NEAR-BOUNDARY EXPANSION OF GREEN'S FUNCTION ASSOCIATED WITH CLAMPED PLATES*

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Abstract. The Green's function $G(P, P')$ associated with a clamped plate of arbitrary shape is considered, when $P'$ is at a distance $O(\varepsilon)$ from a regular point $O$ of the boundary. First an outer expansion of $G$ is described, valid when $P$ is not near $P'$. Then an inner expansion of $G$ is constructed when both $P$ and $P'$ are near $O$. The leading term of the inner expansion is just the Green's function $G$, for the halfplane bounded by the tangent to the boundary at $O$, and $\varepsilon^{-2}G$ differs from $\varepsilon^{-2}G_{e}$ by $O(\varepsilon)$. The first two terms of the inner expansion agree with the first two terms of the expansion of $G_{e}$, the Green's function for the interior of the osculating circle of the boundary at $O$, if the boundary is convex at $O$. If it is concave, $G_{e}$ is the Green's function for the exterior of the osculating circle. Moreover, $\varepsilon^{-2}G$ differs from $\varepsilon^{-2}G_{e}$ by $O(\varepsilon^2)$. A two-term inner expansion is explicitly given.

1. Introduction. Let $(x, y)$ be rectangular Cartesian coordinates and let $D$ be a domain of the $(x, y)$-plane. Suppose that $O$ is a regular point of the boundary $\partial D$ of $D$. We locate the origin of the coordinates at $O$ in such a way that the $z$-axis is tangent to $\partial D$ and the $y$-axis points toward the interior of $D$. For $x$ sufficiently small $\partial D$ has the Taylor expansion

$$y = b(x) = \sum_{n=0}^{\infty} \frac{b_n}{n!} x^n. \quad (1.1)$$

We shall use the radius of curvature of $\partial D$ at $O$ as a length scale so that $b_2 = \pm 1$.

Suppose that $G(x, y, x', y')$ is a function defined on $D$, satisfying the equations

$$\nabla^4 G = \delta(x - x')\delta(y - y'), \quad (x, y) \in D, \quad (1.2)$$

$$G = \partial_{n} G = 0, \quad (x, y) \in \partial D, \quad (1.3)$$

where $\partial_{n}$ is taken along the outward normal $n$ to $\partial D$.

If $D$ is the semi-infinite plane $y > 0$, the associated Green's function $G_{e}$ obtained by Michell [1] is

$$G_{e} = \frac{1}{2}[(x - x')^2 + (y - y')^2] \ln \frac{(x - x')^2 + (y - y')^2}{(x - x)^2 + (y + y')^2} + 2yy'. \quad (1.4)$$

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If \( D \) is the interior of the unit circle \( x^2 + (y - 1)^2 = 1 \), then the Green's function \( G_+ \) obtained by Michell [1] is

\[
G_+ = \frac{1}{4}\{1 - [x'^2 + (y' - 1)^2]\} \{1 - [x^2 + (y - 1)^2]\}
\]

\[
+ \frac{1}{8}[x^2 + 2y(x^2 + y^2) + 2y'x'^2 + 2y'y'^2] \ln[(x - x')^2 + (y - y')^2 - z^2]
\]

where

\[
z^2 = (x - x')^2 + (y + y')^2 - 2y(x'^2 + y'^2) - 2y'(x^2 + y^2) + (x'^2 + y'^2)(x^2 + y^2).
\]

(1.5)

The Green's function \( G_- \) for the exterior of the unit circle \( x^2 + (y + 1)^2 = 1 \) was obtained by Symonds [2]. More general but related problems were investigated by Dundurs and Lee [3] and Amon and Dundurs [4].

For a general domain \( D \), the Green's function \( G \) cannot be obtained explicitly in a closed form. Let \( G(P, P') \) be the Green's function. Then \( G(P, P') \) may be interpreted as the deflection at \( P \) of a plate subjected to a unit load applied at \( P' \). Our objective is to determine \( G(P, P') \), when the unit load is placed near the boundary point \( O \). Let \( \epsilon \) denote the distance \( OP' \). We shall determine \( G \) asymptotically in terms of the parameter \( \epsilon \) as \( \epsilon \to 0 \). Inner and outer expansions will be constructed and matched. Moreover, the first three terms of the inner expansion can be obtained explicitly. Only the first two terms, however, are calculated. The result is

\[
\epsilon^{-2}G(\epsilon x, \epsilon \eta, \epsilon x', \epsilon \eta') \sim \frac{1}{2}(\xi - \xi')^2 + (\eta - \eta')^2 \ln\left[\frac{(\xi - \xi')^2 + (\eta - \eta')^2}{(\xi - \xi')^2 + (\eta + \eta')^2}\right] + 2\eta\eta' - \epsilon^4 b_2 \frac{\eta^2\eta'(\xi^2 + \eta'^2) + \eta\eta'^2(\xi^2 + \eta^2)}{(\xi - \xi')^2 + (\eta + \eta')^2} + O(\epsilon^2)
\]

