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MODIFIED COMPLEMENTARY VARIATIONAL PRINCIPLES FOR A BOUNDARY-VALUE PROBLEM IN TWISTING OF RING SECTORS*

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Abstract. New dual extremum principles and error bounds for the problem of stress in close-coiled helical springs are presented. An accurate variational solution is obtained in an illustrative calculation.

In this note we present new extremum principles and error bounds for variational solutions of the boundary-value problem

$$\frac{\partial^2 \psi}{\partial X^2} + \frac{\partial^2 \psi}{\partial Y^2} + \frac{3}{R - X} \frac{\partial \psi}{\partial X} + 2c = 0 \quad \text{in} \quad V, \tag{1}$$

$$\psi = 0 \quad \text{on} \quad \partial V \tag{2}$$

which arises in the theory of stresses in close-coiled helical springs [1], where ψ is the stress function, c is a constant of the spring, V is the circular region $X^2 + Y^2 \leq a^2$, a being the radius of cross section of the ring, ∂V is the boundary of V, and $R(>a)$ is the radius of the helical coil. This present work extends the recent variational [2] and hypercircle [3] results for the problem in (1) and (2), and in particular provides more accurate estimates for approximate solutions.

It is convenient to rewrite (1) and (2) in terms of dimensionless quantities by setting

$$\psi = ca^2\phi, \qquad X = ax, \qquad Y = ay, \qquad R = ar. \tag{3}$$

Then the problem becomes

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{3}{r - x} \frac{\partial \phi}{\partial x} + 2 = 0 \quad \text{in} \quad V, \tag{4}$$

$$\phi = 0 \quad \text{on} \quad \partial V, \tag{5}$$

where V is the region $x^2 + y^2 \leq 1$ and r is greater than 1. If we now set

$$\phi = (r - x)^{3/2} w, \tag{6}$$

Eqs. (4) and (5) become

$$Aw \equiv -\nabla^2 w + p(x)w = q(x) \quad \text{in} \quad V, \tag{7}$$

$$w = 0 \quad \text{on} \quad \partial V, \tag{8}$$

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with

$$p(x) = 15/4(r - x)^2, \qquad q(x) = 2/(r - x)^{3/2}. \tag{9}$$

To obtain complementary variational principles associated with (7) and (8) we could follow the method given in [4]. This would lead to principles equivalent to those of [2]. An alternative approach, which we shall adopt, is to use some recent results for linear equations [5]. In the present case these results provide an improved minimum principle.

First we define the inner product

$$(u, v) = \int_V uv\, dx\, dy. \tag{10}$$

This representation is chosen because the operator A in (7) is self-adjoint with respect to (10); that is

$$(u, Av) = (Au, v) \tag{11}$$

for all suitable functions u and v which vanish on ∂V. A second important property of A, which we exploit later, is that it is positive bounded below, which means that there is a positive number Λ such that

$$0 < \Lambda \leq (u, Au)/(u, u) \tag{12}$$

for all functions u in the domain of A which vanish on ∂V. In other words, the lowest eigenvalue λ of

$$Au = \lambda u \quad \text{in} \quad V, \qquad u = 0 \quad \text{on} \quad \partial V \tag{13}$$

is strictly positive and Λ in (12) is therefore any positive lower bound to λ. To prove $\lambda > 0$ we note that, from (7), $A = -\nabla^2 + p(x)$ and hence

$$\lambda > \rho + (15/4(r + 1)^2), \tag{14}$$

where ρ is the lowest eigenvalue of

$$-\nabla^2 u = \rho u \quad \text{in} \quad V, \qquad u = 0 \quad \text{on} \quad \partial V. \tag{15}$$

From (15) we find that ρ is determined by the smallest zero of the Bessel function $J_0(\sqrt{\rho})$, which means that $\rho > 144/25$. Hence, by (14),

$$\lambda > \frac{144}{25} + \frac{15}{4(r + 1)^2}, \tag{16}$$

and this proves the positive property of λ asserted above.

