COMPLEMENTARY VARIATIONAL PRINCIPLES FOR LARGE DEFLECTIONS OF A CANTILEVER BEAM*

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Abstract. Recent results on complementary variational principles and error bounds are applied to two problems concerning the large deflection of a horizontal cantilever. The results are illustrated by obtaining accurate variational solutions in the form of simple polynomials.

1. Introduction. Complementary variational principles have recently been devel- [1, 2] for a wide class of linear and nonlinear boundary-value problems including many boundary conditions. In certain problems the basic action functional represents a physical quantity of interest such as energy or capacity, and these principles provide upper and lower bounds for these quantities. In nonlinear problems, when attention is centered upon the boundary-value problem itself, complementary variational principles provide the basis for the construction of approximate solutions and error bounds.

The purpose of this paper is to present complementary variational principles relevant to problems occurring in the large-deflection theory of flexible bars. The results are illustrated by two cases of a horizontal cantilever subjected to a vertical load. For these problems it is possible to transform the differential equation and boundary conditions into a fairly simple integral equation and from these representations two sets of complementary variational principles are obtained. The integral equation approach produces the better lower bound of the action functional while using fewer variational parameters. This fact is used in an error bound for the approximate solution.

2. The class of problems. We consider boundary-value problems with equations of the form

$$L\phi = T^*T\phi = f(\phi) \quad \text{in } V \tag{2.1}$$

and boundary conditions

$$\sigma_T\phi = 0 \quad \text{on } \partial V_1\,, \qquad \sigma_T{}^*T\phi = 0 \quad \text{on } \partial V_2\,. \qquad (2.2,\ 3)$$

Here L is a selfadjoint positive definite operator (see Mikhlin [3]), possibly a differential, integral or matrix operator, V is some region of space and $\partial V_1 + \partial V_2 = \partial V$ makes up the boundary of V.

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The linear operator T and its adjoint T^* satisfy the equation

$$\int_V UT\Phi \, dV = \int_V (T^*U)\Phi \, dV + \int_{\partial V} U\sigma_T\Phi \, dB \tag{2.4}$$

for all admissible U, Φ, where σ_T is a linear operator restricted to ∂V, having an adjoint $\sigma_T{}^*$ defined by

$$U\sigma_T\Phi = (\sigma_T{}^*U)\Phi \quad \text{on } \partial V. \tag{2.5}$$

For an example of the operators T, T^* and σ_T see Sec. 7.

To introduce variational principles associated with Eqs. (2.1)–(2.3) we rewrite them in the canonical form

$$T\phi = u = \partial H/\partial u \quad \text{in} \quad V, \tag{2.6}$$

$$T^*u = f(\phi) = \partial H/\partial \phi \quad \text{in} \quad V \tag{2.7}$$

with

$$\sigma_T\phi = 0 \quad \text{on} \quad \partial V_1 , \tag{2.8}$$

$$\sigma_T{}^*u = 0 \quad \text{on} \quad \partial V_2 . \tag{2.9}$$

A suitable Hamiltonian H in Eqs. (2.6), (2.7) is

$$H(u, \phi) = \tfrac{1}{2}u^2 + F(\phi) \tag{2.10}$$

where

$$F(\phi) = \int^{\phi} f(\psi) \, d\psi. \tag{2.11}$$

3. Variational principles. Following Arthurs [1, 2] we introduce the action functional $I(U, \Phi)$ of the form

$$I(U, \Phi) = \int_V \{UT\Phi - H(U, \Phi)\} \, dV - \int_{\partial V_1} U\sigma_T\Phi \, dB \tag{3.1}$$

$$= \int_V \{(T^*U)\Phi - H(U, \Phi)\} \, dV + \int_{\partial V_2} (\sigma_T{}^*U)\Phi \, dB \tag{3.2}$$

by Eqs. (2.4), (2.5), and $H(U, \Phi)$ is defined by Eq. (2.10).

It then follows that

(a) First variational principle: For arbitrary independent functions U, Φ the functional $I(U, \Phi)$ is stationary at (u, ϕ), the solution pair of the boundary-value problem described in Eqs. (2.6)–(2.9).