(1.7)

where \( b_2 = \pm 1 \), depending on whether \( D \) is convex or concave at \( O \). It can be checked that

\[
G(\epsilon x, \epsilon \eta, \epsilon x', \epsilon \eta') - G_+(\epsilon x, \epsilon \eta, \epsilon x', \epsilon \eta') \sim O(\epsilon^2),
\]

(1.8)

\[
G(\epsilon x, \epsilon \eta, \epsilon x', \epsilon \eta') - G_- (\epsilon x, \epsilon \eta, \epsilon x', \epsilon \eta') \sim O(\epsilon^4).
\]

(1.9)

For the purposes of bringing out the underlying idea and illustrating the method of finding (1.7), we consider a simpler problem, namely, the problem of determining \( G(x, y, \epsilon) \equiv G(x, y, 0, \epsilon) \) as \( \epsilon \to 0 \). The function \( G(x, y, \epsilon) \) may be defined by

\[
G(x, y, \epsilon) = [x^2 + (y - \epsilon)^2] \ln[x^2 + (y - \epsilon)^2]^{1/2} + H(x, y, \epsilon)
\]

(1.10)

where \( H \) is a regular biharmonic function satisfying the boundary condition that \( G \), together with its normal derivative, vanishes on \( \partial D \).

We shall occasionally use polar coordinates \((r, \theta)\). We shall also introduce boundary layer variables \((\xi, \eta, \rho, \theta)\) and intermediate variables \((\sigma, \nu, \lambda, \theta)\). The three sets of variables are related by the equations

\[
(x, y, r) = \epsilon^{1/2}(\sigma, \nu, \lambda) = \epsilon(\xi, \eta, \rho).
\]

(1.11)

In terms of the boundary layer variables, \( \partial D \) defined by (1.1) has the expansion

\[
\eta = \beta(\xi, \epsilon) \equiv \frac{1}{\epsilon} \beta(\xi) = \sum_{m=1}^{\infty} \frac{\beta_m(\xi)}{m!} \epsilon^m, \quad \beta_m = \frac{b_{m+1}}{m + 1} \epsilon^{m+1}.
\]

(1.12)
Our method of analysis is that used by Wu and Keller [5] to obtain the corresponding results for Laplace's equation.

2. Outer expansion. The problem satisfied by \( G(x, y, \epsilon) \) is defined by (1.10). Since \( G(P, P') \) is analytic in \( P' \) except at \( P \), \( G(x, y, \epsilon) \) is analytic in \( \epsilon \). Therefore

\[
\epsilon^{-2} G(x, y, \epsilon) \sim \sum_{n=1}^{\infty} \frac{\epsilon^n}{n!} G_n(x, y).
\]  

We shall call (2.1) the outer expansion of \( G(x, y, \epsilon) \). The function \( G_n(x, y) \) is biharmonic and vanishes, together with its normal derivative, on \( \partial D \) except at the point \( O \). Near the point \( O \),

\[
G_n(x, y) = O(r^{-n}), \quad r \to 0.
\]  

Because \( G_n \) is analytic, we conclude from (2.2) that

\[
G_n(x, y) = \sum_{m=-n}^{n} J_{nm}(\theta) r^m,
\]

where

\[
J_{nm}(\theta) = A'_{nm}' S_m(\theta) + B'_{nm}' C_m(\theta) + \tilde{A}'_{nm}' \tilde{S}_m(\theta) + \tilde{B}'_{nm}' \tilde{C}_m(\theta).
\]  

and

\[
C_m(\theta) = \cos m\theta - \cos (m - 2)\theta, \quad m \neq 1, \quad (2.5)
\]

\[
\tilde{C}_m(\theta) = \cos m\theta + \cos (m - 2)\theta, \quad m \neq 1, \quad (2.6)
\]

