

**THE TRANSVERSE VIBRATION OF A ROTATING BEAM WITH TIP MASS:
THE METHOD OF INTEGRAL EQUATIONS***

By

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Abstract. An integral equation method is used to obtain improvable lower bounds for the second eigenvalue of the second-order "reduced" problem obtained from the problem described in the title by singular perturbation methods. These lower bounds are compared with results obtained directly by invariant embedding. The computational aspects of the integral equation method are stressed. The method is shown to be quite general and can be applied to a variety of boundary-value problems including those in which the eigenvalue parameter appears in the boundary conditions as well as in the differential operator.

I. Introduction. The problem of determining the natural frequencies of the transverse vibration of a uniform rotating beam carrying a tip mass reduces to the study of the following eigenvalue problem:

$$u^{IV} - \frac{1}{2}\alpha^2[u^I(1 - x^2 + 2\gamma^2)]^I = \beta^2 u,$$

$$u(0) = u^I(0) = u^{II}(1) = 0, \quad u^{III}(1) = \alpha^2\gamma^2 u^I(1) - \gamma^2\beta^2 u(1),$$

where one wishes to determine the dependence of the eigenvalues β^2 on the parameters α^2 and γ^2 . This problem possesses two major difficulties: (1) the differential operator is quite complex so that solution in closed form is very difficult [1]; and (2) the eigenvalue β^2 appears in the boundary conditions as well as in the operator. The latter difficulty is responsible for the principal novelty of the problem. It should be noted that the problem is self-adjoint and has previously been studied by variational methods [1, 4].

The present paper is primarily concerned with illustrating the use of an integral equation method previously described in [2] to obtain lower bounds for the eigenvalues

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β^2 and with comparing the computational aspects of this method with those of other, less general methods, which can also be used to determine the eigenvalues of problems with eigenvalue dependence in the boundary conditions. The historical background of the problem is given in Sec. 2; the integral equation method is described in Sec. 3; and numerical results for the "reduced" problem are given in Sec. 4.

2. Historical background. The problem of determining the influence of a tip mass on the natural frequencies of transverse vibration of a uniform rotating beam clamped at one end was first studied in connection with the design of helicopter blades. Handelman, Boyce, and Cohen [4] derived the appropriate partial differential equation describing the motion of a uniform beam, separated it assuming harmonic time dependence, $\exp(i\beta t)$, and put it into nondimensional form. Subsequently, Boyce and Handelman [1] derived the corresponding equations for beams of variable density and flexural rigidity and obtained results for the uniform beam using a variety of singular perturbation methods. Following [1], the eigenvalue problem for a uniform beam is:

$$u^{IV} - \frac{1}{2}\alpha^2[u^I(1-x^2+2\gamma^2)]^I = \beta^2 u, \quad 0 < x < 1 \quad (1)$$

$$u(0) = u^I(0) = u^{II}(1) = 0, \quad u^{III}(1) = \gamma^2 \alpha^2 u^I(1) - \gamma^2 \beta^2 u(1),$$

where u is the transverse deflection of the beam, α^2 is the nondimensional speed of rotation, and γ^2 is the ratio of tip mass to beam mass; Roman numerals indicate differentiation with respect to x ($0 < x < 1$); and β is the eigenvalue parameter which is a function of the parameters α^2 and γ^2 . It should be noted that the parameter γ^2 is defined in [4] as twice the ratio of tip mass to beam mass; this leads to a slightly different form of the eigenvalue problem.

Both Handelman, Boyce, and Cohen [4] and Boyce and Handelman [1] have commented on the difficulty of finding a closed solution to the eigenvalue problem (1) and have used singular perturbation theory to reduce the fourth-order problem to the following second-order problem:

$$[v^I(1-x^2+2\gamma^2)]^I = -\frac{2\nu^2}{\alpha^2} v, \quad 0 < x < 1, \quad (2)$$

$$v(0) = 0, \quad v^I(1) = (\nu^2/\alpha^2)v(1).$$

The eigenvalues of (1) have been shown [1] to be related to the eigenvalues of (2) and to those of the following fourth-order problem which has constant coefficients:

$$w^{IV} = \mu^2 w, \quad 0 < x < 1, \quad (3)$$

$$w(0) = w^I(0), \quad w^{II}(1) = 0, \quad w^{III}(1) + \mu^2 \gamma^2 w(1) = 0.$$

Boyce and Handelman [1] used several different methods to obtain approximate values of the second eigenvalue of (2) and found that ν_2^2/α^2 and also β_2^2 increased with increasing tip mass. Goodwin [3] converted the reduced problem (2) into a Fredholm integral equation having a symmetric kernel which depends on the eigenvalue parameter; however, Goodwin's paper does not contain numerical results.