We denote this value of the functional by

$$I(u, \phi) = I(\phi) = \int_V \{\tfrac{1}{2}\phi f(\phi) - F(\phi)\} \, dV. \tag{3.3}$$

(b) Second variational principle: Using Eq. (3.1) we define a functional $J(\Phi_1)$ by

$$J(\Phi_1) = I(U(\Phi_1), \Phi_1), \qquad U(\Phi_1) = T\Phi_1 \tag{3.4}$$

where Φ_1 is any admissible function which satisfies the condition

$$\sigma_T \Phi_1 = 0 \quad \text{on} \quad \partial V_1 . \tag{3.5}$$

Then we see that

$$J(\Phi_1) = \int_V \{\tfrac{1}{2}(T\Phi_1)^2 - F(\Phi_1)\}\, dV \tag{3.6}$$

$$= \int_V \{\tfrac{1}{2}\Phi_1 L\Phi_1 - F(\Phi_1)\}\, dV + \frac{1}{2}\int_{\partial V_2} (\sigma_T{}^* T\Phi_1)\Phi_1 \, dB \tag{3.7}$$

using Eqs. (2.4), (2.5) and (3.5).

Also

$$J(\Phi_1) = I(\phi) + \Delta J \tag{3.8}$$

where

$$\Delta J = \frac{1}{2}\int_V \{(T(\Phi_1 - \phi))^2 - (\Phi_1 - \phi)^2 \overline{\frac{df}{d\phi}}\}\, dV, \tag{3.9}$$

the bar over the derivative indicating that it is to be evaluated for some function $\phi + \eta(\Phi_1 - \phi)$, $0 < \eta < 1$. Eq. (3.8) shows that $J(\Phi_1)$ is stationary at ϕ and further if

$$-\,(df/d\psi)(\psi) \geq 0 \quad \text{in } V \text{ for all } \psi \tag{3.10}$$

we see from Eq. (3.9) that

$$\Delta J \geq 0 \quad \text{for all } \Phi_1 \tag{3.11}$$

and so from (3.8) we have the global minimum principle

$$I(\phi) \leq J(\Phi_1) \tag{3.12}$$

for all admissible functions Φ_1 satisfying condition (3.5).

(c) Third variational principle: Using Eq. (3.2) we define a functional $G(U)$ as follows:

$$G(U) = I(U, \Phi(U)), \qquad \Phi(U) = f^{-1}(T^*U) \quad \text{in } V \tag{3.13}$$

where U is an admissible function satisfying the condition

$$\sigma_T{}^* U = 0 \quad \text{on} \quad \partial V_2 . \tag{3.14}$$

Then we see that

$$G(U) = \int_V \{(T^*U)f^{-1}(T^*U) - \tfrac{1}{2}U^2 - F[f^{-1}(T^*U)]\}\, dV \tag{3.15}$$

$$= I(\phi) + \Delta G \tag{3.16}$$

where

$$\Delta G = -\frac{1}{2}\int_V \left\{(U - u)^2 - (f^{-1}(T^*U) - \phi)^2 \overline{\frac{gf}{d\phi}}\right\} dV, \tag{3.17}$$

the bar over the derivative again denoting that it is to be evaluated at some function $u + \eta(U - u)$, $0 < \eta < 1$. Eq. (3.16) shows that $G(U)$ is stationary at u and further if

$$-\,(df/d\psi)(\psi) \geq 0 \quad \text{in } V \quad \text{for all } \psi \tag{3.10}$$

we see from (3.17) that

$$\Delta G \leq 0 \quad \text{for all } U, \tag{3.18}$$

and so by equation (3.16) we have the global maximum principle

$$G(U) \leq I(\phi) \tag{3.19}$$

for all admissible functions U satisfying condition (3.14).

Since the functions u and ϕ are related by $u = T\phi$ in Eq. (2.6), we consider U to have the form $U = T\phi_2$, where Φ_2 is an approximation to ϕ. Substituting in (3.15) and changing notation slightly by writing $G(\Phi_2)$ for $G(T\Phi_2)$, we have

$$G(\Phi_2) = \int_V \{(T^*T\Phi_2)f^{-1}(T^*T\Phi_2) - \tfrac{1}{2}(T\Phi_2)^2 - F[f^{-1}(T^*T\Phi_2)]\}\, dV \tag{3.20}$$

$$= \int_V \{(L\Phi_2)[f^{-1}(L\Phi_2) - \tfrac{1}{2}\Phi_2] - F[f^{-1}(L\Phi_2)]\}\, dV - \frac{1}{2}\int_{\partial V_1} T\Phi_2(\sigma_T\Phi_2)\, dB \tag{3.21}$$

where Eqs. (2.4), (2.5) and condition (3.14) have been used.