Now we define the functionals

$$F(w_1) = (w_1, 2q - Aw_1), \tag{17}$$

$$K(w_2) = (w_2, 2q - Aw_2) + \Lambda^{-1}(q - Aw_2, q - Aw_2), \tag{18}$$

where Λ is a positive lower bound to λ and w_1 and w_2 are admissible functions such that

$$w_1 = w_2 = 0 \quad \text{on} \quad \partial V. \tag{19}$$

If w denotes the unique solution of (7) and (8), we see that

$$F(w) = K(w) = (w, q). \tag{20}$$

Further, for any admissible functions w_1 and w_2

$$F(w_1) = F(w) - (w_1 - w, A(w_1 - w)) \tag{21}$$

and

$$K(w_2) = K(w) + \Lambda^{-1}(w_2 - w, A(A - \Lambda)(w_2 - w)). \tag{22}$$

Since A and $A - \Lambda$ are positive operators, it therefore follows from (21) and (22) that the complementary principles

$$F(w_1) \leq F(w) = (w, q) = K(w) \leq K(w_2) \tag{23}$$

hold [5]. The result for F is equivalent to one in [2], whereas the result for K is new.

The extremum principles (23) provide upper and lower bounds for the quantity (w, q). In addition they can be used to generate approximate solutions w_1 and w_2 of the boundary-value problem in (7) and (8). When such solutions are of interest, it is important to have an estimate of their error. In the present problem we can use the method of [6] to derive an upper bound for the error in the variational function w_1. Thus, from (23) we have that for any admissible functions w_1 and w_2

$$\begin{aligned} K(w_2) - F(w_1) &\geq F(w) - F(w_1) \\ &= (w_1 - w, A(w_1 - w)) \quad \text{by} \quad (21), \\ &\geq \Lambda(w_1 - w, w_1 - w) \quad \text{by} \quad (12), \end{aligned} \tag{24}$$

where Λ is any positive lower bound to the lowest eigenvalue of A. Hence

$$(w_1 - w, w_1 - w) \leq E^2(w_1), \tag{25}$$

where

$$E^2(w_1) = \Lambda^{-1}\{K(w_2) - F(w_1)\}. \tag{26}$$

$E(w_1)$ provides a root-mean-square estimate of the error in the variational function w_1.

We have performed calculations of the complenemtary functionals F and K in the case $r = 10$. A suitable value for the parameter Λ in (18) and (26) is

$$\Lambda = \frac{144}{25} + \frac{15}{4(r+1)^2}, \tag{27}$$

using the result in Eq. (16). As trial functions we took

$$w_1 = (r - x)^{-3/2}\phi_1, \tag{28}$$

$$w_2 = (r - x)^{-3/2}\phi_2, \tag{29}$$

where

$$\phi_1 = \sum_{n=1}^{2} (a_n x + b_n)(x^2 + y^2 - 1)^n, \tag{30}$$

$$\phi_2 = \sum_{n=1}^{2} (c_n x + d_n)(x^2 + y^2 - 1)^n. \tag{31}$$

The functions w_1 and w_2 satisfy the required boundary condition (19). The parameters a_n, b_n, c_n and d_n were found by optimizing F and K, and the results, including an error bound for w_1, are given in Table 1. From these results we see that the variational solution

TABLE 1

Variational parameters and error bound for $r = 10$.

a_1	0.37477(−1)	c_1	0.37506(−1)
a_2	0.23462(−4)	c_2	−0.27814(−5)
b_1	−0.49813	d_1	−0.49820
b_2	−0.47034(−3)	d_2	−0.43379(−3)
F	0.1573768(−2)	K	0.1573769(−2)

Here $m(-n)$ means $m \times 10^{-n}$.

w_1 is very accurate. The 4-parameter function ϕ_1 provides a variational solution of the problem in (4) and (5), and it is more accurate than the 12-parameter variational solution given in [2]. We note that

$$\phi_1(0, 0) = 0.49766, \tag{32}$$

which agrees with the value at (0, 0) of Timoshenko's perturbation solution [1].

As noted in [2], one advantage of the variational approach is that it is not restricted, as the perturbation solution of [1] is, to very small values of the parameter a/R.

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