The primary purpose of this paper is to describe an extension of the integral equation method used previously in [2]; to show, by means of some simple counterexamples, some advantages of these methods over those described earlier by Goodwin; and to use

the new methods to compute *improvable* lower bounds for the eigenvalues of the “reduced” problem (2). A secondary purpose is to compare the computational aspects of the integral equation method with those of other, less general methods such as invariant embedding [10].

3. Integral equation methods. It is well known [5] that boundary-value problems such as (1) and (2) can be rewritten as integral equations where the kernel of the integral equation is an appropriate Green’s function. Furthermore, in the case of problems with eigenvalue dependence in the boundary conditions such as (1) and (2), Green’s function can be written as the sum of a function that is independent of the eigenvalue parameter and one that is meromorphic in the eigenvalue parameter. Goodwin’s method [3] for obtaining improvable lower bounds for the eigenvalues of the integral equation

$$u(y) - \lambda \int_0^1 G(x, y; \lambda)u(x) dx = 0, \quad (4)$$

where $G(x, y, \lambda) = G_E(x, y) + G_M(x, y; \lambda)$, involves rewriting (4) (in operator notation) as

$$(I - \lambda A_E(\lambda))u(x) = \lambda A_M(\lambda)u(x), \quad (5)$$

where the Fredholm operator $A(\lambda)$ has been decomposed as $A_E(\lambda)$ and $A_M(\lambda)$ to reflect the dependence of $G(x, y; \lambda)$ on λ ; formally solving (5) to obtain an equivalent Fredholm integral equation with degenerate kernel:

$$(I - \lambda(I - \lambda A_E)^{-1}A_M(\lambda))u(x) = 0; \quad (6)$$

obtaining the first few terms of the eigenvalue equation of (6) as a power series in λ ;

$$1 + \alpha_1\lambda + \alpha_2\lambda^2 + \dots = 0; \quad (7)$$

and using the following formulae given by Spiegel [6] to obtain lower bounds for the smallest positive root of (7):

$$\sum_{h=1}^{\infty} \frac{1}{\lambda_h} = -\alpha_1, \quad (8a)$$

$$\sum_{h=1}^{\infty} \frac{1}{\lambda_h^2} = \alpha_1^2 - 2\alpha_2, \quad (8b)$$

$$\sum_{h=1}^{\infty} \frac{1}{\lambda_h^3} = 3\alpha_1\alpha_2 - 3\alpha_3 - \alpha_1^3. \quad (8c)$$

The operator $A_E(\lambda)$ may be either the Fredholm operator used by Goodwin [3] or the Volterra operator used by Jones and Goodwin [2]. For problems such as (1) and (2) the Volterra operator can be obtained easily by writing Green’s function as:

$$G_1(x, y; \lambda) = \sum_{i=1}^n [c_i(y) + b_i(y, \lambda)]u_i(x), \quad 0 \leq x < y \leq 1,$$

$$G_2(x, y; \lambda) = \sum_{i=1}^n b_i(y, \lambda)u_i(x), \quad 0 \leq y < x \leq 1,$$

where $u_i(x)$, $i = 1, 2, \dots, n$, are a fundamental set for the n th-order operators on the left-hand sides of (1) and (2); the coefficients $c_i(y)$ are determined by the continuity