By Eq. (3.19) we see

$$G(\Phi_2) \leq I(\phi) \tag{3.22}$$

for all functions Φ_2 satisfying the condition

$$\sigma_T{}^*T\Phi_2 = 0 \quad \text{on} \quad \partial V_2 . \tag{3.23}$$

(d) Complementary variational principles: From the results in (b) and (c) we have the global complementary variational principles

$$G(\Phi_2) \leq I(\phi) \leq J(\Phi_1) \tag{3.24}$$

when

$$-(df/d\psi) \geq 0 \quad \text{for all } \psi, \tag{3.25}$$

for all admissible functions Φ_1 and Φ_2 such that

$$\sigma_T\Phi_1 = 0 \quad \text{on} \quad \partial V_1 \tag{3.26}$$

$$\sigma_T{}^*T\Phi_2 = 0 \quad \text{on} \quad \partial V_2 . \tag{3.27}$$

Equality holds in (3.24) when Φ_1 and Φ_2 are both equal to the exact function ϕ.

4. Error bound. For approximate solutions Φ_1 which satisfy the exact boundary conditions (2.2) and (2.3), an estimate of the error is available from the error bound [2],

$$\|\Phi_1 - \phi\|_{L^2} \leq E(\Phi_1), \tag{4.1}$$

where $\|\psi\|_{L^2}$ is the usual L^2 norm $[\int_V \psi^2\, dV]^{1/2}$, and

$$E(\Phi_1) = [2\Lambda^{-1}(J(\Phi_1) - I^-)]^{1/2}. \tag{4.2}$$

Here I^- denotes a lower bound for $I(\Phi)$, and Λ is a lower bound to the lowest eigenvalue of the problem

$$L\psi = \lambda\psi \quad \text{in } V \tag{4.3}$$

with

$$\sigma_T \psi = 0 \quad \text{on} \quad \partial V_1 \tag{4.4}$$

$$\sigma_T^* T \psi = 0 \quad \text{on} \quad \partial V_2 \, . \tag{4.5}$$

5. Integral approach. Hereafter, we take L to be a differential operator satisfying the required conditions and suppose that the sufficient conditions for complementary variational principles are satisfied. Let the Green's operator corresponding to L and its domain of functions be K, an integral operator with kernel $\mathcal{K}(\mathbf{r}, \mathbf{s})$. Then formally we have

$$K = L^{-1} \tag{5.1}$$

and Eqs. (2.1) to (2.3) may be rewritten as

$$\phi = K\{f(\phi)\} \quad \text{in } V \tag{5.2}$$

where the boundary conditions have been absorbed into the operator K. Setting

$$\theta = f(\phi) \tag{5.3}$$

and

$$m(\theta) = f^{-1}(\theta) = \phi, \tag{5.4}$$

we can write Eq. (5.2) as

$$K\theta = m(\theta) \quad \text{in } V \tag{5.5}$$

and since the boundary conditions have been absorbed we take

$$\sigma_T = \sigma_T^* = 0 \quad \text{on } \partial V_1 \quad \text{and} \quad \partial V_2 \, . \tag{5.6}$$

It is shown in [4] that provided the sufficient condition (3.25) is satisfied and certain constants of integration are suitably chosen we obtain the complementary bounds

$$-\tilde{J}(\Theta_1) \leq I(\phi) \leq -\tilde{G}(\Theta_2) \tag{5.7}$$

where

$$\tilde{J}(\Theta_1) = \int_V \{\tfrac{1}{2}\Theta_1 K \Theta_1 - M(\Theta_1)\} \, dV, \tag{5.8}$$

$$\tilde{G}(\Theta_2) = \int_V \{(K\Theta_2)[m^{-1}(K\Theta_2) - \tfrac{1}{2}\Theta_2] - M[m^{-1}(K\Theta_2)]\} \, dV, \tag{5.9}$$

$$M(\theta) = \int^\theta m(\psi) \, d\psi, \tag{5.10}$$

and Θ_1, Θ_2 are approximations to the exact solution θ of Eq. (5.5).

