ASYMPTOTIC SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS WITH DAMPING AND DELAY*

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Abstract. Certain hyperbolic partial differential equations with small nonlinearities involving significant damping as well as time delay are investigated. The Krylov–Bogoliubov–Mitropolskii method is extended and applied. An application is given to longitudinal vibrations of a nonlinear elastic rod.

1. Introduction. Recently the problem of finding asymptotic solutions of almost linear partial differential equations has been investigated [1, 2, 3] using an extension of the asymptotic method given originally by Krylov–Bogoliubov–Mitropolskii (KBM) for ordinary differential equations. In all the cases considered to date, the damping force which takes part in the process described by the partial differential equation was taken to be small, although an attempt to discuss a particular case with significant damping was made by Osinski [4]. For second-order autonomous ordinary differential equations, the KBM method was extended to processes with significant damping by Popov [5]. (It is perhaps noteworthy that because of the importance of physical processes involving damping, Popov's results have later been rediscovered by several authors.) However, the more difficult and no less important case of partial differential equations with large damping has remained almost untouched. The aim of the present paper is in part to fill that gap, and to provide a basis for further generalizations.

In addition to strong damping, we shall allow the differential equations to be considered to involve time delay. Since the appearance of various books on differential equations with retarded arguments, for example the book by Bellman and Cooke [6], many investigations have been devoted to this subject, which has many applications. Thus, in this paper we shall investigate physical systems modeled by a class of hyperbolic partial differential equations involving both time delay and significant damping, by means of an extension of the KBM method. More precisely, in the main body of the paper we shall examine the monofrequent solutions of the equation

$$\rho(x)\left[\frac{\partial^2 u}{\partial t^2} + 2c\,\frac{\partial u}{\partial t} + 2\gamma\,\frac{\partial u_r}{\partial t}\right] - \frac{\partial}{\partial x}\left[k(x)\,\frac{\partial u}{\partial x} + \kappa(x)\,\frac{\partial u_r}{\partial x}\right] = \epsilon F\left(x, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}, u_r, \frac{\partial u_r}{\partial x}, \frac{\partial u_r}{\partial t}, \cdots, \epsilon\right) \qquad (1)$$

where ϵ is a small positive parameter, $\epsilon\Delta = r$ is the time lag, and $u_r = u(x, t - \epsilon\Delta)$; $\rho(x)$, $k(x)$ and $\kappa(x)$ are given positive functions on the interval $0 < x < l$, and $c > 0$

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and $\gamma > 0$ are the damping coefficients. The dots in F indicate that higher derivatives of u may occur, although we shall include explicitly only those of first order. The function F is assumed to have a sufficient number of derivatives with respect to each of its arguments in a suitable region. In addition to Eq. (1), $u(x, t)$ is required to satisfy a pair of homogeneous boundary conditions involving u and its derivatives at $x = 0$ and $x = l$:

$$\mathfrak{B}_j(u) = \beta_{j1}u(0, t) + \beta_{j2}\frac{\partial u(0, t)}{\partial x} + \beta_{j3}u(l, t) + \beta_{j4}\frac{\partial u(l, t)}{\partial x} = 0, \qquad j = 1, 2. \tag{2}$$

The investigation of monofrequent oscillations of equations of the type (1) is of interest in certain problems occurring in physics and mechanics. For instance, such an equation describes the vibrations of certain elastic systems in which there occurs significant damping of viscous type. In Sec. 4 we shall examine in detail the longitudinal vibrations of a rod subject to viscous damping. The material of the rod is taken to be predominantly Hookean but with, in addition, small nonlinear elastic and viscoelastic characteristics. The general method to be developed in Secs. 2 and 3 will be applied to examine the effect of these nonlinearities on the vibration modes of the rod.

2. The generating equation. The generating equation of (1)—i.e., the corresponding equation with $\epsilon = 0$ — does not have purely harmonic solutions when both c and γ are non-zero, but rather decaying harmonic solutions containing an exponential decay factor superimposed on sinusoidal time dependence. These solutions are obtained in the following way.