conditions on Green's function and its first $n-2$ derivatives and the jump discontinuity condition in the n -1st derivative of Green's function; and the coefficients $b_i(y, \lambda)$ are determined by the boundary conditions. When $A_E(\lambda)$ is a Volterra operator, the inverse operator $(I - \lambda A_E(\lambda))^{-1}$ will exist for all values of λ ; however, the inverse of the corresponding Fredholm operator will fail to exist at the roots of $D_{A_E}(\lambda)$. It should be noted that if the kernel of (6) consists of a single term of the form $X(x, \lambda)Y(y, \lambda)/f(\lambda)$, then at least the first four terms of the canonical form of the eigenvalue equations obtained for the two operators are identical [7], in agreement with a result obtained by Brysk [8] for a special case. If, however, the kernel of (6) consists of more than one term, then the canonical form (7) of the eigenvalue equation obtained from the Fredholm formulation by multiplying the eigenvalue equation of (6) by an appropriate power of $D_{A_E}(\lambda)$ will differ from that obtained from the Volterra formulation. This difference is due to differences in the multiplicity with which the eigenvalues of $A_E(\lambda)$ appear in the canonical form of the eigenvalue equation (7) and is illustrated by the examples given in the Appendix. We note also that some care must be taken in handling problems in which $A_M(\lambda)$ consists of more than one term to avoid counting the poles of $A_M(\lambda)$ with artificially high multiplicity. This multiplicity problem occurs for differential operators of order higher than two and can be eliminated by using the trace formulae given by LaGinestra [9].

4. Eigenvalues of the "reduced" problem. The "reduced" problem for the rotation of a beam with tip mass has been derived by Boyce and Handelman [1] and is given by:

$$\begin{aligned} [v'(1 - x^2 + 2\gamma^2)]' &= -2\lambda v(x), & 0 < x < 1 \\ v(0) &= 0, & v'(1) = \lambda v(1), \end{aligned} \quad (2)$$

where $\lambda = v^2/\alpha^2$. Eq. (2) is equivalent to the integral equation

$$v(y) = 2\lambda \int_0^1 G(x, y; \lambda)v(x) dx, \quad (9)$$

where Green's function, $G(x, y; \lambda)$, is given by:

$$G_1(x, y; \lambda) = \frac{1}{2A} \left[\ln \left(\frac{A+x}{A-x} \right) - \ln \left(\frac{A+y}{A-x} \right) \right] + \frac{1}{2A} \ln \left(\frac{A+y}{A-x} \right) \cdot \left[1 + \frac{\lambda \ln \left(\frac{A+x}{A-x} \right)}{\left\{ \frac{2A}{A^2-1} - \lambda \ln \left(\frac{A+1}{A-1} \right) \right\}} \right], \quad 0 \leq x < y \leq 1,$$

$$G_2(x, y; \lambda) = \frac{1}{2A} \ln \left(\frac{A+y}{A-x} \right) \left[1 + \frac{\lambda \ln \left(\frac{A+x}{A-x} \right)}{\left\{ \frac{2A}{A^2-1} - \lambda \ln \left(\frac{A+1}{A-1} \right) \right\}} \right], \quad 0 \leq y < x \leq 1,$$

and $A = (1 + 2\gamma^2)^{1/2}$. Eq. (9) may be considered as a non-homogeneous Volterra integral equation with the kernel

$$k_v(y, x) = \frac{1}{A} \left[\ln \left(\frac{A+x}{A-x} \right) - \ln \left(\frac{A+y}{A-y} \right) \right], \quad (10)$$

which is equivalent to its formal "solution" [5]:

$$u(y) = \lambda \int_0^1 \frac{1}{A} \left[1 + \frac{\lambda \ln \frac{A+x}{A-x}}{\left[\frac{2A}{A^2-1} - \lambda \ln \left(\frac{A+1}{A-1} \right) \right]} \right] \cdot \left[\ln \left(\frac{A+y}{A-y} \right) - \lambda \int_0^y H_k(y, z; \lambda) \ln \left(\frac{A+z}{A-z} \right) dz \right] u(x) dx. \quad (11)$$