6. Problem I: horizontal cantilever with a vertical point load at the free end. By introducing the dimensionless variable $t = s/L$, where s is the distance along the beam, the boundary-value problem satisfied by the angle ϕ, measured between the tangent at a point on the beam, and the horizontal, is as follows:

$$-d^2\phi/dt^2 = (PL^2/EI)\cos\phi, \qquad 0 \le t \le 1, \tag{6.1}$$

$$\phi(0) = 0, \qquad (d\phi/dt)(1) = 0. \tag{6.2}$$

The cantilever is assumed to be inextensible, of length L and flexural rigidity EI. It is fixed at the end $t = 0$, and subjected to a vertical point load P at the free end $t = 1$.

7. Complementary variational principles for Problem I. This problem corresponds to the following choices in the general theory:

$$V = (0, 1), \qquad \partial V_1 = \{t = 0\}, \qquad \partial V_2 = \{t = 1\}, \tag{7.1}$$

$$L = d^2/dt^2, \tag{7.2}$$

$$T = d/dt, \qquad T^* = -d/dt, \qquad \sigma_T = +1 \quad \text{at} \quad t = 1, \qquad -1 \quad \text{at} \quad t = 0 \tag{7.3}$$

$$f(\phi) = q^2 \cos\phi, \tag{7.4}$$

where $q^2 = PL^2/EI$. Then we take

$$F(\phi) = q^2 \sin\phi \tag{7.5}$$

and the Green's operator $\mathcal{K}$ is found to have kernel

$$\mathcal{K}(t, z) = \min(t, z) \qquad 0 \le t, \quad z \le 1. \tag{7.6}$$

From (7.4)

$$-(df/d\psi)(\psi) = q^2 \sin\psi \tag{7.7}$$

which is nonnegative since $0 \le \psi \le \pi/2$ on physical grounds, and therefore the previous theory may be applied to obtain global complementary variational principles.

For the various functionals we find

$$I(\phi) = \int_0^1 q^2\{\tfrac{1}{2}\phi\cos\phi - \sin\phi\}\, dt, \tag{7.8}$$

$$J(\Phi_1) = \int_0^1 \left\{\frac{1}{2}\left(\frac{d\Phi_1}{dt}\right)^2 - q^2 \sin\Phi_1\right\} dt \quad \text{with} \quad \Phi_1(0) = 0, \tag{7.9}$$

$$G(\Phi_2) = \int_0^1 \left\{-\frac{d^2\Phi_2}{dt^2}\cos^{-1}\left[-\frac{1}{q^2}\frac{d^2\Phi_2}{dt^2}\right] - \frac{1}{2}\left(\frac{d\Phi_2}{dt}\right)^2 - q^2 \sin\left(\cos^{-1}\left[-\frac{1}{q^2}\frac{d^2\Phi_2}{dt^2}\right]\right)\right\} dt \quad \text{with} \quad \frac{d\Phi_2}{dt}(1) = 0, \tag{7.10}$$

$$\tilde{J}(\Theta_1) = \int_0^1 \left\{\tfrac{1}{2}\Theta_1 K\Theta_1 - \Theta_1\cos^{-1}\left[\frac{\Theta_1}{q^2}\right] + q^2\left[1 - \left(\frac{\Theta_1}{q^2}\right)^2\right]^{1/2}\right\} dt, \tag{7.11}$$

$$\tilde{G}(\Theta_2) = \int_0^1 \{-\tfrac{1}{2}\Theta_2 K\Theta_2 + q^2 \sin(K\Theta_2)\}\, dt. \tag{7.12}$$

Calculations have been performed for the case $q^2 = 2$. Simple polynomials in t were taken as the trial functions:

$$\Phi_1 = \sum_{r=1}^{6} a_r t^r, \tag{7.13}$$

TABLE I. Differential approach variational parameters for problem I.

a_1	a_2	a_3	a_4	a_5	a_6	$J(\Phi_1)$
1.67868	−9.99995(−1)	4.42960(−3)	2.17196(−1)	−1.50552(−1)	3.19995(−2)	−0.56478528
b_1	b_2	b_3	b_4	b_5	b_6	$G(\Phi_2)$
1.67868	−9.99995(−1)	4.39960(−3)	2.17196(−1)	−1.50542(−1)	3.20060(−2)	−0.56478564

Here $m(-n)$ means $m \times 10^{-n}$.