The generating system from (1) and (2) is

$$\rho(x)\left[\frac{\partial^2 u^{(0)}}{\partial t^2} + 2(c + \gamma)\frac{\partial u^{(0)}}{\partial t}\right] - \frac{\partial}{\partial x}[k(x) + \kappa(x)]\frac{\partial u^{(0)}}{\partial x} = 0,$$

$$\mathfrak{B}_j(u^{(0)}) = 0, \qquad j = 1, 2. \tag{3}$$

The problem (3) has an infinite set of separable solutions, $u^{(0)}(x, t) = \Phi(x)T(t)$, of the form

$$a_n\Phi_n(x)\exp(-\xi t)\cos(\omega_n t + \psi_n), \qquad n = 1, 2, \cdots \tag{4}$$

where a_n and ψ_n are constants and $\Phi_n(x)$ satisfies the differential equation

$$\frac{d}{dx}\{[k(x) + \kappa(x)]\Phi_n'(x)\} + \lambda_n^2\rho(x)\Phi_n(x) = 0 \tag{5}$$

and boundary conditions $\mathfrak{B}_j(\Phi_n) = 0$. Here and further on the prime means derivative with respect to the argument shown. In terms of the eigenvalues $\{\lambda_n\}$ of this Sturm–Liouville problem, the quantities ξ and ω_n are given by

$$\xi = c + \gamma, \qquad \omega_n^2 = \lambda_n^2 - \xi^2, \qquad n = 1, 2, \cdots . \tag{6}$$

We shall assume that damping is less than critical, that is to say $\xi < \lambda_n$ for all n, which means that all ω_n^2 are positive.

It is well known that, provided the boundary conditions $\mathfrak{B}_j(\Phi_n)$ satisfy the self-adjointness condition, the eigenfunctions $\{\Phi_n(x)\}$ form a complete set of functions, orthogonal with respect to the weight function $\rho(x)$. After suitable normalization, therefore, we have that

$$\int_0^l \rho(x)\Phi_n(x)\Phi_m(x)\,dx = \delta_{nm}, \tag{7}$$

where δ_{nm} is the Kronecker symbol.

In the next section, we shall seek the solutions of (1) which correspond to these decaying vibrations, using an extension and modification of the publications cited.

3. General asymptotic solution. Taking the solution (4) corresponding to $n = 1$, for the sake of definiteness, we seek a monofrequent solution of (1) in the form

$$u(x, t) = \sum_{s=0}^{\infty} \epsilon^s u_s(x, \alpha, \psi), \qquad u_0 = \Phi_1(x)\,\mathrm{Re}\,\{\exp(-\xi\alpha + i\psi)\} \tag{8}$$

where α and ψ are functions of t which are assumed to satisfy the ordinary differential equations

$$d\alpha/dt = 1 + \sum_{s=1}^{\infty} \epsilon^s P_s(\alpha), \qquad d\psi/dt = \omega_1 + \sum_{s=1}^{\infty} \epsilon^s S_s(\alpha). \tag{9}$$

Here Re z is used to denote the real part of z. The functions $P_s(\alpha)$, $S_s(\alpha)$ and $u_s(x, \alpha, \psi)$, $s = 1, 2, \cdots$ are determined from the condition that (8) satisfy Eq. (1) to each order in ϵ. The functions $u_1, u_2, \cdots$ are assumed to be 2π-periodic in the variable ψ.

For $\epsilon = 0$ the solution (8) of Eq. (1) returns to the separable solution (4) for $n = 1$ of the generating equation (3).

From (8) and (9) we obtain that

$$\frac{\partial u(x, t)}{\partial t} = \Phi_1(x)\,\mathrm{Re}\,\{[\eta + \epsilon R_1(\alpha)]\exp(-\xi\alpha + i\psi)\} + \epsilon\left(\frac{\partial u_1}{\partial\alpha} + \omega_1\frac{\partial u_1}{\partial\psi}\right) + \epsilon^2\cdots,$$

$$\frac{\partial^2 u(x, t)}{\partial t^2} = \Phi_1(x)\,\mathrm{Re}\,\{[\eta^2 + 2\epsilon\eta R_1(\alpha) + \epsilon R_1'(\alpha)]\exp(-\xi\alpha + i\psi)\} + \epsilon\left(\frac{\partial^2 u_1}{\partial\alpha^2} + 2\omega_1\frac{\partial^2 u_1}{\partial\alpha\,\partial\psi} + \omega_1^2\frac{\partial^2 u_1}{\partial\psi^2}\right) + \epsilon^2\cdots,$$

where $\eta = -\xi + i\omega_1$ and $R_1(\alpha) = -\xi P_1(\alpha) + iS_1(\alpha)$. Substituting these expressions and

$$u_r = u - \epsilon\Delta\frac{\partial u}{\partial t} + \epsilon^2\cdots, \qquad \frac{\partial u_r}{\partial t} = \frac{\partial u}{\partial t} - \epsilon\Delta\frac{\partial^2 u}{\partial t^2} + \epsilon^2\cdots,$$

$$\frac{\partial u_r}{\partial x} = \frac{\partial u}{\partial x} - \epsilon\Delta\frac{\partial^2 u}{\partial x\,\partial t} + \epsilon^2\cdots,$$

