case.

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