Here $H_k(y, z; \lambda)$ is the resolvent kernel for $k_v(y, z; \lambda)$. The eigenvalue equation of (11) is

$$1 - \lambda a_{11} = 0, \quad (12)$$

where

$$a_{11} = \int_0^1 X(x, \lambda) Y(y, \lambda) dx.$$

Normally, only the first few terms of the eigenvalue equation (12) written as a power series in λ ,

$$1 + \alpha_1 \lambda + \alpha_2 \lambda^2 + \alpha_3 \lambda^3 + \dots = 0,$$

are required to obtain lower bounds for the smallest positive root of (12) using Spiegel's formulae [6]. However, since the first eigenvalue of (2) is $\lambda_1 = 1$ (corresponding to the eigenfunction $v(x) = x$), lower bounds for the second eigenvalue, λ_2 , can be obtained from expressions based on simple modifications of Spiegel's formulae:

$$(\lambda_2)_1 \geq \frac{-1}{\alpha_1 + 1}, \quad (13a)$$

$$(\lambda_2)_2 \geq \frac{1}{(\alpha_1^2 - 2\alpha_2 - 1)^{1/2}} \quad (13b)$$

$$(\lambda_2)_3 \geq \frac{1}{(3\alpha_1\alpha_2 - 3\alpha_3 - \alpha_1^3 - 1)^{1/3}}; \quad (13c)$$

where α_1 , α_2 , and α_3 are given by:

$$\alpha_1 = -\left(\frac{A^2-1}{2A}\right) \ln \left(\frac{A+1}{A-1}\right) - \ln \left(\frac{A^2-1}{A^2}\right), \quad (14a)$$

$$\begin{aligned} \alpha_2 = & \left(\frac{A^2-1}{2A^2}\right) \ln \left(\frac{A+1}{A-1}\right) \left[A \ln \left(\frac{A^2-1}{A^2}\right) + \ln \left(\frac{A+1}{A-1}\right) \right] \\ & + \frac{1}{A} \int_0^1 \ln \left(\frac{A+x}{A-x}\right) \ln \left(\frac{A^2-x^2}{A^2}\right) dx \\ & + \frac{1}{2A^2} \int_0^1 (4x - (A^2+1)) \left[\ln \left(\frac{A+x}{A-x}\right) \right]^2 dx, \end{aligned} \quad (14b)$$

and

$$\begin{aligned}
\alpha_3 = & \int_0^1 \frac{1}{A^3} [A^2 + x(1 + A^2)] \left[\ln \left(\frac{A+x}{A-x} \right) \right]^3 dx + \int_0^1 \left(\frac{A^2 + 1 + 2x}{A^2} \right) \left[\ln \left(\frac{A+x}{A-x} \right) \right]^2 \\
& \cdot \ln \left(\frac{A^2 - x^2}{A^2} \right) dx + \frac{1}{A} \int_0^1 \ln \left(\frac{A+x}{A-x} \right) \left[\ln \left(\frac{A^2 - x^2}{A^2} \right) \right]^2 dx \\
& + \left(-\frac{(A^2 + 1)}{2A^2} \ln \left(\frac{A+1}{A-1} \right) - \frac{1}{A} \ln \left(\frac{A^2 - 1}{A^2} \right) \right) \int_0^1 \left[\ln \left(\frac{A+x}{A-x} \right) \ln \left(\frac{A^2 - x^2}{A^2} \right) \right] dx \\
& + \left\{ -\frac{(A^2 + 1)}{A^3} \ln \left(\frac{A+1}{A-1} \right) - \frac{2}{A^2} \ln \left(\frac{A^2 - 1}{A^2} \right) - \frac{2}{A^2} \right\} \int_0^1 x \left[\ln \left(\frac{A+x}{A-x} \right) \right]^2 dx \\
& + \left\{ -\frac{1}{A} \ln \left(\frac{A+1}{A-1} \right) - \left(\frac{A^2 + 1}{2A^2} \right) \ln \left(\frac{A^2 - 1}{A^2} \right) + \frac{2}{A^2} \right\} \int_0^1 \left[\ln \left(\frac{A+x}{A-x} \right) \right]^2 dx.
\end{aligned} \tag{14c}$$

The apparent complexity of (14c) results from the reduction of the iterated integral

$$\int_0^1 \left(\int_0^x \left(\int_z^x k_v(x, y) k_v(y, z) dy \right) \ln \left(\frac{A+z}{A-z} \right) dz \right) dx$$

to a single integral by changing the order of integration with appropriate changes in the limits of integration. This reduction substantially decreases the time and space complexity of the computation. The bound given by (13c) could be further improved by calculating additional terms of the resolvent kernel.