into Eq. (1), we find that the terms of zero order in ϵ are identically zero while the terms of order ϵ give the following equation for u_1:

$$\rho(x)\left[\frac{\partial^2 u_1}{\partial\alpha^2} + 2\omega_1\frac{\partial^2 u_1}{\partial\alpha\,\partial\psi} + \omega_1^2\frac{\partial^2 u_1}{\partial\psi^2} + 2\xi\left(\frac{\partial u_1}{\partial\alpha} + \omega_1\frac{\partial u_1}{\partial\psi}\right)\right] - \frac{\partial}{\partial x}\left\{[k(x) + \kappa(x)]\frac{\partial u_1}{\partial x}\right\} + \rho(x)\Phi_1(x)\,\mathrm{Re}\,\{[2(\eta + \xi)R_1(\alpha) + R_1'(\alpha)]\exp(-\xi\alpha + i\psi)\} - 2\Delta\gamma\rho(x)\Phi_1(x)\,\mathrm{Re}\,\{\eta^2\exp(-\xi\alpha + i\psi)\} + \Delta\,\mathrm{Re}\,\{\eta\exp(-\xi\alpha + i\psi)\}\frac{d}{dx}[\kappa(x)\Phi_1'(x)] = F_1(x, \alpha, \psi). \tag{10}$$

Here we have introduced the abbreviation

$$F_1(x, \alpha, \psi) = F\left(x, u_0, \frac{\partial u_0}{\partial x}, u_0, \frac{\partial u_0}{\partial x}, \frac{\partial u_0}{\partial t}, \cdots, 0\right). \tag{11}$$

In evaluating the derivative $\partial u_0/\partial t$ here using (8) and (9), ϵ must be set equal to zero .

Let us expand u_1 and F_1 as double Fourier series in ψ and x, using the harmonic basis for ψ and the basis $\{\Phi_n(x)\}$ for x:

$$u_1(x, \alpha, \psi) = \sum_{k=1}^{\infty} \sum_{m=-\infty}^{\infty} v_m^{(k)}(\alpha)\Phi_k(x) \exp(im\psi), \tag{12}$$

$$F_1(x, \alpha, \psi)/\rho(x) = \sum_{k=1}^{\infty} \sum_{m=-\infty}^{\infty} F_m^{(k)}(\alpha)\Phi_k(x) \exp(im\psi). \tag{13}$$

From (7) it follows that

$$F_m^{(k)}(\alpha) = \frac{1}{2\pi} \int_0^{2\pi} \int_0^l F_1(x, \alpha, \psi)\Phi_k(x) \exp(-im\psi)\, d\psi. \tag{14}$$

Substituting (12) and (13) into (10) and making use of (5) then gives

$$\sum_{k=1}^{\infty} \sum_{m=-\infty}^{\infty} \left[\frac{d^2 v_m^{(k)}}{d\alpha^2} + 2(\xi + im\omega_1)\frac{dv_m^{(k)}}{d\alpha} + (\lambda_k^2 - m^2\omega_1^2 + 2i\xi m\omega_1)v_m^{(k)}\right]$$
$$\times \Phi_k(x) \exp(im\psi) + \Phi_1(x) \operatorname{Re} \{[2(\eta + \xi)R_1(\alpha) + R_1'(\alpha)] \exp(-\xi\alpha + i\psi)\}$$
$$- 2\Delta\gamma\Phi_1(x) \operatorname{Re} \{\eta^2 \exp(-\xi\alpha + i\psi)\} + \Delta \operatorname{Re} \{\eta \exp(-\xi\alpha + i\psi)\} \frac{1}{\rho(x)}$$
$$\times \frac{d}{dx}[\kappa(x)\Phi_1'(x)] = \sum_{k=1}^{\infty} \sum_{m=-\infty}^{\infty} F_m^{(k)}(\alpha)\Phi_k(x) \exp(im\psi). \tag{15}$$

