MIXED FINITE-ELEMENT APPROXIMATIONS OF LINEAR BOUNDARY-VALUE PROBLEMS*

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Abstract. A theory of mixed finite-element/Galerkin approximations of a class of linear boundary-value problems of the type $T^*Tu + ku + f = 0$ is presented, in which appropriate notions of consistency, stability, and convergence are derived. Some error estimates are given and the results of a number of numerical experiments are discussed.

1. Introduction. A substantial majority of the literature on finite-element approximations concerns the so-called primal or "displacement" approach in which a single (possibly vector-valued) variable is approximated which minimizes a certain quadratic functional (e.g. the total potential energy in an elastic body). A shortcoming of such approximations is that they often lead to very poor approximations of various derivatives of the dependent variable (e.g. strains and stresses). The dual model, also referred to as the "equilibrium" model, employs a maximum principle (complementary energy), and can lead to better approximations of derivatives, but it leads to difficulties in computing the values of the function itself for irregular domains. The alternative is to use so-called mixed or hybrid approximations in which two or more quantities are approximated independently (e.g. displacements and strains are treated independently). Numerical experiments indicate that this alternative can lead to improved accuracies for derivatives in certain cases, but the extremum character of the associated variational statements of the problem is lost in the process. This means that most of the techniques used to establish the convergence of the finite element method in the dual and primal formulations are not valid for the mixed case.

In the mid-1960s, use of mixed finite-element models for plate bending were proposed, independently, by Herrmann [1] and Hellan [2]. These involved the simultaneous approximation of two dependent variables, the bending moments and the transverse deflection of thin elastic plates, and were based on stationary rather than extremum variational principles. Prager [3], Visser [4], and Dunham and Pister [5] employed the idea of Herrmann to construct mixed finite-element models from a form of the Hellinger–Reissner principle for plate bending problems with very good results. Backlund [6] used the mixed plate-bending elements developed by Herrmann and Hellan for the analysis of elastic and elasto-plastic plates in bending, and Wunderlich [7] used the idea of mixed models in a finite-element analysis of nonlinear shell behavior. Parallel to the work on mixed models was the development of the closely related hybrid models by

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Pian and his associates (e.g. [8, 9, 10]). Reddy [11], Johnson [12], and Kikuchi and Ando [13] obtained some error estimates for mixed models of the biharmonic equation; however, their approach is not general and the biharmonic equation has the special feature that it decomposes into uncoupled systems of canonical equations which are themselves elliptic. In all of these studies, results of numerical experiments suggest that mixed models can be developed which not only converge very rapidly but also may yield higher accuracies for stresses than the corresponding displacement-type model. More importantly, the stationary conditions of the mixed formulation are a set of canonical equations involving lower-order derivatives than those encountered in the governing equations. This makes it possible to relax continuity requirements on the trial functions in mixed finite-element models.

It is the purpose of the present paper to describe properties of a broad class of mixed finite-element approximations and to present fairly general procedures for establishing the convergence of the method and, in certain cases, to derive error estimates. Preliminary investigations of the type reported herein were given in [14] and centered around notions of consistency and stability of mixed approximations. The present study utilizes a similar but more general approach, and we are able to obtain the conclusions of [14] as well as those of previous investigators (e.g. [13]) as special cases.

2. A class of linear boundary-value problems. We are concerned with a class of boundary-value problems of the type

$$T^*Tu + ku + f = 0 \text{ in } \Omega, \qquad Mu - \mathbf{g}_1 = 0 \text{ on } \partial\Omega_1, \qquad N(Tu) - g_2 = 0 \text{ on } \partial\Omega_2. \tag{2.1}$$

Here T is a linear operator from a Hilbert space $\mathcal{U}$ into a Hilbert space $\mathcal{V}$, T^* is the adjoint of T and its domain $D_T{}^*$ is in $\mathcal{V}$, the dependent variable $u(\mathbf{x})$ is an element of $\mathcal{U}$ and is a function of points $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ in an open bounded domain $\Omega \subset \mathbf{R}^n$. The boundary $\partial\Omega$ of Ω is divided into two portions, $\partial\Omega_1 \cup \partial\Omega_2 = \partial\Omega$ on which the images of u and Tu under the boundary operators M and N are prescribed, as indicated. If $\{u_1, u_2\}$ and $[\mathbf{v}_1, \mathbf{v}_2]$ denote the inner products associated with spaces $\mathcal{U}$ and $\mathcal{V}$, respectively, then T and T^* are assumed to satisfy a generalized Green's formula of the type

$$[Tu, \mathbf{v}] = \{u, T^*\mathbf{v}\} + \{N\mathbf{v}, u\}_{\partial\Omega_2} + [Mu, \mathbf{v}]_{\partial\Omega_1} \tag{2.2}$$

where $\{\cdot, \cdot\}_{\partial\Omega_1}$ and $[\cdot, \cdot]_{\partial\Omega_2}$ are associated bilinear forms obtained using the extensions of u and $\mathbf{v}$ and Mu and $N\mathbf{v}$ to the indicated portions of the boundary. Clearly, the forms of M and N depend upon T and the definition of the inner products (for a complete picture see [15]).

The boundary-value problem (2.1) can be split into a canonical pair of problems equivalent to (2.1) of the form

$$\begin{aligned} Tu &\equiv \mathbf{v} \quad \text{in} \quad \Omega \qquad Mu - \mathbf{g}_1 = 0 \quad \text{on} \quad \partial\Omega_1, \\ T^*\mathbf{v} + ku &= -f \quad \text{in} \quad \Omega \qquad N\mathbf{v} - g_2 = 0 \quad \text{on} \quad \partial\Omega_2. \end{aligned} \tag{2.3}$$

Our mission is to study finite-element-Galerkin approximations of this pair.

3. Mixed Galerkin projections. We now identify finite, linearly-independent sets of functions $\{\Phi_\alpha(\mathbf{x})\}_{\alpha=1}{}^G \in \mathcal{U}$ and $\{\boldsymbol{\omega}^\Delta(\mathbf{x})\}_{\Delta=1}{}^H \in \mathcal{V}$, which, respectively span the finite-dimensional subspaces $\mathfrak{M}_G^h$ and $\mathfrak{N}_H^l$. Now if $u(\mathbf{x})$ and $\mathbf{v}(\mathbf{x})$ are arbitrary elements in $\mathfrak{U}$ and $\mathfrak{V}$, respectively, their projections into $\mathfrak{M}_G^h$ and $\mathfrak{N}_H^l$ are of the form (see [16])

$$\Pi_h(u) = U(\mathbf{x}) = \sum_{\alpha=1}^{G} a^\alpha \Phi_\alpha(\mathbf{x}, h); \qquad P_l(\mathbf{v}) = \mathbf{V}(\mathbf{x}) = \sum_{\beta=1}^{H} b_\Delta \boldsymbol{\omega}^\Delta(\mathbf{x}, l). \tag{3.1}$$

Here a^α and b_Δ are constants, uniquely determined by u, Φ_α, $\mathbf{v}$, and $\boldsymbol{\omega}^\Delta$. It must be noted here that there is no relation between the spaces $\mathfrak{U}$ and $\mathfrak{V}$, and the biorthogonal bases in $\mathfrak{M}_G^h$ and $\mathfrak{N}_H^l$ are completely independent of each other.

Consider the case in which $\mathfrak{M}_G^h \subset \mathfrak{D}_T$ and $\mathfrak{N}_H^l \subset \mathfrak{D}_T^*$. In general, $T(\mathfrak{M}_G^h)$ is not a subspace of $\mathfrak{N}_H^l$, and $T^*(\mathfrak{N}_H^l)$ is not a subspace of $\mathfrak{M}_G^h$. The operators T and T^* can be approximated by projecting $T(\mathfrak{M}_G^h)$ into $\mathfrak{N}_H^l$ and $T^*(\mathfrak{N}_H^l)$ into $\mathfrak{M}_G^h$. This projection process leads to a number of rectangular matrices of which the following are encountered naturally:

$$\begin{aligned} &P_l T(\mathfrak{M}_G^h): P_l(T\Phi_\alpha) = \sum_\Delta T_\alpha{}^\Delta \boldsymbol{\omega}_\Delta; \qquad P_l(\boldsymbol{\omega}^\Delta) = \sum_\Gamma H^{\Delta\Gamma} \boldsymbol{\omega}_\Gamma ; \\ &\Pi_h T^*(\mathbf{N}_H^l): \Pi_h(T^*\boldsymbol{\omega}^\Delta) = \sum_\alpha T_\alpha{}^{*\Delta} \Phi^\alpha; \qquad \Pi_h(\Phi_\alpha) = \sum_\beta G_{\alpha\beta}\Phi^\beta, \end{aligned} \tag{3.2}$$

where $\{\Phi^\alpha\}$ and $\{\boldsymbol{\omega}_\Delta\}$ are the biorthogonal bases, and

$$T_\alpha{}^{\cdot\Delta} = [T\Phi_\alpha , \boldsymbol{\omega}^\Delta], \quad T_\alpha{}^{*\Delta} = \{\Phi_\alpha , T^*\boldsymbol{\omega}^\Delta\} \quad H^{\Delta\Gamma} = [\boldsymbol{\omega}^\Delta, \boldsymbol{\omega}^\Gamma], \quad G_{\alpha\beta} = \{\Phi_\alpha , \Phi_\beta\}. \tag{3.3}$$

Analogously, the boundary operators M and N can be approximated using projections to yield

$$P_l(M\Phi_\alpha) = \sum_\Delta M_\alpha^{\cdot\Delta} \boldsymbol{\omega}_\Delta ; \qquad \Pi_h(N\boldsymbol{\omega}^\Delta) = \sum_\alpha N_\alpha{}^\Delta \Phi^\alpha \tag{3.4}$$

where

$$M_\alpha{}^{\cdot\Delta} = [M\Phi_\alpha , \boldsymbol{\omega}^\Delta]_{\partial\Omega_1} ; \qquad N_\alpha{}^\Delta = \{N\boldsymbol{\omega}^\Delta, \Phi_\alpha\}_{\partial\Omega_2} . \tag{3.5}$$

In view of Green's formula $T_\alpha{}^{\cdot\Delta}$ can be also written in terms of $T_\alpha{}^{*\Delta}$:

$$T_\alpha{}^{\cdot\Delta} = T_\alpha{}^{*\Delta} + M_\alpha{}^{\cdot\Delta} + N_\alpha{}^\Delta. \tag{3.6}$$

Mixed projections. Primal-dual projection. The primal-dual projection, together with dual-primal projection to be discussed subsequently, give mixed approximation of boundary-value problem (2.1). In primal-dual projection, approximate solutions $U^* = \sum a^\alpha \Phi_\alpha$ and $\mathbf{V}^* = \sum b_\Delta \boldsymbol{\omega}^\Delta$ of (2.3) are sought simultaneously by requiring

$$\Pi_h(T^*\mathbf{V}^* + kU^* + f) = 0 \text{ in } \Omega, \qquad \Pi_h(N\mathbf{V}^* - g_2) = 0 \text{ on } \partial\Omega_2 . \tag{3.7}$$

This yields

$$\sum_\Delta^H (T_\alpha{}^{\cdot\Delta} - M_\alpha{}^{\cdot\Delta})b_\Delta + k\sum_\beta^G a^\beta G_{\alpha\beta} + f_\alpha = 0. \tag{3.8}$$

Since, in general, $\mathfrak{M}_G^h$ and $\mathfrak{N}_H^l$ are of different dimensions, $(T_\alpha{}^\Delta - M_\alpha{}^\Delta)$ is a rectangular matrix; and since (3.8) involves $(G + H)$ unknowns with only G equations, no unique solution to (3.8) exists. The remaining H equations are provided by the dual-primal projection.

Dual-primal projection. Here the approximate solutions U^* and $\mathbf{V}^*$ are obtained by requiring

$$P_l(TU^* - \mathbf{V}^*) = 0 \text{ in } \Omega, \qquad P_l(MU^* - \mathbf{g}_1) = 0 \text{ on } \partial\Omega_1 \tag{3.9}$$

which upon simplification lead to

$$\sum_{\alpha}^{G} (T_{\alpha}{}^{*\Delta} + N_{\alpha}{}^{\cdot\Delta})^T a^{\alpha} - \sum_{\Gamma}^{H} b_{\Gamma} H^{\Gamma\Delta} + f^{\Delta} = 0 \tag{3.10}$$

where $f^{\Delta} = [\mathbf{g}_1, \boldsymbol{\omega}^{\Delta}]_{\partial\Omega_1}$. Note that (3.10) involves H equations in $(G + H)$ unknowns. Eqs. (3.8) and (3.10) combined lead to a determinate system for the approximate solutions U^* and $\mathbf{V}^*$. Solving (3.10) for b_{Γ}, we obtain

$$b_{\Gamma} = \sum_{\Delta} H_{\Gamma\Delta}(\sum_{\alpha} (T_{\alpha}{}^{*\Delta} + N_{\alpha}{}^{\Delta})^T a^{\alpha} - f^{\Delta}), \tag{3.11}$$

where $H_{\Delta\Gamma} = (H^{\Delta\Gamma})^{-1}$. Substitution of (3.11) into (3.8) leads to

$$\sum_{\alpha} K_{\beta\alpha} a^{\alpha} + F_{\beta} = 0 \tag{3.12}$$

where

$$\begin{aligned} K_{\beta\alpha} &= \sum_{\Delta,\Gamma} (T_{\beta}{}^{\cdot\Delta} - M_{\beta}{}^{\cdot\Delta}) H_{\Delta\Gamma} (T_{\alpha}{}^{*\Gamma} + N_{\alpha}{}^{\Gamma})^T + kG_{\alpha\beta}, \\ F_{\beta} &= f_{\beta} - \sum_{\Delta,\Gamma} (T_{\beta}{}^{\cdot\Delta} - M_{\beta}{}^{\cdot\Delta}) H_{\Delta\Gamma} f^{\Gamma}. \end{aligned} \tag{3.13}$$

Since $T_{\beta}{}^{\cdot\Delta} - M_{\beta}{}^{\cdot\Delta} = T_{\beta}{}^{*\Delta} + N_{\beta}{}^{\cdot\Delta}$, clearly $K_{\beta\alpha}$ is symmetric. Eq. (3.12) determines the coefficients a^{α}, and hence leads to the approximate solution U^*. The local form of (3.12) can be generated using usual finite-element approximations (see [16]); techniques for connecting elements together to obtain the global model are well known (see [17]).