Lower bounds for the first and second eigenvalues of the “reduced” problem were computed in single precision with a Burroughs B-6700 computer using an adaptive form of Simpson’s rule with the error parameter set at 10^{-8} . The lower bounds for λ_1 converged rapidly to 1 for all values of γ^2 . Fig. 1 is a graph of λ_2 vs. γ^2 showing the lower bounds $(\lambda_2)_1$, $(\lambda_2)_2$, and $(\lambda_2)_3$ and the “approximative” (second) eigenvalue obtained by Boyce and Handelman [1] using the Rayleigh–Ritz method. It should be noted that $(\lambda_2)_1$, the first lower bound for λ_2 , is quite poor but that $(\lambda_2)_2$ and $(\lambda_2)_3$ agree quite well with the Rayleigh–Ritz results. A complete analysis of the errors in $(\lambda_2)_3$ requires knowledge of the spectrum of (2); however, numerical experiments with functions with well-separated zeros, such as $\cos x$, indicate that when the first root is known exactly, Spiegel’s formulae can be used to obtain lower bounds for the second eigenvalue which are precise to about one decimal digit less than the number of terms of the series expansion used in computation, providing the ill-conditioned expressions (13 a, b, and c) are evaluated in such a way that the round-off errors are reasonably small.

In general, it has been found that the lower bound for the first zero of these functions is precise to about the same number of digits as terms used in computation [6]. Thus, the expected precision of the values of $(\lambda_2)_3$ given in Table I is about two decimal digits.

In the case of the self-adjoint “reduced” problem (2), the precision of the results in Table I could be verified independently using an invariant embedding technique described by Scott, Shampine, and Wing [10]. A Ricatti-like transformation was used to obtain the following equations from (2):

$$\frac{dR(x)}{dx} = \frac{1}{(1 - x^2 + 2\gamma^2)} + 2\lambda R^2(x), \tag{15a}$$

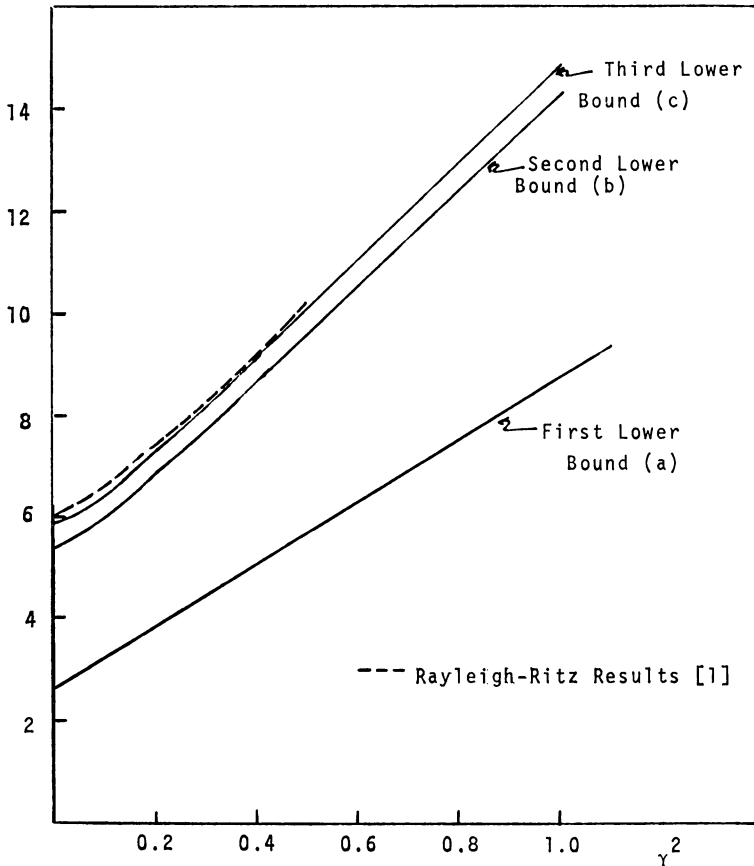


FIG. 1. The second eigenvalue of the "reduced" equation (2) as a function of tip mass.