Returning to (8), we see that we may take $v_{\pm 1}^{(1)}(\alpha) = 0$ since any terms in $u(x, t)$ proportional to $\Phi_1(x) \exp(\pm i\psi)$ can be included in with the first term in the expansion and do not need to be included in u_1 . Thus comparing the coefficients of $\Phi_1(x) \exp(\pm i\psi)$ in (15) gives that

$$R_1'(\alpha) + 2i\omega_1 R_1(\alpha) - \Delta[2\gamma(\xi^2 - \omega_1^2 - 2i\xi\omega_1) - M_1(-\xi + i\omega_1)] = 2 \exp(\xi\alpha)F_1^{(1)}(\alpha)$$

after setting $\eta + \xi = i\omega_1$. Here we have used the notation M_1 for the first member of the sequence defined by

$$M_k = \int_0^l \Phi_k(x) \frac{d}{dx}[\kappa(x)\Phi_1'(x)]\, dx.$$

The solution of this equation is

$$R_1(\alpha) = 2 \exp(-2i\omega_1\alpha) \int_{\alpha_0}^{\alpha} \exp[(2i\omega_1 + c + \gamma)\beta]F_1^{(1)}(\beta)\, d\beta$$
$$+ \frac{\Delta}{2i\omega_1}[2\gamma(\xi^2 - \omega_1^2 - 2i\xi\omega_1) - M_1(-\xi + i\omega_1)]. \tag{16}$$

If we recall that $R_1(\alpha) = -\xi P_1(\alpha) + iS_1(\alpha)$, the real and imaginary parts of (16) give explicit expressions for $P_1(\alpha)$ and $S_1(\alpha)$. Thus the solution of the differential equations (9) enables α and ψ to be found to first order in ϵ.

From (15) we also obtain a differential equation for $v_m^{(k)}(\alpha)$ after comparing the coefficients of $\Phi_k(x)$ exp $(im\psi)$. This is

$$\frac{d^2v_m^{(k)}}{d\alpha^2} + 2(\xi + im\omega_1)\frac{dv_m^{(k)}}{d\alpha} + [(\xi + im\omega_1)^2 + \omega_k^2]v_m^{(k)} = F_m^{(k)}(\alpha)$$

$$- \tfrac{1}{2}\Delta M_k \exp(-\xi\alpha)[(-\xi + i\omega_1)\,\delta_{m,1} + (-\xi - i\omega_1)\,\delta_{m,-1}], \tag{17}$$

and a particular solution is

$$v_m^{(k)}(\alpha) = \frac{1}{\omega_k}\int_{\alpha_0}^{\alpha} \exp[-(c + \gamma + im\omega_1)(\alpha - \beta)] \sin \omega_k(\alpha - \beta)F_m^{(k)}(\beta)\,d\beta$$

$$- \frac{\Delta M_k \exp(-\xi\alpha)}{2(\omega_k^2 - \omega_1^2)}[(-\xi + i\omega_1)\,\delta_{m,1} + (-\xi - i\omega_1)\,\delta_{m,-1}]. \tag{18}$$

Hence the first approximate solution of (1) corresponding to the frequency ω_1 is given by

$$u(x, t) = \Phi_1(x)\,\mathrm{Re}\,\{\exp[-\xi\alpha + i\psi]\} + \epsilon \sum_{k=1}^{\infty}\sum_{m=-\infty}^{\infty} v_m^{(k)}(\alpha)\Phi_k(x) \exp(im\psi) \tag{19}$$

where α and ψ are determined by $d\alpha/dt = 1 + \epsilon P_1(\alpha)$, $d\psi/dt = \omega_1 + \epsilon S_1(\alpha)$.

In a similar manner the functions $u_2(x, \alpha, \psi)$, $P_2(\alpha)$ and $S_2(\alpha)$ can be found which gives the second improved approximation.

4. Vibrations of a nonlinear elastic rod. In [3] we have considered the longitudinal vibrations of an elastic rod which displays a small nonlinearity in its stress-strain behavior as well as small nonlinear viscoelastic and viscous damping. An important feature of the extension of this analysis to be presented here is that it relates to the case when large damping effects are present. When the viscous damping is large (but linear) the longitudinal displacement $u(x, t)$ of such a rod satisfies the equation

$$\rho(\partial^2 u/\partial t^2) + c(\partial u/\partial t) = \partial\sigma/\partial x, \tag{20}$$

where σ is the axial tension and ρ the mass per unit length. The term $c(\partial u/\partial t)$ represents the viscous damping. The stress-strain relation is assumed to be given by

$$\sigma = ke + \epsilon[(1/2)be^2 + (1/3)de^3] + \epsilon[f\dot{e} + (1/3)g\dot{e}^3] \tag{21}$$

where $\dot{e} = \partial e/\partial t$, $e = \partial u/\partial x$ is the axial strain, and b, d, f, g are constants. The b and d terms represent nonlinear elastic behavior and the f and g terms provide for linear and nonlinear viscoelasticity. The quantities ρ and k are respectively the mass per unit length and elastic modulus of the rod, and for a homogeneous rod—which is the case of most interest and which is the one considered in this section—these are constant. Making use of (21), we can write Eq. (20) in the form (1) ($\Delta = 0, \gamma = 0$)

$$\rho\frac{\partial^2 u}{\partial t^2} + c\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + \epsilon F\left(x, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x\,\partial t}, \frac{\partial^3 u}{\partial x^2\,\partial t}\right) \tag{22}$$

where

$$F = b\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} + d\left(\frac{\partial u}{\partial x}\right)^2\frac{\partial^2 u}{\partial x^2} + f\frac{\partial^3 u}{\partial x^2\,\partial t} + g\left(\frac{\partial^2 u}{\partial x\,\partial t}\right)^2\frac{\partial^3 u}{\partial x^2\,\partial t}.$$