4. Some basic properties of mixed finite-element approximations. The proof of convergence and the establishment of error estimates for conventional primal and dual finite-element approximations follow easily from extremum properties of the associated variational principles, and concrete results are available for a number of different approximations of this type (see, for example, [18–24]). While a great deal of numerical evidence has accumulated on the utility of mixed models, rigorous studies of their advantages or disadvantages as compared to traditional formulations have not heretofore been made. Indeed, the true utility of mixed models can only be determined when answers to a number of basic questions concerning their intrinsic properties are resolved. The main objective of this section is to examine some of these questions for linear boundary-value problems of the type (2.3).

Let U and $\mathbf{V}$ denote the typical elements of $\mathfrak{M}_G{}^h$ and $\mathfrak{N}_H{}^l$ respectively, and U^* and $\mathbf{V}^*$ denote the mixed finite-element (or Galerkin) approximations of the weak solutions u^* and $\mathbf{v}^*$ of the boundary-value problem (2.1):

$$[Tu - \mathbf{v}, \bar{\mathbf{v}}] = 0 \text{ in } \Omega, \qquad [Mu - \mathbf{g}_1, \bar{\mathbf{v}}] = 0 \text{ on } \partial\Omega_1, \tag{4.1}$$

$$\{T^*\mathbf{v} + ku + f, \bar{u}\} = 0 \text{ in } \Omega, \qquad \{N\mathbf{v} - g_2, \bar{u}\} = 0 \text{ on } \partial\Omega_2. \tag{4.2}$$

By construction of the mixed models (3.7) and (3.9), U^* and $\mathbf{V}^*$ satisfy the following orthogonality relations:

$$[TU^* - \mathbf{V}^*, \mathbf{V}] = 0 \text{ in } \Omega, \qquad [MU^* - \mathbf{g}_1, \mathbf{V}] = 0 \text{ on } \partial\Omega_1, \tag{4.3}$$

$$\{T^*\mathbf{V}^* + kU^* + f, U\} = 0 \text{ in } \Omega, \qquad \{N\mathbf{V}^* - g_2, U\} = 0 \text{ on } \partial\Omega_2. \tag{4.4}$$

These relations can be expressed in a more general form [see (3.7) and (3.9)] by employing the projection operators Π_h and P_l of (3.1):

$$P_l T U^* - \mathbf{V}^* = 0 \text{ in } \Omega; \qquad P_l M U^* - P_l \mathbf{g}_1 = 0 \text{ on } \partial\Omega_1 \tag{4.5}$$

$$\Pi_h(T^*\mathbf{V}^* + f) + kU^* = 0 \text{ in } \Omega; \qquad \Pi_h(N\mathbf{V}^* - g_2) = 0 \text{ on } \partial\Omega_2 \tag{4.6}$$

where advantage is taken of the fact $P_l \mathbf{V}^* = \mathbf{V}^*$ and $\Pi_h U^* = U^*$.

THEOREM 4.1. Let $(u^*, \mathbf{v}^*)$ be the weak solution of (4.1) and (4.2) and let U^* and $\mathbf{V}^*$ be the corresponding mixed finite-element solutions satisfying (4.5) and (4.6). Then the following relations hold:

$$(P_l T \Pi_h T^* + k\mathbf{I})\mathbf{V}^* + P_l T \Pi_h f = 0, \tag{4.7}$$

$$(\Pi_h T^* P_l T + kI)U^* + \Pi_h f = 0, \tag{4.8}$$

where I and $\mathbf{I}$ are identity operators.

Proof: The relation (4.7) is obtained from (4.6) by eliminating U^* and (4.8) is obtained from (4.5) by eliminating $\mathbf{V}^*$. Indeed, operating with $P_l T$ on (4.6) and substituting for $P_l T U^*$ from (4.5) yields (4.7). Similarly, operating with $\Pi_h T^*$ on (4.5) and substituting for $\Pi_h T^* \mathbf{V}^*$ from (4.7) lead to (4.8).

At first glance at (4.7) and (4.8), it may seem that the approximate solutions U^* and $\mathbf{V}^*$ are required to satisfy a greater degree of differentiability, equal to that of exact solutions u^* and $\mathbf{v}^*$. However, no extra smoothness of U^* and $\mathbf{V}^*$ is required since projections of $T^*\mathbf{V}^*$ and $T\Pi_h T^*\mathbf{V}^*$ are always continuous, even if $T^*\mathbf{V}^*$ is piecewise continuous. Now define

$$R_{lh} = (P_l T \Pi_h T^* + kI), \tag{4.9}$$

$$Q_{hl} = (\Pi_h T^* P_l T + kI). \tag{4.10}$$

Note that Q_{hl} and R_{lh} are mixed discrete approximations of

$$Q = (T^*T + kI), \qquad R = (TT^* + kI), \tag{4.11}$$

respectively. For the sake of simplicity, define

$$\begin{aligned} e_u &= u^* - U^* = \text{approximation error in } u^*, \\ e_{\mathbf{v}} &= \mathbf{v}^* - \mathbf{V}^* = \text{approximation error in } \mathbf{v}^*, \\ E_{\mathbf{v}} &= u - \Pi_h u = \text{interpolation error in } u, \\ E_{\mathbf{v}} &= \mathbf{v} - P_l \mathbf{v} = \text{interpolation error in } \mathbf{v}, \\ E_u^* &= u^* - \Pi_h u^* = \text{interpolation error in } u^*, \\ E_{\mathbf{v}}^* &= \mathbf{v}^* - P_l \mathbf{v}^* = \text{interpolation error in } \mathbf{v}^*. \end{aligned} \tag{4.12}$$

The following theorem establishes some fundamental properties of the approximation errors e_u and $e_{\mathbf{v}}$ in terms of the interpolation errors E_u^* and $E_{\mathbf{v}}^*$.

THEOREM 4.2. Let the conditions of Theorem 4.1 hold. Then the approximation errors e_u and $e_{\mathbf{v}}$ satisfy

$$P_l T e_u - e_{\mathbf{v}} + E_{\mathbf{v}}^* = 0, \tag{4.13}$$

$$\Pi_h T^* e_{\mathbf{v}} + k e_u - k E_u^* = 0, \tag{4.14}$$

$$\Pi_h (T^* e_{\mathbf{v}} + k e_u) = 0, \tag{4.15}$$

$$P_l (T e_u - e_{\mathbf{v}}) = 0, \tag{4.16}$$

$$P_l (M e_u)|_{\partial \Omega_1} = 0, \tag{4.17}$$

$$\Pi_h (N e_{\mathbf{v}})|_{\partial \Omega_2} = 0. \tag{4.18}$$

Proof: Proof of this theorem is straightforward and can be found in [14].

By using the relations (4.13) and (4.14), equations of the type in (4.7) and (4.8) can be obtained for e_u and $e_{\mathbf{v}}$.

COROLLARY 4.1. The approximation errors e_u and $e_{\mathbf{v}}$ satisfy the relations

$$Q_{hl} e_u = k E_u^* - T^* E_{\mathbf{v}}^*, \qquad R_{lh} e_{\mathbf{v}} = k E_{\mathbf{v}}^* + k P_l T E_u^* \tag{4.19}$$

5. Consistency of mixed variational approximations. The notion of consistency of approximation of a differential equation is fundamental to conventional methods of numerical analysis. It is a measure of how well the problem is discretized and whether the discretized operators Q_{hl} and R_{lh} approach the exact operators Q and R, respectively, as the mesh parameters h and l approach zero. Consistency of a discrete model assures that the discretization error goes to zero as the associated mesh parameters approach zero. For primal and dual problems, the notion of consistency of variational approximation is studies by Aubin [22], and differs from the consistency of difference approximations defined in Isaacson and Keller [25]. In the present analysis, notions of consistency which are appropriate for the problems considered here are introduced.

Suppose that it is required to obtain approximate solutions of (2.3). Variational methods of approximation involve seeking solutions to the weak problem (4.1) and (4.2). The discrete approximations of (4.1) and (4.2) are obtained by replacing u by $\Pi_h u$ and $\mathbf{v}$ by $P_l \mathbf{v}$:

$$[T(\Pi_h u) - P_l \mathbf{v}, \mathbf{V}]_\Omega\,; \qquad [M(\Pi_h u) - \mathbf{g}_1\,, \mathbf{V}]_{\partial\Omega_1} \tag{5.1}$$

$$\{T^*(P_l \mathbf{v}) + k(\Pi_h u), U\}_\Omega\,; \qquad \{N(P_l \mathbf{v}) - g_2\,, U\}_{\partial\Omega_2} \tag{5.2}$$

for every $U \in \mathfrak{M}_G{}^h$ and $\mathbf{V} \in \mathfrak{N}_H{}^l$.

Weakly consistent approximations. The discrete system (5.1) and (5.2) shall be referred to as *weakly consistent* with the variational equations (4.1) and (4.2) if

$$\lim_{h,l \to 0} [T(\Pi_h u) - P_l \mathbf{v}, P_l \mathbf{v}_1] = [Tu - \mathbf{v}, \mathbf{v}_1], \tag{5.3}$$

$$\lim_{h,l \to 0} \{T^*(P_l \mathbf{v}) + k\Pi_h u, \Pi_h u_1\} = \{T^* \mathbf{v} + ku, u_1\}, \tag{5.4}$$

$$\lim_{h,l \to 0} [M(\Pi_h u), \; P_l \mathbf{v}_1]_{\partial \Omega_1} = [Mu, \mathbf{v}_1]_{\partial \Omega_1}\,, \tag{5.5}$$

$$\lim_{h,l \to 0} \{N(P_l \mathbf{v}), \; \Pi_h u_1\}_{\partial \Omega_2} = \{N\mathbf{v}, u_1\}_{\partial \Omega_2}\,. \tag{5.6}$$

The quantities

$$A_{hl}(u, \mathbf{v}) = |\{T^*(P_l \mathbf{v}) + k\Pi_h u, \Pi_h u_1\} - \{T^* \mathbf{v} + ku, u_1\}|, \tag{5.7}$$

$$B_{lh}(u, \mathbf{v}) = |[T(\Pi_h u) - P_l \mathbf{v}, P_l \mathbf{v}_1] - \{Tu - \mathbf{v}, \mathbf{v}_1\}|, \tag{5.8}$$

$$C_{hl}(u, \mathbf{v}) = |\{N(P_l\mathbf{v}), \Pi_h u_1\}_{\partial\Omega_2} - \{N\mathbf{v}, u_1\}_{\partial\Omega_2}|, \tag{5.9}$$

$$D_{lh}(u, \mathbf{v}) = |[M(\Pi_h u), P_l\mathbf{v}_1]_{\partial\Omega_1} - [Mu, \mathbf{v}_1]_{\partial\Omega_1}|, \tag{5.10}$$

shall be referred to as the *lack-of-consistency* of the approximate problem (4.3) and (4.4). Then (5.3)–(5.6) are equivalent to the conditions

$$\lim_{h,l\to 0} A_{hl}(u, \mathbf{v}) = 0, \qquad \lim_{h,l\to 0} C_{lh}(u, \mathbf{v}) = 0, \tag{5.11}$$

$$\lim_{h,l\to 0} B_{lh}(u, \mathbf{v}) = 0, \qquad \lim_{h,l\to 0} D_{lh}(u, \mathbf{v}) = 0. \tag{5.12}$$

LEMMA 5.1. Let $A_{hl}(u, \mathbf{v})$, $B_{lh}(u, \mathbf{v})$, $C_{hl}(u, \mathbf{v})$, and $D_{lh}(u, \mathbf{v})$ be as defined in (5.7)–(5.10). Then the following relations hold:

$$A_{hl}(u, \mathbf{v}) = |\{T^*E_\mathbf{v} + kE_u, \Pi_h u_1\} - \{T^*\bar{e}_\mathbf{v} + k\bar{e}_u, \bar{E}_u\}|, \tag{5.13}$$

$$B_{lh}(u, \mathbf{v}) = |[TE_u - E_\mathbf{v}, P_l\mathbf{v}_1] - [T\bar{e}_u - \bar{e}_\mathbf{v}, \bar{E}_\mathbf{v}]|, \tag{5.14}$$

$$C_{hl}(u, \mathbf{v}) = |\{NE_\mathbf{v}, \Pi_h u_1\}_{\partial\Omega_2} - \{N\bar{e}_\mathbf{v}, \bar{E}_u\}_{\partial\Omega_2}|, \tag{5.15}$$

$$D_{lh}(u, \mathbf{v}) = |[ME_u, P_l\mathbf{v}_1]_{\partial\Omega_1} - [Me_u, \bar{E}_\mathbf{v}]\partial_{\Omega_1}|, \tag{5.16}$$

where $\bar{e}_u = u^* - u$, $\bar{e}_\mathbf{v} = \mathbf{v}^* - \mathbf{v}$, $\bar{E}_u = u_1 - \Pi_h u_1$, and $\bar{E}_\mathbf{v} = \mathbf{v}_1 - P_l\mathbf{v}_1$, and E_u and $E_\mathbf{v}$ as defined in (4.12).