$$\Phi_2 = \sum_{r=1}^{6} b_r t^r, \tag{7.14}$$

$$\Phi_3 = \sum_{r=1}^{3} c_r t^r, \qquad \Theta_1 = q^2 \cos \Phi_3 , \tag{7.15}$$

$$\Phi_4 = \sum_{r=1}^{3} d_r t^r, \qquad \Theta_2 = q^2 \cos \Phi_4 , \tag{7.16}$$

the boundary conditions in (6.2) were imposed on all the trial functions Φ_1 to Φ_4 , and then the functionals were optimized over the remaining free parameters. To estimate the error in Φ_1 from (4.2) we require Λ which, on using Eqs. (7.1) to (7.3), was found to be

$$\Lambda = \pi^2/4. \tag{7.17}$$

The optimum functional and parameter values are given in Tables I and II, and from these we see that $-\tilde{J}(\Theta_1)$ produced a better lower bound for $I(\phi)$ than $G(\Phi_2)$, using fewer parameters. This value was substituted for I^- in the error estimate, giving

$$E(\Phi_1) = 3.37 \times 10^{-4}, \tag{7.18}$$

indicating that the trial function Φ_1 is quite accurate. This is borne out by calculating the vertical and horizontal displacements by numerical integration of sin Φ_1 and cos Φ_1 and comparing these with the corresponding values found from an approach given in Frisch–Fay [5], based upon complete and incomplete elliptic integrals. The comparisons are given in Table III.

TABLE II. Integral approach variational parameters for Problem I.

c_1	c_2	c_3	$-\tilde{J}(\theta_1)$
1.67030	−9.96970(−1)	1.07880(−1)	−0.56478541
d_1	d_2	d_3	$-\tilde{G}(\theta_2)$
1.65152	−9.54492(−1)	8.58211(−2)	−0.56478522

TABLE III. Comparison of the coordinates of the deflected shape for Problem I.

t	x/L using Φ_1	x/L from [5]	y/L using Φ_1	y/L from [5]
0.25	0.244229	0.244230	0.046764	0.046764
0.5	0.464534	0.464536	0.163650	0.163647
0.75	0.659356	0.659358	0.319999	0.319996
1.0	0.839355	0.839358	0.493462	0.493458

8. Problem II: horizontal cantilever with uniformly distributed load. It is not possible to use the elliptic integral theory for this problem and recourse to techniques supplying an accurate numerical approximation must be made.

The same assumptions hold as in Problem I, but the vertical point load is replaced by uniformly distributed load per unit length w, and the free end occurs at $t = 0$, the fixed end at $t = 1$. In this case the nonlinear boundary-value problem takes the form [6]:

$$-d^2\phi/dt^2 = (wL^3/EI)t \cos \phi, \qquad 0 \leq t \leq 1; \tag{8.1}$$

$$d\phi/dt(0) = 0, \qquad \phi(1) = 0. \tag{8.2}$$

9. Complementary variational principles for Problem II. We have the following cases of the basic theory:

$$V = (0, 1), \qquad \partial V_1 = \{t = 1\}, \qquad \partial V_2 = \{t = 0\}, \tag{9.1}$$

$$L = -d^2/dt^2, \tag{9.2}$$

$$T = d/dt, \qquad T^* = -d/dt, \qquad \sigma_T = +1 \quad \text{at} \quad t = 1, \qquad -1 \quad \text{at} \quad t = 0, \tag{9.3}$$

$$f(\phi) = r^2 t \cos \phi, \tag{9.4}$$

where $r^2 = wL^3/EI$. We take

$$F(\phi) = r^2 t \sin \phi \tag{9.5}$$

and the Green's operator K has kernel

$$\mathcal{K}(t, z) = 1 - \max (t, z), \qquad 0 \leq t, \quad z \leq 1. \tag{9.6}$$

From Eq. (9.4)

$$-(df/d\psi)(\psi) = r^2 t \sin \psi, \tag{9.7}$$

which again is nonnegative on physical grounds, and the global complementary variational principles hold.