Let us take boundary conditions as follows:

$$u(0,\, t) = 0, \qquad h\,\frac{\partial u(l,\, t)}{\partial x} + u(l,\, t) = 0. \tag{23}$$

For $\epsilon = 0$ Eq. (22) has separable solutions

$$\Phi_n(x) \exp(-\xi t) \cos(\omega_n t + \psi_n).$$

Applying (23), we get

$$\Phi_n(x) = C_n \sin \lambda_n x, \qquad \omega_n^{\,2} = \frac{k}{\rho}\,\lambda_n^{\,2} - \xi^2, \qquad \xi = c/2\rho, \qquad n = 1, 2, \cdots, \tag{24}$$

where $\{\lambda_n\}$ are roots of equation $\tan \lambda l = -h\lambda$ and the constants $\{C_n\}$ satisfy the equality

$$\frac{\rho C_n^{\,2}}{2}\left[l + \frac{h}{1 + h^2\lambda_n^{\,2}}\right] = 1.$$

From (11) for this case we obtain that

$$\begin{aligned} F_1(x, \alpha, \psi) = {} & b\Phi_1'(x)\Phi_1''(x) \exp(-2\xi\alpha) \cos^2 \psi + d[\Phi_1'(x)]^2\Phi_1''(x) \exp(-3\xi\alpha) \cos^3 \psi \\ & - f\Phi_1''(x) \exp(-\xi\alpha)(\xi \cos \psi + \omega_1 \sin \psi) \\ & - g[\Phi'(x)]^2\Phi_1''(x) \exp(-3\xi\alpha)(\xi \cos \psi + \omega_1 \sin \psi)^3. \end{aligned}$$

It follows from (14) therefore that the non-zero quantities $F_m^{(k)}(\alpha)$ are given in the following list

$$\begin{aligned} & F_0^{(k)}(\alpha) = (1/2)D_k \exp(-2\xi\alpha), \qquad F_2^{(k)}(\alpha) = (1/4)D_k \exp(-2\xi\alpha) \\ & F_1^{(k)}(\alpha) = (3/8)[E_k + (\xi - i\omega_1)(\xi^2 + \omega_1^{\,2})H_k] \exp(-3\xi\alpha) \\ & \qquad\qquad + (1/2)(\xi - i\omega_1)G_k \exp(-\xi\alpha) \\ & F_3^{(k)}(\alpha) = (1/8)[E_k + (\xi - i\omega_1)^3 H_k] \exp(-3\xi\alpha) \\ & F_{-m}^{(k)}(\alpha) = \overline{F_m^{\,k}(\alpha)}, \qquad m = 1, 2, 3. \end{aligned} \tag{25}$$

The constants appearing in (25) are

$$\begin{aligned} & D_k = b \int_0^l \Phi_1'(x)\Phi_1''(x)\Phi_k(x)\, dx, \qquad E_k = d \int_0^l [\Phi_1'(x)]^2\Phi_1''(x)\Phi_k(x)\, dx, \\ & G_k = -f \int_0^l \Phi_1''(x)\Phi_k(x)\, dx = (f\lambda_1^{\,2}/k)\ \delta_{k1}, \quad H_k = -g \int_0^l [\Phi_1'(x)]^2\Phi_1''(x)\Phi_k(x)\, dx. \end{aligned} \tag{26}$$

Substituting the expression for $F_1^{(1)}(\alpha)$ from (25) into (16) ($\Delta = \gamma = 0$), we find the particular solution

$$R_1(\alpha) = \frac{\xi - i\omega_1}{2i\omega_1}\, G_1 - \frac{3}{8(\xi - i\omega_1)}\,[E_1 + (\xi - i\omega_1)(\xi^2 + \omega_1^{\,2})H_1] \exp(-2\xi\alpha),$$

from which we find $P_1(\alpha) = -\xi^{-1}\,\mathrm{Re}\, R_1(\alpha)$ and $S_1(\alpha) = \mathrm{Im}\, R_1(\alpha)$, where $\mathrm{Im}\, z$ denotes the imaginary part of z. Substituting these into (9), we obtain the differential equation for α

$$d\alpha/dt = A + B \exp(-2\xi\alpha)$$

where

$$A = 1 + \epsilon \frac{G_1}{2\xi}, \qquad B = \epsilon \frac{3}{8}\left[\frac{E_1}{\xi^2 + \omega_1^{\,2}} + \frac{(\xi^2 + \omega_1^{\,2})H_1}{\xi}\right]. \tag{27}$$