Proof: Relations (5.13)–(5.16) easily follow from the observation

$$u_1 = \Pi_h u_1 + u_T, \qquad \mathbf{v}_1 = P_l\mathbf{v}_1 + \mathbf{v}_T \quad \text{for every} \quad u_1 \in \mathfrak{U}, \mathbf{v}_1 \in \mathfrak{V}, \tag{5.17}$$

where u_T and $\mathbf{v}_T$ are elements of spaces orthogonal to $\mathfrak{U}$ and $\mathfrak{V}$ respectively.

COROLLARY 5.1. Definitions (5.7)–(5.10) are also equivalent to

$$A_{hl}(u, \mathbf{v}) = |\{T^*E_\mathbf{v} + kE_u, u_1\} + \{T^*P_l\mathbf{v} + k\Pi_h u, \bar{E}_u\}|, \tag{5.18}$$

$$B_{lh}(u, \mathbf{v}) = |[TE_u - E_\mathbf{v}, \mathbf{v}] + [T\Pi_h u - P_l\mathbf{v}, \bar{E}_v]|, \tag{5.19}$$

$$C_{hl}(u, \mathbf{v}) = |\{NE_\mathbf{v}, u_1\}_{\partial\Omega_2} + \{NP_l\mathbf{v}, \bar{E}_u\}_{\partial\Omega_2}|, \tag{5.20}$$

$$D_{lh}(u, \mathbf{v}) = |[ME_u, \mathbf{v}_1]_{\partial\Omega_1} + [M\Pi_h u, \bar{E}_\mathbf{v}]_{\partial\Omega_1}|. \tag{5.21}$$

From (5.13)–(5.16), it is clear that the *lack-of-consistency* $A_{hl}(u, \mathbf{v})$ and $B_{lh}(u, \mathbf{v})$ depend on the interpolation errors. This fact is emphasized in Theorem 5.1.

THEOREM 5.1. Let the interpolation errors E_u and $E_\mathbf{v}$ be such that E_u, $E_\mathbf{v}$, TE_u, and $T^*E_\mathbf{v}$ approach zero with h and l. Then the approximations (4.3) and (4.4) are *weakly consistent* with (4.1) and (4.2).

Proof: It is sufficient to show that the lack-of-consistency (5.7)–(5.10) are bounded by the interpolation errors. Indeed, by replacing u_1 by $\Pi_h u_1$ and $\mathbf{v}_1$ by $P_l\mathbf{v}_1$ in (5.7) and (5.8) (i.e. $\bar{E}_u = 0, \bar{E}_\mathbf{v} = 0$), we obtain

$$A_{hl}(u, \mathbf{v}) = |\{T^*E_\mathbf{v} + kE_u, U\}|, \qquad U \in \mathfrak{M}_G{}^h, \tag{5.22}$$

$$B_{lh}(u, \mathbf{v}) = |[TE_u - E_\mathbf{v}, \mathbf{V}]|, \qquad \mathbf{V} \in \mathfrak{N}_H{}^l. \tag{5.23}$$

Now, by using the Schwarz inequality, (5.22) and (5.23) become

$$A_{hl}(u, \mathbf{v}) \le (|||T^*E_\mathbf{v}||| + k\,|||E_u|||)\,|||U|||, \tag{5.24}$$

$$B_{lh}(u, \mathbf{v}) \le (||TE_u|| + ||E_\mathbf{v}||)\,||\mathbf{V}||. \tag{5.25}$$

Since U and $\mathbf{V}$ are arbitrary elements of $\mathfrak{M}_G{}^h$ and $\mathfrak{N}_H{}^l$ respectively, choose $||U|| \leq M$, and $||\mathbf{V}|| \leq N$. This completes the proof.

Close examination of (5.22) and (5.23) suggests the following set of lack-of-consistency functions:

$$\bar{A}_{hl}(u, \mathbf{v}) = \Pi_h(T^*E_{\mathbf{v}} + kE_u) = \Pi_h T^* E_{\mathbf{v}} , \tag{5.26}$$

$$\bar{B}_{lh}(u, \mathbf{v}) = P_l(TE_u - E_{\mathbf{v}}) = P_l T E_u . \tag{5.27}$$

It is clear that

$$|||\bar{A}_{hl}(u, \mathbf{v})||| = |||\bar{A}_{hl}(\mathbf{v})||| \leq |||T^*E_v|||, \tag{5.28}$$

$$||\bar{B}_{lh}(u, \mathbf{v})|| = ||\bar{B}_{lh}(u)|| \leq ||TE_u||. \tag{5.29}$$

By comparing the approximate problems (4.7) and (4.8) with the strongly-posed boundary-value problem (2.1) and its adjoint, an alternate definition of consistency can be given.

Strongly consistent approximation. The discrete system (5.1) and (5.2) shall be referred to as *strongly consistent* with (2.3) if

$$\lim_{h,l\to 0} E_{hl}(u) = \lim_{h,l\to 0} |||Q_{hl}(\Pi_h u) - \Pi_h Q u||| = 0, \qquad u \in \mathfrak{U}, \tag{5.30}$$

$$\lim_{h,l\to 0} F_{lh}(\mathbf{v}) = \lim_{h,l\to 0} ||R_{lh}(P_l \mathbf{v}) - P_l R \mathbf{v}|| = 0, \qquad \mathbf{v} \in \mathfrak{V}. \tag{5.31}$$

Here Q_{hl} , Q, R_{lh} , and R are the operators defined in (4.9)–(4.11) and $E_{hl}(u)$ and $F_{lh}(\mathbf{v})$ are *lack-of-consistency functions.* Here it must be pointed out that the discrete operators Q_{hl} and R_{lh} are associated with the mixed problem (2.3), and quite different from the discrete operators associated with the primal and dual problems.

It is convenient to define

$$\tilde{Q}_{hl} = \Pi_h T^* P_l T, \qquad \tilde{R}_{lh} = P_l T \Pi_h T^*, \tag{5.32}$$

$$\tilde{Q} = T^*T, \qquad \tilde{R} = TT^*. \tag{5.33}$$

THEOREM 5.2. Suppose that the interpolation errors $E_u = u - \Pi_h u$, and $E_{\mathbf{v}} = \mathbf{v} - P_l \mathbf{v}$, and the operators $\tilde{Q}_{\partial hl}$ and $\tilde{R}_{lh}$ are such that

$$\lim_{h\to 0} T^* E_{Tu} = 0, \qquad \lim_{l\to 0} TE_{T^*\mathbf{v}} = 0, \tag{5.34}$$

$$\lim_{h,l\to 0} \tilde{Q}_{hl} E_u = 0, \qquad \lim_{h,l\to 0} \tilde{R}_{lh} E_{\mathbf{v}} = 0, \tag{5.35}$$

uniformly, where $E_{Tu} = Tu - P_l Tu$, and $E_{T^*\mathbf{v}} = T^*\mathbf{v} - \Pi_h(T^*\mathbf{v})$. Then the approximations (4.3) and (4.4) are *strongly consistent* with (4.1) and (4.2).

Proof: Note that

$$\begin{aligned} E_{hl}(u) &= |||Q_{hl}\Pi_h u - \Pi_h Q u||| = |||\tilde{Q}_{hl}\Pi_h u - \Pi_h \tilde{Q} u||| \\ &= |||-\tilde{Q}_{hl}E_u - \Pi_h T^* E_{Tu}||| \leq |||\tilde{Q}_{hl}E_u||| + |||T^*E_{Tu}||| \end{aligned} \tag{5.36}$$

and

$$\begin{aligned} F_{lh}(\mathbf{v}) &= ||R_{lh}P_l\mathbf{v} - P_l R\mathbf{v}|| = ||\tilde{R}_{lh}P_l\mathbf{v} - P_l\tilde{R}\mathbf{v}|| \\ &= ||-\tilde{R}_{lh}E_{\mathbf{v}} - P_l T E_{T^*\mathbf{v}}|| \leq ||\tilde{R}_{lh}E_{\mathbf{v}}|| + ||TE_{T^*\mathbf{v}}||. \end{aligned} \tag{5.37}$$

Now, by hypothesis, the right-hand sides of (5.36) and (5.37) vanish as h and l approach zero, imploying (5.10) and (5.31).

6. Stability, existence and uniqueness of mixed approximations. The growth of round-off errors in the numerical solution of (4.3) and (4.4) is related to the notion of *stability*. For arbitrary choices of the mesh parameters h and l, it may not be possible to bound the round-off errors. This suggests that there be some criteria to select the mesh parameters h and l so that the numerical scheme is stable. In this section the concept of stability as applied to mixed approximation is discussed.

Guided by the form of the approximate equations (4.7) and (4.8), the following definitions of stability are introduced:

Weak stability. The mixed approximation scheme in (4.7) and (4.8) shall be referred to a *weakly stable* if positive constants γ_1 and μ_1 exist such that

$$|||Q_{hl}(\Pi_h u)||| \geq \gamma_1 \, |||\Pi_h u||| \qquad u \in \mathfrak{U}, \tag{6.1}$$

$$||R_{lh}(P_l \mathbf{v})|| \geq \mu_1 \, ||P_l \mathbf{v}|| \qquad \mathbf{v} \in \mathfrak{V}, \tag{6.2}$$

where Q_{hl} and R_{lh} are given by (4.9) and (4.10).

The approximate scheme (4.5) and (4.6) suggest another definition of stability.

Strong stability. The mixed approximation scheme (4.5) and (4.6) shall be referred to *strongly stable* if there exist positive constants γ_2 and μ_2 such that

$$|||\Pi_h T^* P_l \mathbf{v}||| \geq \gamma_2 \, ||P_l \mathbf{v}|| \qquad \mathbf{v} \in \mathfrak{V}, \tag{6.3}$$

$$||P_l T \Pi_h u|| \geq \mu_2 \, |||\Pi_h u||| \qquad u \in \mathfrak{U}. \tag{6.4}$$

Define

$$T_{hl}^* = \Pi_h T^* P_l \, , \qquad T_{lh} = P_l T \Pi_h \tag{6.5}$$

Now suppose that $P_l \mathbf{v} = \sum_\Delta b_\Delta \boldsymbol{\omega}^\Delta$, and $\Pi_h u = \sum_\alpha a^\alpha \varphi_\alpha$. Then

$$\Pi_h T^* P_l \mathbf{v} = \sum_\Delta b_\Delta \sum_\alpha \{T^* \boldsymbol{\omega}^\Delta, \varphi_\alpha\} \varphi^\alpha = \sum_\alpha \sum_\Delta T_\alpha{}^{*\Delta} \varphi^\alpha b_\Delta$$

and

$$|||\Pi_h T^* P_l \mathbf{v}|||^2 = \sum_{\alpha,\beta} \sum_{\Delta,\Gamma} b_\Delta b_\Gamma T_\alpha{}^{*\Delta} G^{\alpha\beta} T_\beta{}^{*\Gamma}, \tag{6.6}$$

where $T_\alpha{}^{*\Delta}$ is given by (3.3). Also,

$$||P_l T \Pi_h u||^2 = \sum_{\alpha,\beta} \sum_{\Delta,\Gamma} a^\alpha a^\beta T_\alpha{}^{\cdot\Delta} H_{\Delta\Gamma} T_\beta{}^{\cdot\Gamma} \tag{6.7}$$

$$||P_l \mathbf{v}||^2 = \sum_{\Delta,\Gamma} b_\Delta b_\Gamma H^{\Delta\Gamma}; \qquad |||\Pi_h u|||^2 = \sum_{\alpha,\beta} a^\alpha a^\beta G_{\alpha\beta} \tag{6.8}$$

where $T_\alpha{}^{\cdot\Delta}$, $H^{\Delta\Gamma}$ and $G_{\alpha\beta}$ are defined in (3.3). Similarly,

$$\begin{aligned} Q_{hl}\Pi_h u &= \Pi_h T^* P_l T \Pi_h u + k \Pi_h u \\ &= \sum_{\alpha,\beta} \sum_\Delta a^\alpha [T\varphi_\alpha \, , \boldsymbol{\omega}_\Delta] \{T^* \boldsymbol{\omega}^\Delta, \varphi_\beta\} \varphi^\beta + k \sum_\alpha a^\alpha \varphi_\alpha \\ &= \sum_{\alpha,\beta} \sum_{\Delta,\Gamma} a^\alpha [T\varphi_\alpha \, , \boldsymbol{\omega}^\Delta] H_{\Delta\Gamma} \{T^* \boldsymbol{\omega}^\Gamma, \varphi_\beta\} \varphi^\beta + k \sum_{\alpha,\beta} a^\alpha G_{\alpha\beta} \varphi^\beta \\ &= \sum_{\alpha,\beta} \Big(\sum_{\Delta,\Gamma} T_\alpha{}^{\cdot\Delta} H_{\Delta\Gamma} T_\beta{}^{*\Gamma} + k G_{\alpha\beta}\Big) a^\alpha \varphi^\beta \end{aligned} \tag{6.9}$$

and

$$R_{lh}P_l\mathbf{v} = \sum_{\Delta,\Gamma} \left(\sum_{\alpha,\beta} T_\alpha{}^{*\Delta} G^{\alpha\beta} T_\beta{}^{\cdot\Gamma} + kH^{\Delta\Gamma}\right) b_\Delta \boldsymbol{\omega}_\Gamma . \tag{6.10}$$