$$-\frac{dS(x)}{dx} = 2\lambda + \frac{1}{(1-x^2+2\gamma^2)^2} S^2(x), \quad (15b)$$

with the boundary conditions $R(0) = 0$ and $R(1, \lambda) = -1/2\gamma^2$ and the additional condition $S(x) = 1/R(x)$. Eq. (15a) was integrated using a standard fourth-order Runge-Kutta method with step size of 0.01 until $|R(x)| > 2$; at this point Eq. (15b) was integrated until $|S(x)| > 2$. Integration of (15a) or (15b) with appropriate switching was continued up to $x = 1.00$; at this point Newton's method was applied to the boundary condition equation, $R(1, \lambda) + (1/2\gamma^2) = 0$, to obtain a new value of λ . This process was repeated until the value of λ was stable to four decimal digits.

Results obtained by invariant embedding are compared with $(\lambda_2)_3$ in Table I. These results confirm the conjecture that the lower bounds $(\lambda_2)_3$ obtained by the integral equation method are precise to about two decimal digits. However, it should be noted that the invariant embedding technique is restricted to problems which are self-adjoint while the integral equation method is more generally applicable.

5. Eigenvalues for the fourth-order problem. Lower bounds for the first eigenvalue of the fourth order problem (1) can be obtained from the results given in Sec. 4. It

TABLE I.
Comparison of integral equation and invariant embedding result

<i>Lower Bounds—Integral Equation Method</i>		<i>Invariant Embedding Result</i>	
λ	$(\lambda_2)_2$	$(\lambda_2)_3$	λ_2
0.1	5.399	5.865	6.000
0.2	5.551	5.987	6.100
0.3	5.911	6.321	6.408
0.4	6.492	6.890	7.000
0.5	7.288	7.691	7.799
0.6	8.291	8.712	8.800
0.7	9.496	9.947	10.020
0.8	10.898	11.389	11.500
0.9	12.495	13.035	13.116
1.0	14.286	14.883	15.000

has been shown by variational methods [1] that

$$\beta_1^2 \geq \mu_1^2 + \nu_1^2 = \mu_1^2 + \alpha^2, \quad (16)$$

where μ_1^2 is the first eigenvalue of the boundary-value problem (3). The eigenvalue equation of (3) can easily be determined as a power series in μ^2 , and Spiegel's formulae (8a, b, c) used to compute lower bounds for μ_1^2 as a function of γ^2 . In addition, Spiegel's formulae can be modified to give lower bounds for the second eigenvalue, μ_2^2 , but this calculation is extremely ill-conditioned. Lower bounds for μ_1^2 ($\gamma^2 = 0$) obtained from (8a), (8b) and (8c), respectively, are

$$(\mu_1^2)_1 \geq 12.000, (\mu_1^2)_2 \geq 12.358, \quad \text{and} \quad (\mu_1^2)_3 \geq 12.362$$

which compares favorably with the exact value of 12.362. Lower bounds for μ_2^2 ($\gamma^2 = 0$) obtained from modified Spiegel formula are

$$(\mu_2^2)_1 \geq 409.39, (\mu_2^2)_2 \geq 481.26; \quad \text{and} \quad (\mu_2^2)_3 \geq 483.24$$

which again compares favorably with an exact value of 485.52. Lower bounds for μ_1^2 and μ_2^2 for $\gamma^2 \neq 0$ are given in Fig. 2.

It is clear that the lower bound for β_1^2 given by (16) decreases with increasing tip mass for all values of α^2 . It is also possible to calculate lower bounds for β_2^2 , from lower bounds for both the first and second eigenvalues of (2) and (3) and an upper bound for β_1^2 [1].

The values of $(\lambda_2)_3$ agree well with results given previously and it is unlikely that new results will be obtained in this way.