The solution is

$$\alpha = (1/2\xi) \ln \{(1/A)[\exp(2\xi A(t - t_0)) - B]\} \tag{28}$$

where t_0 is the constant of integration. From (9) we also obtain a differential equation for ψ which may be integrated directly to provide the solution

$$\psi = \left(\omega_1 - \epsilon \frac{\xi G_1}{2\omega_1}\right)t - \epsilon \frac{3\omega_1 E_1}{16\xi(\xi^2 + \omega_1^{\,2})B} \ln \{1 - B \exp[-2\xi A(t - t_0)]\} + \psi_0 . \tag{29}$$

With these two results, the first approximate solution from (8) is simply given by $\Phi_1(x) \exp(-\xi\alpha) \cos \psi$.

Finally, substituting the values of $F_m^{(k)}$ from (25) into (18) enables the coefficients $v_m^{(k)}(\alpha)$ to be found. If we ignore the arbitrary solution of the homogeneous equation corresponding to (17), the non-zero $v_m^{(k)}(\alpha)$ are as follows:

$$\begin{aligned}
v_0^{(k)} &= \frac{D_k \exp(-2\xi\alpha)}{2(\xi^2 + \omega_k^{\,2})}, \qquad v_2^{(k)} = \frac{D_k \exp(-2\xi\alpha)}{4(\xi^2 + \omega_k^{\,2})},\\
v_1^{(k)} &= \tfrac{3}{8}[E_k + (\xi - i\omega_1)(\xi^2 + \omega_k^{\,2})H_k] \frac{\exp(-3\xi\alpha)}{(2\xi + i\omega_1)^2 + \omega_k^{\,2}},\\
v_3^{(k)} &= \tfrac{1}{8}[E_k + (\xi - i\omega_1)^3 H_k] \frac{\exp(-3\xi\alpha)}{(2\xi + i\omega_1)^2 + \omega_k^{\,2}},\\
v_{-m}^{(k)} &= \overline{v_m^{(k)}}, \qquad m = 1, 2, 3.
\end{aligned} \tag{30}$$

Then the first improved approximate solution of Eq. (20) is

$$\begin{aligned}
u(x, t) = {} & \Phi_1(x) \exp(-c\alpha/2\rho) \cos \psi \\
& + \epsilon \sum_{k=1}^{\infty} \Phi_k(x)\{v_0^{(k)} + 2 \operatorname{Re} v_1^{(k)} \cos \psi - 2 \operatorname{Im} v_1^{(k)} \sin \psi \\
& + 2v_2^{(k)} \cos 2\psi + 2 \operatorname{Re} v_3^{(k)} \cos 3\psi - 2 \operatorname{Im} v_3^{(k)} \sin 3\psi\}, \qquad v_1^{(1)} = 0,
\end{aligned} \tag{31}$$

where α and ψ are given by (26) and (27), v_0, v_1, v_2 and v_3 by (30) and $\Phi_k(x)$ by (24).

5. Discussion of the results. We shall examine these results from a qualitative point of view with the aim of demonstrating their principal features of physical significance. The first approximation according to the KBM method is provided by the first term in (31), and in the customary way one can confine one's attention to this term alone, which after making use of (27) can be written in the form

$$u_0(x, t) = a \cos \psi \, \Phi_1(x), \tag{32}$$

where

$$a = A^{1/2} \exp[-\xi A(t - t_0)]\{1 - B \exp[-2\xi A(t - t_0)]\}^{-1/2}$$

and ψ is given by (29). In terms of the constants in (20), $\xi = c/2\rho$.

The solution (32) has the general nature of an oscillatory part cos ψ, superimposed on which is a decaying amplitude a. The decay is dominantly exponential with decay constant ξ, but its exponential form is modified by additional terms of order ϵ. The phase ψ of the oscillatory part for large times approaches a linearly increasing function of t with an angular frequency of $\omega_1 - \epsilon(\xi G_1/2\omega_1)$. It is apparent then from the definitions (26) that only the linear viscoelastic term in the stress-strain relation (21) influences the eventual frequency of oscillation. However, for short times ($|t - t_0| \ll \xi^{-1}$) the phase ψ in (29) may be approximated as

$$\psi \approx \text{const} + \left\{\omega_1 + \epsilon\left[\frac{\xi G}{2\omega_1} - \frac{3\omega_1 E_1}{8(\xi^2 + \omega_1^2)}\right]\right\}t,$$

showing that the initial value of the frequency is affected also by the cubic elastic term $(1/3)de^3$ in (21).