Thus the stability conditions (6.1) and (6.2) are equivalent to

$$\sum_{\alpha,\beta} \left(\sum_{\Delta,\Gamma} T_\alpha{}^{\cdot\Delta} H_{\Delta\Gamma} T_\beta{}^{*\Gamma} + (k - \gamma_1) G_{\alpha\beta}\right) a^\alpha \varphi^\beta \geq 0, \tag{6.11}$$

$$\sum_{\Delta,\Gamma} \left(\sum_{\alpha,\beta} T_\alpha{}^{*\Delta} G^{\alpha\beta} T_\beta{}^{\cdot\Gamma} + (k - \mu_1) H^{\Delta\Gamma}\right) b_\Delta \boldsymbol{\omega}_\Gamma \geq 0, \tag{6.12}$$

and the stability conditions (6.3) and (6.4) are equivalent to

$$\sum_{\Delta,\Gamma} \left(\sum_{\alpha,\beta} T_\alpha{}^{*\Delta} G^{\alpha\beta} T_\beta{}^{*\Gamma} - \gamma_2{}^2 H^{\Delta\Gamma}\right) b_\Delta b_\Gamma \geq 0, \tag{6.13}$$

$$\sum_{\alpha,\beta} \left(\sum_{\Delta,\Gamma} T_\alpha{}^{\cdot\Delta} H_{\Delta\Gamma} T_\beta{}^{\cdot\Gamma} - \mu_2{}^2 G_{\alpha\beta}\right) a^\alpha a^\beta \geq 0. \tag{6.14}$$

It is convenient to define the following matrices:

$$\mathbf{H} = [\mathrm{H}^{\Delta\Gamma}],\ \mathbf{G} = [\mathrm{G}_{\alpha\beta}],\ \mathbf{M} = [\mathrm{T}_\alpha{}^{\cdot\Delta}],\ \mathbf{N} = [\mathrm{T}_\alpha{}^{*\Delta}]. \tag{6.15}$$

Then (6.11)–(6.14) imply the following fundamental result.

THEOREM 6.1. Let the following matrices be positive definite:

$$\mathbf{M}^T\mathbf{H}^{-1}\mathbf{N} + (k - \gamma_1)\mathbf{G}, \qquad k > 0, \tag{6.16}$$

$$\mathbf{N}^T\mathbf{G}^{-1}\mathbf{M} + (k - \mu_1)\mathbf{H}, \qquad k > 0. \tag{6.17}$$

Then the mixed approximations (4.7) and (4.8) are *weakly stable.* Moreover, if the matrices

$$\mathbf{N}^T\mathbf{G}^{-1}\mathbf{N} - \gamma_2\mathbf{H} \tag{6.18}$$

$$\mathbf{M}^T\mathbf{H}^{-1}\mathbf{M} - \mu_2{}^2\mathbf{G} \tag{6.19}$$

are positive definite, the mixed approximations (4.7) and (4.8) are *strongly stable.*

Since **G** and **H** are the fundamental (Gram) matrices, they are always positive definite. Consequently, from (6.16) and (6.17) it is clear that k has the stabilizing effect on the system.

Existence and uniqueness of solutions. The stability conditions (6.1)–(6.4) can be used to establish the existence and uniqueness proofs for approximate solutions of (4.5) and (4.6). We shall prove here the existence and uniqueness in the case of weak stability.

THEOREM 6.2. Let the mixed approximation (4.7) and (4.8) be weakly stable in the sense of (6.1) and (6.2). Then the approximate scheme (5.8) is uniquely solvable. Moreover, if the operator $T_{lh} = P_l T \Pi_h$ is bounded above

$$c\,||P_l T \Pi_h f|| \leq |||\Pi_h f||| \leq |||f|||, \qquad c = \text{constant} \tag{6.20}$$

then the approximate scheme (4.7) is uniquely solvable.

Proof: From (4.8) and the assumed stability condition (6.1),

$$|||f||| \geq |||\Pi_h f||| = |||Q_{hl} U^*||| \geq \gamma_1\, |||U^*|||. \tag{6.21}$$

Thus, Q_{hl} is bounded and hence (see Naylor and Sell [26, p. 244]) invertible. This implies that (4.8) has at least one solution. Note from (6.21) that (4.8) has only the trivial solution if f is identically zero. This proves unique solvability of (4.8).

To prove unique solvability of (4.7) note from (6.2) and (6.20) that

$$|||f||| \geq c\,||P_l T \Pi_h f|| = c\,||R_{lh} \mathbf{V}^*|| \geq c\mu_1\,||\mathbf{V}^*|| \tag{6.22}$$

This completes proof of the theorem.

7. Convergence of mixed finite-element solutions. Thus far the notions of consistency and stability of mixed approximations are discussed. Now the more important issue of convergence is to be resolved based on the knowledge of previous sections. Convergence proof based on the assumption of stability will be given. A more direct proof of convergence, without using the stability concept, is given in [27].

THEOREM 7.1 (Convergence Theorem I). The mixed finite-element approximations (4.3) and (4.4) are convergent; that is, $|||e_u|||$ and $||e_{\mathbf{v}}||$ approach zero as h and l tend to zero in some manner, if the interpolation errors E_u^*, $E_{\mathbf{v}}^*$, TE_u^* and $T^*E_{\mathbf{v}}^*$ vanish as h and l approach zero and the following sets of conditions hold:

Case $k = 0$. The approximate scheme is strongly stable in the sense of (6.3) and (6.4).

Case $k > 0$. The approximate scheme is weakly stable in the sense of (6.1) and (6.2) and the operators $\tilde{Q}_{hl}$ and $\tilde{R}_{lh}$ of (5.32) are continuous in the topologies induced by the norms $|||\cdot|||$ and $||\cdot||$ respectively.

Proof: *Case* $k = 0$. Using the triangular inequality,

$$\begin{aligned} |||e_u||| &= |||u^* - \Pi_h u^* + \Pi_h u^* - U^*|||, \\ &\leq |||E_u^*||| + |||\Pi_h u^* - U^*|||, \end{aligned} \tag{7.1}$$

and

$$||e_{\mathbf{v}}|| \leq ||E_{\mathbf{v}}^*|| + ||P_l \mathbf{v}^* - \mathbf{V}^*||. \tag{7.2}$$

In view of stability conditions (7.3) and (7.4),

$$\begin{aligned} \gamma_2\,||P_l \mathbf{v}^* - \mathbf{V}^*|| &\leq |||\Pi_h T^*(P_l \mathbf{v}^* - \mathbf{V}^*)||| = |||\Pi_h T^*(P_l \mathbf{v}^* - \mathbf{V}^*)||| \\ &\leq |||T^* E_{\mathbf{v}}^*|||. \end{aligned}$$

Hence,

$$||e_{\mathbf{v}}|| \leq ||E_{\mathbf{v}}^*|| + \frac{1}{\gamma_2}\,|||T^* E_{\mathbf{v}}^*|||. \tag{7.3}$$

Similarly,

$$\begin{aligned} |||e_u||| &\leq |||E_u^*||| + \frac{1}{\mu_2}\,||P_l T(\Pi_h u^* - U^*)|| \\ &= |||E_u^*||| + \frac{1}{\mu_2}\,||P_l(T\Pi_h u^* - \mathbf{V}^*)|| \\ &\leq |||E_u^*||| + \frac{1}{\mu_2}\,(||e_{\mathbf{v}}|| + ||TE_u^*||) \\ &\leq |||E_u^*||| + \frac{1}{\mu_2}\,||E_{\mathbf{v}}^*|| + \frac{1}{\mu_2\gamma_2}\,|||T^* E_{\mathbf{v}}^*||| + \frac{1}{\mu_2}\,||TE_u^*||. \end{aligned} \tag{7.4}$$

Eqs. (7.3) and (7.4) imply, in view of the assumptions on the interpolation errors, that $|||e_u|||$ and $||e_{\mathbf{v}}||$ approach zero as h and l tend to zero.

Case $k > 0$. From the stability condition (6.1),

$$\gamma_1 |||\Pi_h u^* - U^*||| \leq |||Q_{hl}\Pi_h(u^* - U^*)|||.$$

From (4.8) $Q_{hl}\Pi_h U^* = -\Pi_h f = \Pi_h(T^*Tu + ku) = \Pi_h Qu^*$,

$$\gamma_1 |||\Pi_h u^* - U^*||| \leq |||Q_{hl}\Pi_h u^* - \Pi_h Qu^*||| = E_{hl}(u^*), \tag{7.5}$$

and by assumed continuity of $\tilde{Q}_{hl}$ (see (5.36)), we have

$$\gamma_1 |||\Pi_h u^* - U^*||| \leq \alpha |||E_u^*||| + |||T^*E_{\mathbf{v}}^*|||$$

and

$$|||e_u||| \leq \left(1 + \frac{\alpha}{\gamma_1}\right) |||E_u^*||| + \frac{1}{\gamma_1} |||T^*E_{\mathbf{v}}^*|||. \tag{7.6}$$

Now, using (6.2),

$$\mu_1 ||P_l\mathbf{v}^* - \mathbf{V}^*|| \leq ||R_{lh}P_l(\mathbf{v}^* - \mathbf{V}^*)||.$$

Again, from (4.7),

$$R_{lh}\mathbf{V}^* = -P_lT\Pi_h f = P_lT\Pi_h(T^*T + k)u^*,$$

so that

$$\begin{aligned}\mu_1 ||P_l\mathbf{v}^* - \mathbf{V}^*|| &\leq ||(P_lT\Pi_hT^* + k)P_l\mathbf{v}^* - P_lT\Pi_h(T^*\mathbf{v}^* + ku^*)||\\ &= ||P_lT\Pi_hT^* E_{\mathbf{v}}^* + kP_lTE_u^*||\\ &\leq \beta ||E_{\mathbf{v}}^*|| + k ||TE_u^*||\end{aligned}$$

and

$$||e_{\mathbf{v}}|| \leq \left(1 - \frac{\beta}{\mu_1}\right) ||E_{\mathbf{v}}^*|| + \frac{k}{\mu_1} ||TE_u^*||. \tag{7.7}$$

Eqs. (7.6) and (7.7) prove convergence of e_u and $e_{\mathbf{v}}$.

COROLLARY 7.1. Let the assumptions of Theorem 7.1 hold. Then the mixed approximations (4.3) and (4.4) are *weakly consistent* if $k = 0$, and *strongly consistent* for $k > 0$.

The proof of this corollary follows directly from Theorem 5.1. In conventional methods of numerical analysis (for example, finite-differences), for consistent schemes stability implies convergence. With the particular definitions of consistency and stability given here for mixed finite-element schemes, it seems such conclusions cannot be drawn. However, for consistent mixed finite-element schemes, stability implies the following inequalities:

THEOREM 7.2. Let the mixed approximation scheme (4.3) and (4.4) be weakly consistent for $k = 0$ and strongly consistent for $k > 0$. Then strong stability implies convergence of $|||e_u|||$ and $||e_{\mathbf{v}}||$ for $k = 0$, and weak stability implies convergence of $|||e_u|||$ for $k > 0$.

Proof: Consider the case $k = 0$. From the strong stability condition (6.3) and (6.4),

$$\begin{aligned}\mu_2 |||\Pi_h u^* - U^*||| \leq ||P_lT(\Pi_h u^* - U^*)|| &= ||P_l(T\Pi_h u^* - \mathbf{V}^*)||\\ &= ||P_l(T\Pi_h u^* - Tu^*) + P_l\mathbf{v}^* - \mathbf{V}^*||\\ &= ||-P_lTE_u^* + P_l\mathbf{v}^* - \mathbf{V}^*||\end{aligned}$$

and

$$\gamma_2 \,||P_l \mathbf{v}^* - \mathbf{v}^*|| \leq |||\Pi_h T^*(P_l \mathbf{v}^* - \mathbf{v}^*)||| = |||\Pi_h T^* E_{\mathbf{v}}^*|||.$$

In view of (5.28) and (5.29),

$$\gamma_2 \,||P_l \mathbf{v}^* - \mathbf{V}^*|| \leq ||\bar{A}_{hl}(\mathbf{v})^*|| \tag{7.8}$$

and

$$\mu_2 \,|||\Pi_h u^* - U^*||| \leq ||\bar{B}_{lh}(u)^*|| + \frac{1}{\gamma_2}\,||\bar{A}_{hl}(\mathbf{v})^*|| \tag{7.9}$$

which proves the assertion.