It is also possible to rewrite (1) as an integral equation using Green's function and to compute lower bounds directly using the integral equation techniques. This procedure requires determination of a fundamental set of solutions for the operator specified on the lefthand side and the boundary conditions of (1). Although power series representations of this fundamental set have been found [11], these series converge only for relatively small values of α^2 . Since the results attainable by this route are limited, we have not attempted to obtain lower bounds for β_1 and β_2 directly. It is important to note that in this case the multiplicity problem discussed in the Appendix will produce poor lower bounds unless the trace formulae developed by La Ginestra [9] are used.

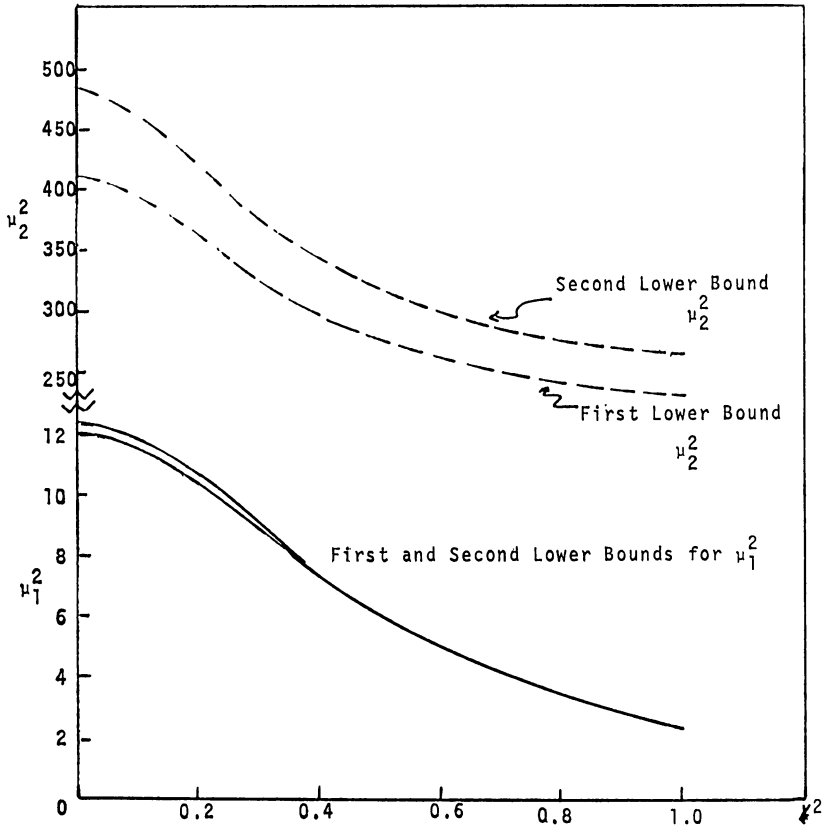


FIG. 2. Lower bounds for μ_1^2 and μ_2^2 as a function of tip mass.

6. Conclusion. Improvable lower bounds have been calculated for the second eigenvalue of the "reduced" problem derived from the boundary-value problem describing the transverse vibration of a rotating beam carrying tip mass. The integral equation method used is based on an unsymmetric decomposition of Green's function which leads to a Volterra integral equation. The lower bounds agree with approximate results obtained previously [1] by a variety of methods including the Rayleigh-Ritz method. The current method offers significant conceptual and computational advantages over these methods in producing improvable lower bounds which are uniform for all values of the parameters. The precision of the current results (about two decimal) digits has been confirmed independently by means of an invariant embedding method.

The integral equation method is general in scope and can be applied to many problems; it is not restricted to self-adjoint operators as are variational and invariant embedding methods. Furthermore, the current Volterra equation method offers substantial computational advantages not found in a related Fredholm equation method described previously [3]; these advantages are derived from the differences in representation between the Volterra and Fredholm resolvent kernels. Counterexamples which are given in the Appendix show that care must be taken in generalizing the procedure given in [3] for problems where the meromorphic part of Green's function consists of more than one term in order to obtain good lower bounds.

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Appendix. The purpose of this Appendix is to provide examples showing that care must be taken in extending the procedure given in [3] to integral equations in which the meromorphic part of the kernel consists of two or more terms.