More generally, we note that the amplitude a and phase ψ are influenced by the coefficients G_1, E_1 and H_1, or, according to (26), by d, f, and g, representing the cubic nonlinear elastic behavior and the viscoelastic character of the material. The coefficient b, providing the quadratic nonlinear elastic behavior, does not affect a and ψ in the first approximation (32). This leads to the conclusion that from the point of view of the vibrations of the material, the quadratic part of the stress-strain relation (21) is of less significance than the linear and cubic terms. All the coefficients b, d, f and g affect higher approximations, such as the improved first approximation (31), (30).

The effect of the cubic terms in the stress-strain relation, both elastic and viscoelastic, appears in the amplitude exp $(-\xi\alpha)$ of the first-order solution (32) through the single constant B defined in (27_2). Using the definitions (26), we find that the ratio of the contributions of the viscoelastic and elastic terms is $g(\xi^2 + \omega_1^2)^2/d\xi$. As we increase ω_1, this ratio increases, showing that the viscoelastic terms assume greater importance the higher the vibration mode being considered. Furthermore, let us suppose that the two cubic terms in (21) make comparable contributions to the stress in the fundamental mode, so that the quantities d and $g(\xi^2 + \omega_1^2)^{3/2}$ are of the same order of magnitude. Then the ratio of the viscoelastic to elastic contributions to B is roughly $(\xi^2 + \omega_1^2)^{1/2}/\xi$. For strong viscous damping ($\xi \approx \omega_1$) this ratio is of order unity, but as the viscous damping becomes weak ($\xi \ll \omega_1$), the effect of the cubic viscoelastic terms on the amplitude of vibrations assumes greater and greater dominance over the effect of the cubic elastic terms. On the other hand, bearing in mind that B is a small quantity, it is clear from (29) that the phase of the vibration is considerably more strongly affected by the cubic elastic terms than by the cubic viscoelastic ones.

These conclusions may be emphasized by examining the behavior of the solution in the limit $\xi \to 0$, corresponding to zero viscous damping. Returning to the differential equation for α, using the expressions (27), and letting $a = \exp(-\xi\alpha)$ denote the amplitude of vibration, we can see that the solution in the limit is

$$a = G_1^{1/2}\{\exp[\epsilon G_1(t - t_1)] - \tfrac{3}{4}\omega_1^2 H_1\}^{-1/2}, \qquad \psi = \left(\omega - \frac{3E_1}{8\omega_1}\right)t + \psi_0 .$$

In this limit, the viscoelastic terms affect only the amplitude and the elastic terms affect only the phase of the solution. (We note that this solution for small damping can also be obtained using the simpler method given in [3].)

To illustrate more fully the roles of the physical coefficients b, d, f and g let us consider now some particular cases.

(i) *Nonlinear elastic material* ($f = g = 0$). In this case (21) becomes $\sigma = ke + \epsilon[(b/2)e^2 + (d/3)e^3]$. From (26) we get $G_k = H_k = 0$ which implies that $A = 1$ and $B = 3\epsilon E_1/[8(\xi^2 + \omega_1^2)]$. The amplitude and phase depend on the coefficient d only, i.e. on the cubic elastic behavior of the material.

As remarked above, changing the coefficient b does not affect the first-order solution. However, as can be seen from (30), the even harmonics in the improved first approximation do depend on b, and if $b = 0$ these harmonics do not appear in the solution.

If we assume $d = 0$, then the first approximation returns to the solution (4) of the generating equation (with $n = 1$).

(ii) *Viscoelastic material* ($b = d = 0$). In this case (21) becomes $\sigma = ke + \epsilon(f\dot{e} + (g/3)\dot{e}^3]$. Since $E_1 = 0$, the phase (29) assumes the form $\psi = [\omega_1 - \epsilon(\xi G_1/2\omega_1)]t + \psi_0$, corresponding to a purely harmonic vibration. The superposed decaying amplitude is still given by (32_2).

If there is no linear viscoelasticity ($f = 0$), then $G_1 = 0$ and the angular frequency of the solution is not changed from its unperturbed value ω_1 : on its own cubic viscoelasticity affects only the amplitude a, which is given by

$$a = \exp[-\xi(t - t_0)]\{1 - B \exp[-2\xi(t - t_0)]\}^{-1/2},$$

where $B = 3\epsilon H_1(\xi^2 + \omega_1^2)/8\xi$.