For $k > 0$ the result follows from (7.5).

A more interesting result can be obtained using (4.13)–(4.18) and some additional assumptions, which are stated in the hypothesis of the following theorem.

THEOREM 7.3 (Convergence Theorem II). Let U^* and $\mathbf{V}^*$ be the mixed finite-element solutions satisfying (4.3) and (4.4), and suppose that there exist positive constants γ and μ independent of h and l, such that

$$[P_l T e_u \,,\, T e_u] \geq \gamma \,||T e_u||^2, \tag{7.10}$$

$$\{\Pi_h T^* e_{\mathbf{v}} \,,\, T^* e_{\mathbf{v}}\} \geq \mu \,|||T^* e_{\mathbf{v}}|||^2. \tag{7.11}$$

Then the mixed approximations (4.3) and (4.4) are convergent for all $k \geq 0$, provided the interpolation errors E_u^*, $E_{\mathbf{v}}^*$, TE_u^*, and $T^*E_{\mathbf{v}}^*$ vanish as h and l approach zero.

Proof: Since U^* and $\mathbf{V}^*$ satisfy (4.3) and (4.4), relations (4.13)–(4.18) hold for $k \geq 0$. Now suppose that $k = 0$. Then (4.15) becomes

$$\Pi_h(T^* e_u) = 0. \tag{7.12}$$

From (7.12) and (4.13) it follows that

$$\Pi_h[T^*(P_l T e_u + E_{\mathbf{v}}^*)] = 0, \qquad [P_l T e_u \,,\, TU] + [E_{\mathbf{v}}^*, TU] = 0$$

where $U \in \mathfrak{M}_G{}^h$ such that

$$MU = 0 \text{ on } \partial\Omega_1 \quad \text{and} \quad U = 0 \text{ on } \partial\Omega_2\,. \tag{7.13}$$

Then

$$\gamma \,||T e_u||^2 \leq [P_l T e_u \,,\, T_{eu}] = [E_{\mathbf{v}}^*, TE_u^*] + [P_l T e_u \,,\, TE_u^*] - [E_{\mathbf{v}}^*, T e_u]$$

$$\leq \epsilon \,||T e_u|| + \frac{1}{4\epsilon}\,(||E_{\mathbf{v}}^*|| + ||TE_u^*||)^2 + ||E_{\mathbf{v}}^*||\cdot||TE_u^*||$$

where ϵ is an arbitrary positive constant. Choose C_1 such that $C_1 = \gamma - \epsilon > 0$ and $D_1 = 1/4\epsilon > 0$ .Then

$$\begin{aligned} C_1 \,||T e_u||^2 &\leq D_1(||E_{\mathbf{v}}^*|| + ||TE_u^*||)^2 + ||E_{\mathbf{v}}^*||\cdot||TE_u^*|| \\ &\leq (D_1 + 1)(2\,||E_{\mathbf{v}}^*||\cdot||TE_u^*|| + ||E_{\mathbf{v}}^*||^2 + ||TE_u^*||^2) \\ &= (D_1 + 1)(||E_{\mathbf{v}}^*|| + ||TE_u^*||)^2, \end{aligned}$$

or

$$||T e_u|| \leq \left(\frac{D + 1}{C}\right)^{1/2}(||E_{\mathbf{v}}^*|| + ||TE_u^*||). \tag{7.14}$$

To prove convergence of $e_{\mathbf{v}}$, note that

$$\begin{aligned} ||P_l\mathbf{v}^* - \mathbf{V}^*||^2 &= [P_l\mathbf{v}^* - \mathbf{V}^*, P_l\mathbf{v}^* - \mathbf{V}^*] \\ &= [P_l\mathbf{v}^* - \mathbf{V}^*, P_l\mathbf{v}^* - TU^*] \\ &= ||P_l\mathbf{v}^* - TU^*|| \leq ||E_{\mathbf{v}}^*|| + ||Te_u|| \end{aligned}$$

Hence,

$$||e_{\mathbf{v}}|| \leq 2\,||E_{\mathbf{v}}|| + \left(\frac{D_1 + 1}{C_1}\right)^{1/2}(||E_{\mathbf{v}}^*|| + ||TE_u^*||). \tag{7.15}$$

Now suppose that $k > 0$. From (4.13) and (4.13) and (4.15), it can be shown that $[P_l Te_u, Te_u] + k\{e_u, e_u\} = k\{e_u, E_u^*\} + [P_l Te_u, TE_u^*] + [E_{\mathbf{v}}^*, TE_u^*] - [E_{\mathbf{v}}^*, Te_u]$ and

$$\gamma\,||Te_u||^2 + k\,|||e_u|||^2 \leq k\left(\delta\,|||e_u|||^2 + \frac{1}{4\delta}\,|||E_u^*|||^2\right)$$
$$+ \epsilon\,||Te_u||^2 + \frac{1}{4\epsilon}(||E_{\mathbf{v}}^*|| + ||TE_u^*||)^2 + ||E_{\mathbf{v}}^*||\cdot||TE_u^*||$$

where δ and ϵ are arbitrary positive constants. Choose $\delta = \frac{1}{2}$, and $\epsilon = \gamma/2$.

$$\gamma\,||Te_u||^2 + k\,|||e_u|||^2 \leq k\,|||E_u^*|||^2 + \frac{1}{\gamma}(||E_{\mathbf{v}}^*|| + ||TE_u^*||)^2 + 2\,||E_{\mathbf{v}}^*||\cdot||TE_u^*||.$$

Let

$$C_2 = \min(\gamma, 1); \qquad D_2 = 1 + 1/\gamma. \tag{7.16}$$

Then

$$||Te_u|| + k\,|||e_u||| \leq ((2D_2/C_2))^{1/2}(k\,|||E_u^*||| + ||E_{\mathbf{v}}^*|| + ||TE_u^*||). \tag{7.17}$$

Similarly, from (4.14) and (4.16), the following result can be obtained:

$$|||T^*e_{\mathbf{v}}||| + k\,||e_{\mathbf{v}}|| \leq ((2D_3/C_3))^{1/2}(k\,||E_{\mathbf{v}}^*|| + k\,|||E_u^*||| + |||T^*E_{\mathbf{v}}^*|||) \tag{7.18}$$

where

$$C_3 = \min(\mu, 1); \qquad D = 1 + 1/\mu \tag{7.19}$$

Thus, $|||e_u|||$, $||Te_u||$, $||e_{\mathbf{v}}||$, and $|||T^*e_{\mathbf{v}}|||$ approach zero as h and l tend to zero. This completes the proof of the theorem.

It must be observed that Theorem 7.3 assures convergence of not only e_u and $e_{\mathbf{v}}$ but also of Te_u and $T^*e_{\mathbf{v}}$. This indicates that Te_u and $T^*e_{\mathbf{v}}$ converge at the same rate as TE_u^* and $TE_{\mathbf{v}}^*$, respectively. Intuitively, the errors e_u and $e_{\mathbf{v}}$ may approach zero at the rate of E_u^* and $E_{\mathbf{v}}^*$, respectively. In that case, faster convergence of e_u and $e_{\mathbf{v}}$ is established by Theorem 7.3.

8. Some error estimates. Consider the case in which

$$\mathfrak{U} = W_2^{\,k}(\Omega), \qquad \mathfrak{V} = W_2^{\,q}(\Omega) \tag{8.1}$$

where $W_2^{\,k}(\Omega)$ is the Hilbert space of order k, and let

$\mathcal{P}_s$ = the space of polynomial of degree s on $\Omega \subset E^n$; h, ρ = finite-element mesh parameters (see [28]) of approximations $U^*(\mathbf{x}) \in \mathfrak{M}_G{}^h$ of $u^*(\mathbf{x})$; l, a = finite-element mesh parameters of approximations $\mathbf{V}^*(\mathbf{x}) \in \mathfrak{N}_H{}^l$ of $\mathbf{v}^*(\mathbf{x})$. (8.2)

Let the inner products in $\mathfrak{U} = W_2{}^k(\Omega)$ and $\mathfrak{V} = W_2{}^q(\Omega)$ be defined by

$$\{u_1, u_2\} = (u_1, u_2)_{W_2{}^k(\Omega)} = \int_\Omega \sum_{|\alpha| \le k} D^\alpha u_1 D^\alpha u_2 \, d\mathbf{x}, \tag{8.3}$$

$$[\mathbf{v}_1, \mathbf{v}_2] = (\mathbf{v}_1, \mathbf{v}_2)_{W_2{}^q(\Omega)} = \int_\Omega \sum_{|\alpha \le q} D^\alpha \mathbf{v}_1 \cdot D^\alpha \mathbf{v}_2 \, \mathbf{bx}. \tag{8.4}$$

Assume that

$$\mathfrak{M}_G{}^h = \mathcal{P}_s(\Omega), \qquad \mathfrak{N}_H{}^l = \mathcal{P}_r(\Omega), \qquad s < k, \ r < q. \tag{8.5}$$

In most of the applications the operators T and T^* are differential operators of the form

$$T^{(m)} = \sum_{|\alpha| \le m} a_\alpha(\mathbf{x}) D^\alpha, \qquad T_{(m)}{}^* = \sum_{|\alpha| \le m} (-1)^{|\alpha|} D^\alpha(a_\alpha(\mathbf{x})) \tag{8.6}$$

where $m < s, r$.

Now assume that there exist interpolants $\tilde{U} = \Pi_h u$ and $\tilde{V} = P_l \mathbf{v}$

$$||\Pi_h u - u||_{W_p{}^m(\Omega)} \le \tilde{C} \, |u|_{W_p{}^{k+1}(l)} (h^{k+1}/\rho^m), \tag{8.7}$$

$$||P_l \mathbf{v} - \mathbf{v}||_{W_p{}^m(\Omega)} \le \tilde{D} \, |\mathbf{v}|_{W_p{}^{r+1}(\Omega)} (l^{r+1}/a^m), \tag{8.8}$$

where Π_h and P_l are projection operators such that $\Pi_h u = u$ and $P_l \mathbf{v} = \mathbf{v}$ for all $u \in \mathcal{P}_s$ and $\mathbf{v} \in \mathcal{P}_r$, and $\tilde{C}$ and $\tilde{D}$ are constants independent of the mesh parameters. Interpolation formulas of the type (8.7) and (8.8) are derived by Ciarlet and Raviant [28, 29]. If the coefficients a satisfy the condition

$$\sum_{|\alpha| \le m} ||a_\alpha D^\alpha u||_{\mathbf{L}_2(\Omega)} \le \bar{C} \sum_{|\alpha| \le m} ||D^\alpha u||_{\mathbf{L}_2(\Omega)}, \tag{8.9}$$

then

$$||T^{(m)} u||_{L_2(\Omega)} \le \hat{C} \, ||u||_{W_2{}^m}, \qquad m < k \tag{8.10}$$

for every $u \in W_2{}^k(\Omega)$. Moreover, if the estimate (8.7) holds, then

$$||T^{(m)} E_u||_{L_2(\Omega)} \le \bar{C} \frac{h^{s+1}}{\rho^m} |u|_{W_2{}^{s+1}(\Omega)} \tag{8.11}$$

holds for any $u \in W_2{}^{s+1}(\Omega)$, $m < s + 1$.

Similar results can be obtained for T^*:

$$||T_{(m)}{}^* E_\mathbf{v}||_{W_2{}^0(\Omega)} \le \bar{D} \frac{l^{r+1}}{a^m} |\mathbf{v}|_{W_2{}^{r+1}(\Omega)}. \tag{8.12}$$

It can be shown, in a similar way, that

$$||T^{(m)} E_u||_{W_2{}^q(\Omega)} \le C^* \frac{h^{s+1}}{\rho^{q+m}} |u|_{W_2{}^{s+1}(\Omega)}, \tag{8.13}$$

$$||T_m{}^* E_\mathbf{v}||_{W_2{}^k(\Omega)} \le D^* \frac{l^{r+1}}{a^{k+m}} |\mathbf{v}|_{W_2{}^{r+1}(\Omega)}. \tag{8.14}$$

Now error estimates for mixed finite-element solutions can be derived using (8.11)–(8.14).

THEOREM 8.1. Consider a mixed finite-element approximation based on polynomial bases for which the relations (8.7) and (8.8) hold for the spaces defined in (8.5). Then the following error estimates hold, if the conditions of Theorem 7.1 are satisfied.