It should be noted that the meromorphic part of Green's function for the second-order "reduced" problem (2) consists of only one term. However, meromorphic parts of the Green's functions for higher-order problems such as (1) and the fourth-order boundary value problem describing the transverse vibration of a pipe containing flowing fluid consist of two or more terms [2], and poor results will be obtained unless the multiplicity problem is treated appropriately. Example A shows that incorrect results may be obtained from a Fredholm formulation due to inclusion of the roots of $D_{KE}(\lambda)$ with multiplicity greater than one; this difficulty can always be eliminated by using the Volterra formulation. Example B shows that incorrect results can be obtained even with the Volterra formulation if the reciprocals of the pole are counted more than once; this difficulty can be avoided by using the trace formulae developed by LaGinestra [9]. It should be noted that improved lower bounds for the first eigenvalue of a pipe containing flowing fluid could be by the use of LaGinestra's formulae in [2].

Example A: Consider the boundary-value problem

$$u^{IV}(x) = -\lambda^2 u(x),$$

$$u(0) = u(1) = 0,$$

which has eigenvalues $n^2\pi^2$, $n = 1, 2, \dots$. Green's function can be written either as

$$G_1(x, y) = (x - y) + [y(1 - x)], \quad 0 \leq x < y \leq 1,$$

$$G_2(x, y) = [y(1 - x)] \quad 0 \leq y < x \leq 1,$$

or as

$$\begin{aligned} G_1(x, y) &= -y + [y + x(1 - y)], & 0 \leq x < y \leq 1, \\ G_2(x, y) &= -x + [y + x(1 - y)], & 0 \leq y < x \leq 1. \end{aligned}$$

When the portion of Green's function in square brackets is taken as the operator $A_M(\lambda)$, the first (Volterra) formulation leads to the eigenvalue equation

$$1 - \lambda^2 \left(\frac{1}{6}\right) + \lambda^4 \left(\frac{1}{120}\right) + \dots = 0 \quad (a)$$

while the second equivalent (Fredholm) formulation leads to the eigenvalue equation

$$1 + \lambda^2 \left(\frac{1}{3}\right) + \lambda^4 \left(\frac{9}{8}\right) + \dots = 0, \quad (b)$$

Spiegel's formulae applied to (a) lead to well-known results; however, Spiegel's formulae applied to (b) lead to

$$\sum \frac{1}{\lambda_h^2} = -\frac{1}{3}, \quad \sum \frac{1}{\lambda_h^4} = \frac{-77}{36}.$$

Example B: Consider the boundary-value problem

$$\begin{aligned} u''(x) &= -\lambda u(x) \\ u(0) &= 0, \quad u'(1) = \lambda u(1), \end{aligned}$$

with eigenvalue equation

$$\cos \sqrt{\lambda} - \sqrt{\lambda} \sin \sqrt{\lambda} = 0. \quad (c)$$

The first few terms of the series expansion of (c) are

$$1 - 3/2\lambda + \lambda^2 \left(\frac{5}{4!}\right) - \lambda^3 \left(\frac{7}{6!}\right) + \dots = 0, \quad (d)$$

$$G_1(x, y : \lambda) = (x - y) + y \frac{(1 + \lambda(x - 1))}{1 - \lambda}, \quad 0 \leq x < y \leq 1,$$

$$G_2(x, y : \lambda) = y \frac{(1 + \lambda(x - 1))}{1 - \lambda}, \quad 0 \leq y < x \leq 1,$$

If the meromorphic part of Green's function is treated as a single term, the canonical form of the eigenvalue equation obtained by the Goodwin procedure is identically (d). However, if the meromorphic part of $G(x, y; \lambda)$ is treated as two terms, $y/(1 - \lambda) + (\lambda/(1 - \lambda))(x - 1)$, then the first few terms of the eigenvalue equation are:

$$1 - 5/2\lambda + \frac{41}{24}\lambda^2 + \dots = 0. \quad (e)$$

From (d) we find that $\sum 1/\lambda_h = 3/2$ and $\sum 1/\lambda_h^2 = 11/6$ while from (e) we find that $\sum 1/\lambda_h = 5/2 = 3/2 + 1/1$ and $\sum 1/\lambda_h^2 = 17/6 = 11/6 + 1/12$.