Finally, if $g = 0, f \neq 0$, then $A = 1 + \epsilon G_1/2\xi$, $B = 0$ and the solution takes the form

$$u(x, t) = \Phi_1(x)A^{1/2} \exp[-\xi A(t - t_0)] \cos\{[\omega_1 - \epsilon(\xi G_1/2\omega_1)]t + \psi_0\}. \qquad (33)$$

In this case Eq. (22) degenerates into a purely linear equation, and its exact solutions may readily be found. It is easily verified that (33) gives the correct first approximation to the appropriate exact solution.

6. A third-order damped equation. We can apply the same asymptotic method to find the monofrequent solutions of the following partial differential equation with delay:

$$\rho \frac{\partial^2 u}{\partial t^2} - 2c \frac{\partial^3 u}{\partial x^2 \partial t} - k \frac{\partial^2 u}{\partial x^2} = \epsilon F\left(x, u, u_r, \frac{\partial u}{\partial t}, \frac{\partial u_r}{\partial t}, \cdots, \epsilon\right) \qquad (34)$$

where ρ, c and k are positive constants, $0 < x < l$. We suppose in addition that $u(x, t)$ satisfy the self-adjoint boundary conditions (2).

Equations of this type can be used to describe such longitudinal vibrations of an elastic, viscoelastic rod as would arise if the term $\epsilon f\dot{e}$ in (21) were not small. The function F contains terms responsible for small nonlinear elastic and viscoelastic behavior of the material of which the rod is composed.

The generating equation of (34) has separable solutions $u^{(0)}(x, t) = \Phi(x)T(t)$, where $\Phi(x)$ and $T(t)$ satisfy differential equations

$$\Phi''(x) + \mu\Phi(x) = 0, \qquad \rho T''(t) + 2c\mu T'(t) + k\mu T(t) = 0.$$

The function $\Phi(x)$ also satisfies the same boundary condition at $x = 0, l$ as does $u(x, t)$. The set of eigenfunctions $\{\Phi_n(x)\}$ form a complete set of functions which can be appropriately normalized, and the corresponding eigenvalues $\{\mu_n\}$ are positive. We can write the set of separable solutions in the form (4), but now $\xi = \mu_n c/\rho$ and $\omega_n^2 = (-\mu_n^2 c^2 + \rho k\mu_n)/\rho^2$. It is assumed that $\omega_1^2 > 0$.

For the nonlinear equation (32), following the method in Sec. 3, we seek a monofrequent solution of the form (8) and (9). The equation for $u_1(x, \alpha, \psi)$ is

$$\rho\left\{\frac{\partial^2 u_1}{\partial \alpha^2} + 2\omega_1 \frac{\partial^2 u_1}{\partial \alpha\, \partial \psi} + \omega_1{}^2 \frac{\partial^2 u_1}{\partial \psi^2} - \frac{2c}{\rho}\left(\frac{\partial^3 u_1}{\partial \alpha\, \partial x^2} + \omega_1 \frac{\partial^3 u_1}{\partial \psi\, \partial x^2}\right) - k\frac{\partial^2 u_1}{\partial x^2}\right.$$

$$\left. + \rho\Phi_1(x) \operatorname{Re}\left\{\left[2\left(\eta + \frac{c\mu_1}{\rho}\right)R_1(\alpha) + R_1'(\alpha)\right] \exp(-\xi\alpha + i\psi)\right\} = F_1(x, \alpha, \psi), \qquad (35)$$

where F_1 is given by (11). Substituting (12) and (13) into (35) and comparing the coefficients of $\Phi_1(x) \exp(\pm i\psi)$ gives a differential equation for $R_1(\alpha)$ whose solution is

$$R_1(\alpha) = 2 \exp(-2i\omega_1\alpha) \int_{\alpha_0}^{\alpha} \exp[(\mu_1 c/\rho + 2i\omega_1)\beta] F_1{}^{(1)}(\beta)\, d\beta.$$

We also obtain a differential equation for $v_m{}^{(k)}(\alpha)$ and a particular solution is

$$v_m{}^{(k)}(\alpha) = \frac{1}{\omega_k} \int_{\alpha_0}^{\alpha} \exp[-(c\mu_k/\rho + i\omega_1 m)(\alpha - \beta)] \sin \omega_k(\alpha - \beta) F_m{}^{(k)}(\beta)\, d\beta.$$

The first approximate solution of (34) is given again by formula (19).

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