Case $k = 0$.

$$||e_u||_{W_2^k(\Omega)} \le \left(\tilde{C}\rho^{-k} + \frac{C^*}{\mu_2}\rho^{-m-q}\right)h^{s+1}|u^*|_{W_2^{s+1}(\Omega)} + \frac{1}{\mu_2}\left(\tilde{D}a^{-q} + \frac{D^*}{\gamma_2}a^{-m-k}\right)|\mathbf{v}^*|_{W_2^{r+1}(\Omega)}, \tag{8.15}$$

$$||e_\mathbf{v}||_{W_2^{q+1}(\Omega)} \le \left(\tilde{D}a^{-q} + \frac{D^*}{\gamma_2}a^{-m-k}\right)l^{r+1}|\mathbf{v}^*|_{W_2^{r+1}(\Omega)}. \tag{8.16}$$

Case $k > 0$.

$$||e_u||_{W_2^k(\Omega)} \le \left(1 + \frac{\alpha}{\gamma_1}\right)\tilde{C}\frac{h^{s+1}}{\rho^k}|u^*|_{W_2^{s+1}(\Omega)} + \frac{D^*}{\gamma_1}\frac{l^{r+1}}{a^{k+m}}|\mathbf{v}^*|_{W_2^{r+1}(\Omega)}, \tag{8.17}$$

$$||e_\mathbf{v}||_{W_2^q(\Omega)} \le \left(1 + \frac{\beta}{\mu_1}\right)\tilde{D}\frac{l^{r+1}}{a^q}|\mathbf{v}^*|_{W_2^{r+1}(\Omega)} + \frac{k}{\mu_1}C^*\frac{h^{s+1}}{\rho^{q+m}}|u^*|_{W_2^{s+1}(\Omega)}. \tag{8.18}$$

Proof: The proof is straightforward. These estimates can be derived directly from (7.3), (7.4), (7.6) and (7.7) with interpolation errors (8.7), (8.8), (8.13), and (8.14).

It is clear from above estimates that the errors depend on both sets of mesh parameters.

COROLLARY 8.1. Let the conditions of Theorem 8.1 hold, and let $\rho = \nu_1 h$ and $a = \nu_2 l$. Then

Case $k = 0$.

$$||e_u||_{W_2^k(\Omega)} \le \nu_1\left(\tilde{C}h^{-k} + \frac{C^*\nu_1}{\mu_2}h^{-m-q}\right)h^{s+1}|u^*|_{W_2^{s+1}(\Omega)} + \frac{\nu_2}{\mu_2}\left(\tilde{D}l^{-q} + \frac{D^*}{\gamma_2}\nu_2 l^{-m-k}\right)l^{r+1}|\mathbf{v}^*|_{W_2^{r+1}(\Omega)}, \tag{8.19}$$

$$||e_\mathbf{v}||_{W_2^q(\Omega)} \le \nu_2\left(\tilde{D}l^{-q} + \frac{D^*\nu_2}{\gamma_2}l^{-m-k}\right)l^{r+1}|\mathbf{v}^*|_{W_2^{r+1}(\Omega)}, \tag{8.20}$$

rate of convergence for u

$$= \min\,(s + 1 - q - m, r + 1 - k - m, s + 1 - k, r + 1 - q), \tag{8.21}$$

rate of convergence for $\mathbf{v} = \min\,(r + 1 - m - k, r + 1 - q)$.

Case $k > 0$.

$$||e_u||_{W_2^k(\Omega)} \le \left(1 + \frac{\alpha}{\gamma_1}\right)\tilde{C}\nu_1 h^{s+1-k}|u^*|_{W_2^{s+1}(\Omega)} + \frac{D^*\nu_2}{\gamma_1}l^{r+1-k-m}|\mathbf{v}^*|_{W_2^{r+1}(\Omega)}, \tag{8.22}$$

$$||e_\mathbf{v}||_{W_2^q(\Omega)} \le \left(1 + \frac{\beta}{\mu_1}\right)\tilde{D}\nu_2 l^{r+1-q}|\mathbf{v}^*|_{W_2^{r+1}(\Omega)} + \frac{kC^*\nu_1}{\mu_1}h^{s+1-q-m}|u^*|_{W_2^{r+1}(\Omega)}, \tag{8.23}$$

$$\begin{aligned}\text{rate of convergence for } u &= \min\,(s + 1 - k, r + 1 - k - m),\\ \text{rate of convergence for } \mathbf{v} &= \min\,(r + 1 - q, s + 1 - q - m).\end{aligned} \tag{8.24}$$

THEOREM 8.2. Let the conditions of Theorem 7.3 hold. Then the following error estimates hold:

Case $k = 0$.

$$||Te_u||_{W_2^q(\Omega)} \leq \left(\frac{(D_1 + 1)}{C_1}\right)^{1/2}\left[\tilde{D}\,\frac{l^{r+1}}{a^q}\,|\mathbf{v}^*|_{W_2^{r+1}(\Omega)} + \frac{C^* h^{s+1}}{\rho^{q+m}}\,|u^*|_{W_2^{s+1}(\Omega)}\right], \tag{8.25}$$

$$||T^*e_{\mathbf{v}}||_{W_2^k(\Omega)} \leq 2\tilde{D}\,\frac{l^{r+1}}{a^q}\,|\mathbf{v}^*|_{W_2^{r+1}(\Omega)} + ||Te_u||_{W_2^q(\Omega)}\,. \tag{8.26}$$

Case $k > 0$.

$$\begin{aligned}||Te_u||_{W_2^q(\Omega)} \leq \left(\frac{2D_2}{C_2}\right)^{1/2}\Big[(k\tilde{C}\rho^{-k} + C^*\rho^{-q-m})h^{s+1}\,|u^*|_{W_2^{s+1}(\Omega)}\\ + \tilde{D}\,\frac{l^{r+1}}{a^q}\,|\mathbf{v}^*|_{W_2^{r+1}(\Omega)}\Big],\end{aligned} \tag{8.27}$$

$$\begin{aligned}||T^*e_{\mathbf{v}}||_{W_2^k(\Omega)} \leq \left(\frac{2D_3}{C_3}\right)^{1/2}[(k\tilde{D}a^{-q} + D^*a^{-k-m})l^{r+1}\,|\mathbf{v}^*|_{W_2^{r+1}(\Omega)}\\ + k\tilde{C}\rho^{-k}h^{s+1}\,|u^*|_{W_2^{s+1}(\Omega)}].\end{aligned} \tag{8.28}$$

The rates of convergence from (8.25) and (8.26) for Tu and T^*v are, respectively (for $k > 0$),

$$\begin{aligned}\sigma &= \min\,(r + 1 - q, s + 1 - q - m, s + 1 - k),\\ \epsilon &= \min\,(r + 1 - q, r + 1 - k - m, s + 1 - k).\end{aligned} \tag{8.29}$$

The convergence rates for Te_u and $T^*e_{\mathbf{v}}$ seem to be of the same order as compared to those of e_u and $e_{\mathbf{v}}$ in (8.24). Thus, the error estimates obtained from Theorem 7.3 are sharper.

9. Numerical results. There exists ample literature on numerical analysis of mixed finite-element models. For example, Herrmann [1, 30] and Hellan [2] have developed mixed plate bending elements, and later Bäcklund [6] (see also Conner [31] and Visser [4]) used these elements in the analysis of elastic and elastoplastic plates in bending. Dunham and Pister [5] employed the Hellinger–Reissner (mixed) variational principle to construct mixed finite-element models of linear elastic problems. It was observed that the mixed models are particularly effective in capturing steep stress or displacement gradients that can occur near singularities in boundary-value problems. In recent times there has appeared a vast literature on the closely related idea of the hybrid finite-element method [8, 9, 10] applied to stress concentration problems. In all these works, numerical examples have been presented with extremely good results; however, these do not contain any information on the behavior of the error (in energy).

The primary purpose of the examples presented here is to demonstrate, numerically, that the mixed models yield higher accuracies for certain quantities (e.g., stresses), and to give precise rates of convergence for the mixed finite-element solutions. In particular, the error in energy norm (or L_2-norm) is computed in each example to find the rates of convergence. For simplicity, only one-dimensional, second- and fourth-order equations are considered here.

1. *Second-order differential equation.* Consider the boundary-value problem

$$(-d^2u/dx^2) + u + x = 0, \qquad 0 < x < 1 \tag{9.1}$$

subjected to the boundary conditions $u(0) = u'(1) = 0$. Clearly, (9.1) is of the general form (2.1) with T and T^* given by

$$T = -T^* = d/dx, \qquad M = N = 1, \qquad g_1 = g_2 = 0 \tag{9.2}$$

and the inner products $\{\cdot\,,\,\cdot\}$ and $[\cdot\,,\,\cdot]$ are defined

$$\{u_1\,,u_2\} = \int_0^1 u_1u_2\,dx; \qquad [v_1\,,v_2] = \int_0^1 v_1v_2\,dx. \tag{9.3}$$

The following sets of basis functions are selected for the problem at hand:

$$\varphi_1(x) = 1 - (x/h), \qquad 0 \le x \le h$$

$$\left.\begin{aligned}\varphi_\alpha(x) &= (x/h) - (\alpha - 2), \qquad && (\alpha - 2)h \le x \le (\alpha - 1)h \\ &= \alpha - (x/h), && (\alpha - 1)h \le x \le \alpha h\end{aligned}\right\} \quad (\alpha = 2, \cdots, N_e - 1),$$

$$\varphi_N(x) = \frac{x}{h} - (N_e - 2), \qquad (N_e - 2)h \le x \le (N_e - 1)h, \tag{9.4}$$

and similar expressions for ω^Δ, $\Delta = 1, \cdots, M$, are taken with h replaced by l. Here N_e is the number of nodes (or $(N_e - 1)$ is the number of elements) in approximation by φs and M_e is the the number of nodes in approximation by ωs; i.e., $h = 1/(N_e - 1)$ and $l = 1/(M_e - 1)$. Let the ratio of the meshes be denoted by $\nu = h/l$. The fundamental matrices $G_{\alpha\beta}$ and $H^{\Delta\Gamma}$ are given by

$$[G_{\alpha\beta}] = h[G], \qquad [H^{\Delta\Gamma}] = l[G] \tag{9.5}$$

where

$$[G] = \frac{1}{6}\begin{bmatrix} 2 & 1 & & & & & \\ 1 & 4 & 1 & & & \mathbf{0} & \\ & 1 & 4 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 1 & 4 & 1 & \\ & \mathbf{0} & & & 1 & 4 & 1 \\ & & & & & 1 & 2 \end{bmatrix}.$$

Also, f_α and g^Δ are given by

$$\{f_\beta\} = \frac{h^2}{6}\begin{Bmatrix} 1 \\ 6 \\ 12 \\ \vdots \\ 6(\beta - 1) \\ \vdots \\ 6(N_e - 2) \\ 3N_e - 4 \end{Bmatrix}, \qquad g^\Delta = \frac{l}{2}\begin{Bmatrix} 1 \\ 2 \\ 2 \\ \vdots \\ 2 \\ 2 \\ 1 \end{Bmatrix}. \tag{9.6}$$

The matrices $(T_\alpha{}^{\cdot\Delta} - M_\alpha{}^{\cdot\Delta})$ and $(T_\alpha{}^{*\Delta} + N_\alpha{}^{\cdot\Delta})$ of (3.8) and (3.10) are given in Table I. Note that $(T_\alpha{}^{\cdot\Delta} - M_\alpha{}^{\cdot\Delta}) = (T_\alpha{}^{*\Delta} + N_\alpha{}^{\cdot\Delta})$. Here it is assumed that ν is an integer. When ν is not an integer, it is not possible to compute the matrices $(T_\alpha{}^{\cdot\Delta} - M_\alpha{}^{\cdot\Delta})$ and $(T_\alpha{}^{\Delta*} + N_\alpha{}^{\cdot\Delta})$ for arbitrary N_e and M_e. Mixed finite-element solutions u^* and v^*, for different values of ν, are obtained and compared with the exact solutions (see Figs.