The functionals take the forms

$$I(\phi) = \int_0^1 r^2 t\{\tfrac{1}{2}\phi \cos \phi - \sin \phi\}\, dt, \tag{9.8}$$

$$J(\Phi_1) = \int_0^1 \left\{\frac{1}{2}\left(\frac{d\Phi_1}{dt}\right)^2 - r^2 t \sin \Phi_1\right\} dt \quad \text{with} \quad \Phi_1(1) = 0, \tag{9.9}$$

$$G(\Phi_2) = \int_0^1 \left\{-\frac{d^2\Phi_2}{dt^2} \cos^{-1}\left[-\frac{1}{(r^2 t)}\frac{d^2\Phi_2}{dt^2}\right] - \frac{1}{2}\left(\frac{d\Phi_2}{dt}\right)^2 \right.$$
$$\left. - r^2 t \sin\left(\cos^{-1}\left[-\frac{1}{(r^2 t)}\frac{d^2\Phi_2}{dt^2}\right]\right)\right\} dt \quad \text{with} \quad \frac{d\Phi_2}{dt}(0) = 0, \tag{9.10}$$

$$\tilde{J}(\Theta_1) = \int_0^1 \left\{\tfrac{1}{2}\Theta_1 K\Theta_1 - \Theta_1 \cos^{-1}\left[\frac{\Theta_1}{r^2 t}\right] + r^2 t\left[1 - \left(\frac{\Theta_1}{r^2 t}\right)^2\right]^{1/2}\right\} dt, \tag{9.11}$$

$$\tilde{G}(\Theta_2) = \int_0^1 \{-\tfrac{1}{2}\Theta_2 K\Theta_2 + r^2 t \sin (K\Theta_2)\}\, dt. \tag{9.12}$$

TABLE IV. Differential approach variational parameters for Problem II.

a_0	a_3	a_4	a_5	$J(\Phi_1)$
3.21552(−1)	−3.14132(−1)	−2.35000(−3)	−5.07000(−3)	−0.097848772
b_0	b_3	b_4	b_5	$G(\Phi_2)$
3.21880(−1)	−3.10680(−1)	−1.10100(−2)	−1.90000(−4)	−0.097878123

Calculations have been performed for the case $r^2 = 2$. Again simple polynomials in t were used for trial functions:

$$\Phi_1 = \sum_{r=0}^{5}{}' a_r t^r, \tag{9.13}$$

$$\Phi_2 = \sum_{r=0}^{5}{}' b_r t^r, \tag{9.14}$$

$$\Phi_3 = \sum_{r=0}^{3}{}' c_r t^r, \qquad \Theta_1 = r^2 t \cos \Phi_3 , \tag{9.15}$$

$$\Phi_4 = \sum_{r=0}^{3}{}' d_r t^r, \qquad \Theta_2 = r^2 t \cos \Phi_4 , \tag{9.16}$$

where the prime on the summation denotes that we omit the terms t and t^2. The former are omitted in order that $\Phi_1 - \Phi_4$ satisfy the first boundary condition in (8.2), the latter omitted to avoid the term $-(d^2\Phi_2/dt^2)/(r^2t)$ in $G(\Phi_2)$ becoming infinite at $t = 0$. After imposing the second boundary condition of (8.2) on Φ_1—Φ_4, the functionals were optimized over the remaining free parameters. The value of Λ is the same as in Problem I, as given in Eq. (7.17), and the optimum functional and parameter values for this problem are given in tables IV and V. Again $-\tilde{J}(\theta_1)$ produced the better lower bound of $I(\phi)$ and on substitution for I^- in the error bound we obtain

$$E(\Phi_1) = 1.76 \times 10^{-4}. \tag{9.17}$$

The vertical and horizontal displacements for points along the beam were calculated as in Problem I and compared with those from a power series approximation used by Rohde [6]. The comparisons are given in Table VI. Given the values of EI, L and w, this seems to be a more logical approach to the problem.

TABLE V. Integral approach variational parameters for Problem II.

c_0	c_3	$-\tilde{J}(\theta_1)$
3.22300(−1)	−3.22300(−1)	−0.097848809
d_0	d_3	$-\tilde{G}(\theta_2)$
3.22300(−1)	−3.22300(−1)	−0.097848780

TABLE VI. Comparison of the coordinates of the deflected shape for Problem II.

t	x/L using Φ_1	x/L from [6]	y/L using Φ_1	y/L from [6]
0.25	0.237283	0.237280	0.078718	0.078727
0.5	0.475880	0.475877	0.153319	0.153327
0.75	0.718572	0.718570	0.212931	0.212937
1.0	0.966895	0.966893	0.238500	0.238507

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