TABLE I. The matrix $(T_\alpha{}^{\cdot\Delta} - M_\alpha{}^{\cdot\Delta})$ for any integer value of ν.

$$\underset{(M_e \times N_e)}{(T_\alpha{}^{\cdot\Delta} - M_\alpha{}^{\cdot\Delta})}$$

$$\begin{array}{ccccccccccccccccccc}
 & & & \alpha=\nu & \alpha=\nu+1 & & & & 2\nu & 2\nu+1 & & & & 3\nu & 3\nu+1 & & & & N_e \\
 & & & \vdots & \vdots & & & & \vdots & \vdots & & & & \vdots & \vdots & & & &
\end{array}$$

$$= \left[\begin{array}{ccccccccccccccccccc}
\frac{h}{2l} & \frac{h}{l} & \cdots & \frac{h}{l} & \frac{h}{2l} & 0 & \cdots & & 0 & \cdot & & \cdots & & 0 & 0 & & \cdot\cdots & & 0 \\
\frac{-h}{2l} & \frac{-h}{l} & \cdots & \frac{-h}{l} & 0 & \frac{h}{l} & \cdots & & \frac{h}{l} & \frac{h}{2l} & 0 & \cdots & & 0 & 0 & & \cdot\cdots & & 0 \\
0 & 0 & \cdots & 0 & \frac{-h}{2l} & \frac{-h}{l} & \cdots & & \frac{-h}{l} & 0 & \frac{}{l} & \cdots & & \frac{h}{l} & \frac{h}{2l} & 0 & \cdots & & 0 \\
 & & & \vdots & & \vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots & & \vdots & & & \\
 & & & & & \cdot & \cdots & \frac{-h}{l} & 0 & \frac{h}{l} & \cdot & \cdots & \frac{h}{l} & \frac{h}{2l} & 0 & & \cdot\cdots & & 0 \\
 & & & & & \vdots & \cdots & 0 & \frac{-h}{2l} & \frac{-h}{l} & \cdot & \cdots & \frac{-h}{l} & 0 & \frac{h}{l} & & \cdot\cdots & \frac{h}{l} & \frac{h}{2l} \\
 & & & & & \cdot & \cdots & 0 & 0 & 0 & \cdot & \cdots & 0 & \frac{-h}{2l} & \frac{-h}{l} & & \cdot\cdots & \frac{-h}{l} & \frac{h}{2l}
\end{array}\right]$$

9.1 and 9.2). It must be noticed that the mixed solutions are less stable and more inaccurate as the mesh ratio $\nu = l/h$ increases. This can be explained in view of the stability conditions (6.1)–(6.4). A close examination of the matrix in Table I reveals that as

ν increases the sum of the off-diagonal terms increases, and consequently the matrix $K_{\alpha\beta}$ of (3.13) becomes ill-conditioned. Moreover, the error on the boundary is more sensitive to the mesh ratio, with the error increasing with the mesh ratio. For $\nu = 1$, the solutions U^* and V^* are plotted against the primal and exact solutions in Fig. 10.3 and against the dual and exact solutions in Fig. 9.4. For $\nu = 1$ the matrix in Table I takes the simple form

$$(T_\alpha^{*\Delta} + N_\alpha^{\cdot\Delta}) = (T_\alpha^{\cdot\Delta} - M_\alpha^{\cdot\Delta}) = \frac{1}{2}\begin{bmatrix} 1 & -1 & 0 & & & & \\ & \ddots & \ddots & & & & \\ 1 & 0 & -1 & & 0 & & \\ & \ddots & \ddots & \ddots & & & \\ 0 & 1 & 0 & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & & \\ & 0 & & \ddots & 0 & -1 & \\ & & & \ddots & \ddots & \ddots & \\ & & & & 1 & 0 & -1 \\ & & & & & \ddots & \ddots \\ & & & & 0 & 1 & 1 \end{bmatrix}.$$

The broken line in Fig. 9.3 is the solution $V = dU^*/dx$ obtained by differentiating U^*; the broken curve in Fig. 9.4 is the solution $U = \int_0^1 V^*\,dx$ obtained by integrating V^*. Thus, the solution V, in the case of the primal problem, is discontinuous and the solution U, in the case of the dual problem, is in error.

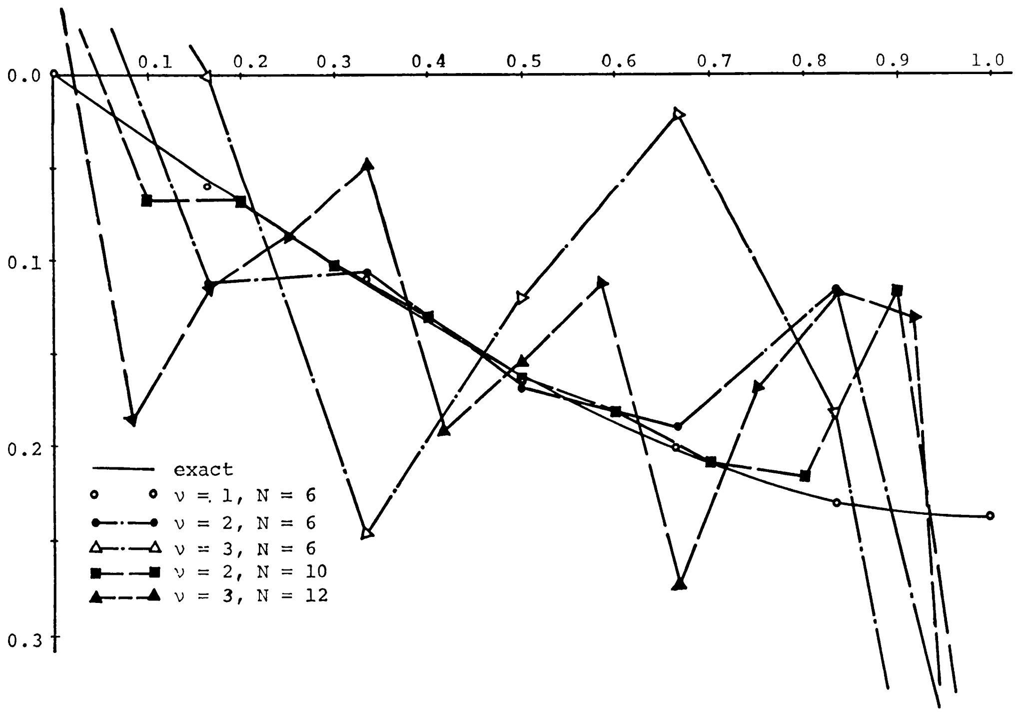

FIG. 9.1. Mixed finite-element solutions u for various values of the mesh ratio ν.

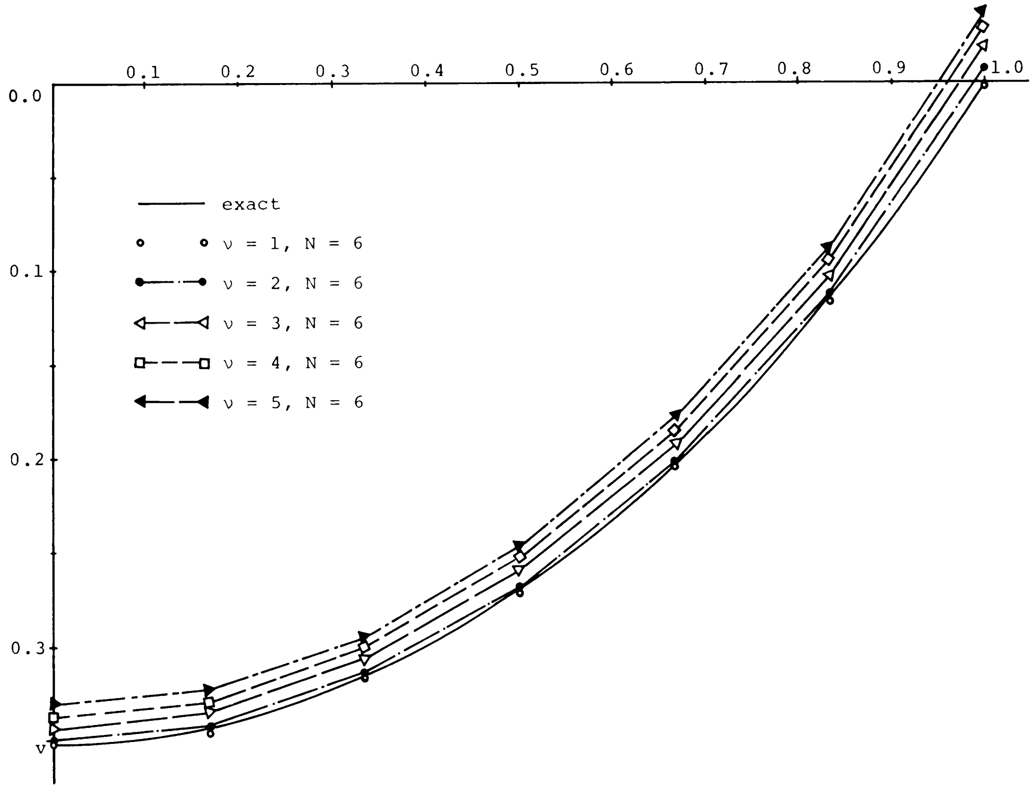

FIG. 9.2. Mixed finite-element solutions v for various values of the mesh ratios ν.

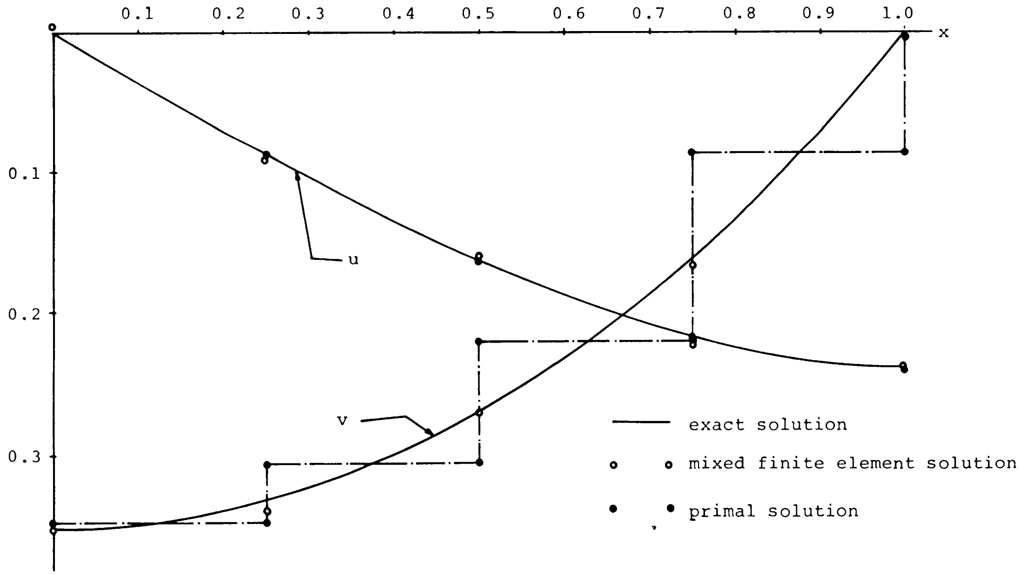

FIG. 9.3. Comparison of mixed and primal solutions with exact solutions.

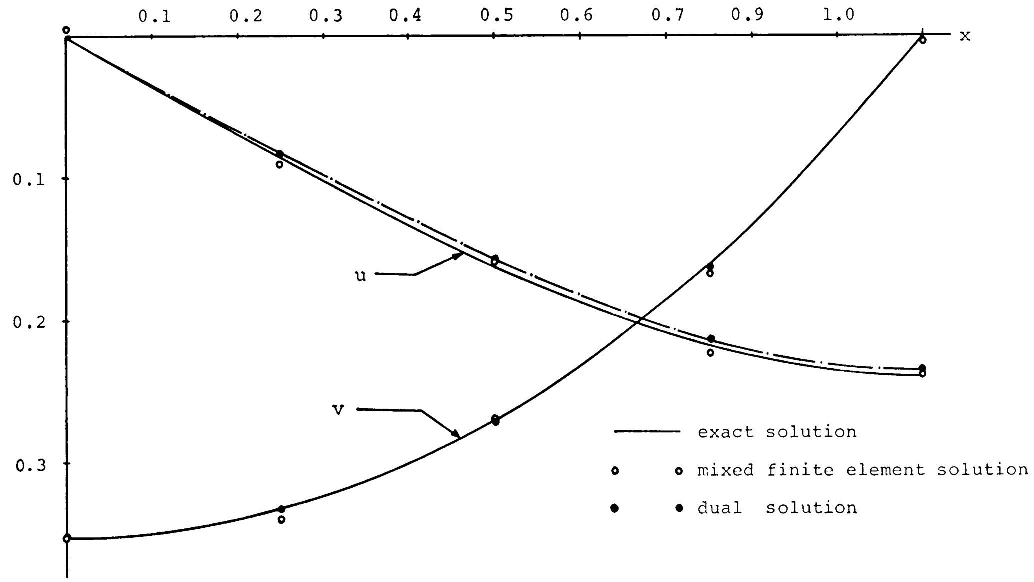

FIG. 9.4. Comparison of mixed and dual solutions with exact solutions.

The error in energy is computed for $\nu = 1$ case and plotted against the mesh size h. In this case, where same basis (or trial) functions (linear polynomials) are employed to approximate u and v, the rates of convergence for U^* as well as for V^* is 2. In Fig. 9.1, the value of k is 1. The same problem is solved with $k = 0$, and same rates of convergence are obtained in this case also (with the same basic functions).

2. *Fourth-order differential equation.* Consider the fourth-order equation

$$(d^4u/dx^4) + x^2 = 0, \qquad 0 \le x \le 1, \tag{9.7}$$

$$u(0) = u(1) = 0; \qquad (d^2u/dx^2)(0) = (d^2u/dx^2)(1) = 0.$$

In this case the basis functions are cubic polynomials:

$$\varphi_1{}^0 = 1 - 3\left(\frac{x}{h}\right)^2 + 2\left(\frac{x}{h}\right)^3, \qquad 0 \le x \le h,$$

$$\varphi_\alpha{}^0 = 3\left[\frac{x}{h} - (\alpha - 2)\right]^2 - 2\left[\frac{x}{h}(\alpha - 2)\right]^3, \qquad (\alpha - 2)h \le x \le (\alpha - 1)h$$

$$= 1 - 3\left[\frac{x}{h} - (\alpha - 1)\right]^2 + 2\left[\frac{x}{h} - (\alpha - 1)\right]^3, \qquad (\alpha - 1)h \le x \le \alpha h$$

$$(\alpha = 2, 3, \cdots, N_e - 1)$$

$$\varphi_{N_e}{}^0 = 3\left[\frac{x}{h} - (N_e - 2)\right]^2 - 2\left[\frac{x}{h} - (N_e - 2)\right]^3, \qquad (N_e - 2)h \le x \le (N_e - 1)h,$$

$$\varphi_1{}^1 = h\left[\frac{x}{h} - 2\left(\frac{x}{h}\right)^2 + \left(\frac{x}{h}\right)^3\right], \qquad 0 \le x \le h \tag{9.8a}$$

$$\varphi_\alpha{}^1 = h\left[\left\{\frac{x}{h} - (\alpha - 1)\right\} - 2\left\{\frac{x}{h} - (\alpha - 1)\right\}^2 + \left\{\frac{x}{h} - (\alpha - 1)\right\}^3\right], \quad (\alpha - 1)h \le x \le \alpha h$$

$$= h\left[\left\{\frac{x}{h} - (\alpha - 2)\right\}^3 - \left\{\frac{x}{h} - (\alpha - 2)\right\}^2\right], \quad (\alpha - 2)h \le x \le (\alpha - 1)h$$

$$\varphi_N{}^1 = h\left[\left\{\frac{x}{h} - (N_e - 2)\right\}^3 - \left\{\frac{x}{h} - (N_e - 2)^2\right\}\right],$$

$$(N_e - 2)h \leq x \leq (N_e - 1)h. \tag{9.8b}$$

The fundamental matrices $G_{\alpha\beta}$ and $H^{\Delta\Gamma}$ in this case (for $\nu = 1$) are given by

$$[G_{\alpha\beta}] = \frac{h}{420} \cdot \left[\begin{array}{cc}
\begin{array}{cccccc}
156 & 54 & & & & \\
54 & 312 & 54 & & & \\
 & 54 & \ddots & \ddots & & \\
 & & \ddots & \ddots & \ddots & \\
 & & & 54 & 312 & 54 \\
 & & & & 54 & 156
\end{array} &
\begin{array}{cccccc}
22h & -13h & & & & \\
13h & 0 & -13h & & & \\
 & 13h & \ddots & \ddots & & \\
 & & \ddots & \ddots & \ddots & \\
 & & & 13h & 0 & -13h \\
 & & & & 13h & -22h
\end{array} \\
\begin{array}{cccccc}
22h & 13h & & & & \\
-13h & 0 & 13h & & & \\
 & -13h & \ddots & \ddots & & \\
 & & \ddots & \ddots & \ddots & \\
 & & & -13h & 0 & 13h \\
 & & & & -13h & -22h
\end{array} &
\begin{array}{cccccc}
4h^2 & -3h^2 & & & & \\
-3h^2 & 8h^2 & -3h^2 & & & \\
 & -3h^2 & \ddots & \ddots & & \\
 & & \ddots & \ddots & \ddots & \\
 & & & -3h^2 & 8h^2 & -3h^2 \\
 & & & & -3h^2 & 4h^2
\end{array}
\end{array}\right]$$

$$= [H^{\Delta\Gamma}]. \tag{9.9}$$

The matrices $(T_\alpha{}^{*\Delta} + N_\alpha{}^{\cdot\Delta})$, and f_α are computed to be

$$(T_\alpha{}^{*\Delta} + N_\alpha{}^{\cdot\Delta}) = \frac{1}{30h^2} \cdot \left[\begin{array}{cc}
\begin{array}{cccccc}
36h & -36h & & & & \\
-36h & 72h & -36h & & & \\
 & -36h & \ddots & \ddots & & \\
 & & \ddots & \ddots & \ddots & \\
 & & & -36h & 72h & -36h \\
 & & & & -36h & 36h
\end{array} &
\begin{array}{cccccc}
33h^2 & 3h^2 & & & & \\
-3h^2 & 0 & 3h^2 & & & \\
 & -3h^2 & \ddots & 3h^2 & & \\
 & & \ddots & \ddots & \ddots & \\
 & & & -3h^2 & 0 & 3h^2 \\
 & & & & -3h^2 & -33h^2
\end{array} \\
\begin{array}{cccccc}
33h^2 & -3h^2 & & & & \\
3h^2 & 0 & -3h^2 & & & \\
 & 3h^2 & \ddots & -3h^2 & & \\
 & & \ddots & \ddots & \ddots & \\
 & & & 3h^2 & 0 & -3h^2 \\
 & & & & 3h^2 & -33h^2
\end{array} &
\begin{array}{cccccc}
4h^3 & -h^3 & & & & \\
-h^3 & 8h^3 & -h^3 & & & \\
 & -h^3 & \ddots & -h^3 & & \\
 & & \ddots & \ddots & \ddots & \\
 & & & -h^3 & 8h^3 & -h^3 \\
 & & & & -h^3 & 4h^3
\end{array}
\end{array}\right] \tag{9.10}$$

and

$$f_\beta = \begin{pmatrix} f^0{}_\beta \\ f^1{}_\beta \end{pmatrix} \tag{9.11}$$

where

$$\begin{aligned} f^0{}_\beta &\equiv \frac{h^3}{15}, & \beta &= 1, \\ &= \frac{h^3}{30}(30\beta^2 - 60\beta + 34), & \beta &= 2, 3, \cdots, N_e - 1, \\ &= \frac{h^3}{30}(15N_e{}^2 - 39N_e + 26), & \beta &= N_e, \\ f^1{}_\beta &= \frac{h^4}{60}, & \beta &= 1, \\ &= \frac{h^4}{15}(-15\beta^2 + 62\beta - 62), & \beta &= 2, 3, \cdots, N_e - 1, \\ &= \frac{h^4}{60}(15N_e{}^2 - 39N_e + 26), & \beta &= N_e. \end{aligned}$$

The mixed finite-element solutions U^* and V^* are plotted against the exact solutions in Fig. 9.5. The rates of convergence in this case, where the same basis (cubic)

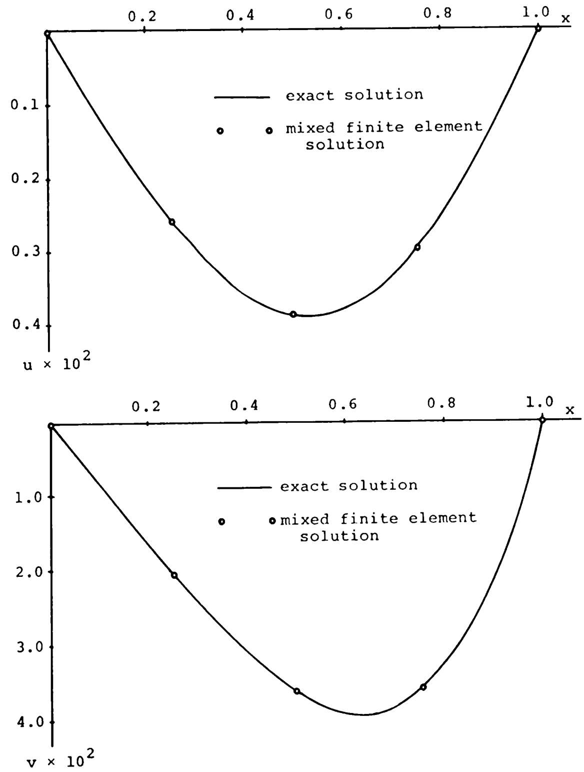

FIG. 9.5. Comparison of mixed finite-element solutions with exact solutions.

functions are employed, are 4. It is also noted that the first derivatives of U^* and V^* are approximated very closely to the exact derivatives.

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References

[1] L. R. Herrmann, *A bending analysis of plates, Proceedings of the Conference on Matrix Methods in Structural Mechanics*, Wright-Patterson AFB, Ohio, AFFDL-TR66-80, pp. 577–604, 1966

[2] K. Hellan, *Analysis of elastic plates in flexure by a simplified finite element method,* Acta Polytechnica Scandinavica, Civil Engineering Series No. 46, Trondheim, 1967

[3] W. Prager, *Variational principles for elastic plates with relaxed continuity requirements,* Int. J. Solids Structures **4**, 837–844 (1968)

[4] W. Visser, *A refined mixed type plate bending element,* AIAA J. **7**, 1801–1803 (1969)

[5] R. S. Dunham and K. S. Pister, *A finite element application of the Hellinger-Resissner variational theorem, in Proceedings of the Conference on Matrix Methods in Structural Mechanics,* Wright-Patterson AFB, Ohio, AFFDL-TR68-150, pp. 471–487, 1968

[6] J. Bäcklund, *Mixed finite element analysis of plates in bending,* Chalmers Tekniska Hogskola Institutionen for byggnadsstatik Publication 71.4, Goteburg, 1972

[7] W. Wünderlich, *Discretisation of structural problems by a generalized variational approach,* Papers presented at International Association for Shell Structures, Pacific Symposium on Hydrodynamically Loaded Shells-Part I, Honolulu, Hawaii, Oct. 10–15, 1971

[8] T. H. H. Pain, *Formulations of finite element methods for solid continua,* in *Recent advances in matrix methods of structural analysis and Design,* R. H. Gallagher, Y. Yamada, and J. T. Oden (eds.), University of Alabama Press, University, 1971

[9] P. Tong, *New Displacement hybrid finite element model for solid continua,* Int. J. Numer. Meth. Eng. **2**, 73–83 (1970)

[10] T. H. H. Pian and P. Tong, *Basis of finite element methods for solid Continua,* Int. J. Numer. Meth. Eng. **1**, 3–28 (1969)

[11] J. N. Reddy, *Accuracy and convergence of mixed finite-element approximations of thin bars, membranes, and plates on elastic foundations, in Proceedings of the Graduate Research Conference in Applied Mechanics,* Las Cruces, New Mexico, paper 1B5, March 1973

[12] C. Johnson, *On the convergence of a mixed finite-element method for plate bending problems,* Numer. Math. **21**, 43–62 (1973)

[13] F. Kikuchi and Y. Ando, *On the convergence of a mixed finite element scheme for plate bending,* Nucl. Eng. Design 24, 357–373 (1973)

[14] J. T. Oden, *Some contributions to the mathematical theory of mixed. finite element approximations,* in Tokyo Seminar on Finite Elements, Tokyo, Japan, The University of Tokyo press, 1973

[15] J. T. Oden and J. N. Reddy, *On dual-complementary variational principles in mathematical physics,* Int. J. Eng. Sci. **12**, 1–29 (1974)

[16] J. N. Reddy, and J. T. Oden, *Convergence of mixed finite element approximations of a class of linear Boundary-Value problems,* Struct. Mech. **2**, 83–108 (1973)

[17] J. T. Oden, *Finite elements of nonlinear continua,* McGraw-Hill, New York, 1972

[18] S. W. Key, *A convergence investigation of the direct stiffness method,* Doctoral Dissertation, University of Washington, Seattle, 1966

[19] R. W. McLay, *Completeness and convergence properties of finite-element displacement functions—a general treatment,* AIAA 5th Aerospace Science Meeting AIAA Paper 67–143, New York, 1967

[20] M. W. Johnson, and R. W. McLay, *Convergence of the finite element method in the theory of elasticity,* J. Appl. Mech. E**35**, 274–278 (1968)

[21] I. Babuska and A. K. Aziz, *Survey lectures on the mathematical foundations of the finite element method, in the mathematical foundation of the finite element method with applications to partial differential equations,* A. K. Aziz (ed.), Academic Press, New York, pp. 3–345, 1972

[22] J. P. Aubin, *Approximation of elliptic boundary-value problems*, Wiley-Interscience, New York, 1972
[23] M. H. Schultz, *Spline analysis*, Prentice-Hall, 1973
[24] G. Strang, and G. Fix, *An analysis of the finite element method*, Prentice-Hall, New York, 1973
[25] E. Isaacson, and H. B. Keller, *Analysis of numerical methods*, John Wiley, New York, 1966
[26] A. W. Naylor, and G. R. Sell, *Linear operator theory in engineering and science*, Holt, Rinehart and Winston, New York, 1971
[27] J. N. Reddy, *A mathematical theory of complementary-dual variational principles and mixed finite-element approximations of linear boundary-value problems in continuum mechanics*, Doctoral Dissertation, University of Alabama in Huntsville, May, 1974
[28] P. G. Ciarlet, and P. A. Raviart, *General Lagrange and Hermite Interpolation in R^n with applications to finite-element methods*, Arch. Rat. Mech. Anal. **46**, 177–199 (1972)
[29] P. G. Ciarlet, and P. A. Raviart, *Interpolation theory over curved elements, with applications to finite element methods*, Computes Meth. Appl. Mech. Eng. **1**, 217–249 (1972)
[30] L. R. Herrmann, "Finite-Element Bending Analysis for Plates," J. Eng. Mech. Div. ASCE **93**, 13–26 (1967)
[31] J. J. Connor, *Mixed models for plates, in Proceedings of a Seminar on Finite-Element Techniques in Structural Mechanics*, University of Southampton, pp. 125–151, 1970