\[
S_m(\theta) = (m - 2) \sin m\theta - m \sin (m - 2)\theta, \quad m \neq 0, 1, 2 \quad (2.7)
\]

\[
\tilde{S}_m(\theta) = (m - 2) \sin m\theta + m \sin (m - 2)\theta, \quad m \neq 0, 1, 2 \quad (2.8)
\]

\[
C_1(\theta) = \cos \theta, \quad S_1(\theta) = \sin \theta, \quad \tilde{C}_1(\theta) = \theta \cos \theta, \quad \tilde{S}_1(\theta) = \theta \sin \theta, \quad (2.9)
\]

\[
S_0(\theta) = S_2(\theta) = \sin 2\theta, \quad S_0(\theta) = \tilde{S}_0(\theta) = \theta. \quad (2.10)
\]

We note that \( J_{nm}(\theta)r^m \) are biharmonic functions. Moreover, the functions \( C_m(\theta) \) and \( S_m(\theta) \) defined by (2.5) and (2.7) vanish, together with their derivatives, for \( \theta = 0 \) and \( \pi \).

The constants \( A', B', \tilde{A}' \) and \( \tilde{B}' \) appearing in (2.4) are so far undetermined. We shall show that some of the constants can be determined by matching. The rest can be determined by solving a series of well-defined boundary-value problems.

For the purpose of matching, we need to know the inner expansion of the outer expansion. First we use (2.3) and (2.1) to obtain the expansion of \( \epsilon^{-2} G(x, y, \epsilon) \) for \( r \) small. It is

\[
\epsilon^{-2} G(x, y, \epsilon) \sim \sum_{s=1}^{\infty} \frac{\epsilon^s}{s!} \sum_{m=-s}^{s} J_{sm}(\theta) r^m.
\]  

Next we set \((x, y, r) = \epsilon^{1/2}(\sigma, \nu, \lambda)\) in (2.11). Then, upon rearranging the series, we get

\[
\epsilon^{-2} G(\epsilon^{1/2}\sigma, \epsilon^{1/2}\nu, \epsilon) \sim \sum_{k=1}^{\infty} \epsilon^{k/2} \sum_{n=1}^{k} \frac{1}{n!} \left[ A_{n,k-2n}' S_{k-2n}(\theta) + B_{n,k-2n}' C_{k-2n}(\theta) + \tilde{A}_{n,k-2n}' \tilde{S}_{k-2n}(\theta) + \tilde{B}_{n,k-2n}' \tilde{C}_{k-2n}(\theta) \right] \lambda^{k-2n}.
\]

This is the inner expansion of the outer expansion.
3. Inner expansion. We consider the case \( x \) and \( y \) are \( O(\varepsilon) \). It follows from (1.10) that we may seek \( G \) in the form

\[
\varepsilon^{-2}G(\xi, \eta, \varepsilon) \sim \frac{1}{2}[\xi^2 + (\eta - 1)^2] \ln[\xi^2 + (\eta - 1)^2] + \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} h_n(\xi, \eta). \tag{3.1}
\]

We shall refer to (3.1) as the inner expansion. The functions \( h_n \) are regular and biharmonic. The boundary conditions \( G = \partial G / \partial n = 0 \) must be satisfied on \( \partial D \) defined by (1.12). We have

\[
\varepsilon^{-2}G(\xi, \eta, \varepsilon) |_{\eta = \beta(\xi, \varepsilon)} \sim 0 \tag{3.2}
\]

\[
\frac{\partial}{\partial \mu} \varepsilon^{-2}G(\xi, \eta, \varepsilon) |_{\eta = \beta(\xi, \varepsilon)} \sim 0 \tag{3.3}
\]

where

\[
\frac{\partial(\varepsilon)\varepsilon^{-2}G(\xi, \eta, \varepsilon)}{\partial \mu} \equiv \left\{1 + \frac{\partial \beta(\xi, \varepsilon)}{\partial \xi}\right\}^{-1/2} \left\{\frac{\partial(\varepsilon)}{\partial \xi} \frac{\partial \beta(\xi, \varepsilon)}{\partial \xi} - \frac{\partial(\varepsilon)}{\partial \eta}\right\} \right|_{\eta = \beta(\xi, \varepsilon)} \tag{3.4}
\]

Substituting (3.1) into (3.2), expanding for small \( \varepsilon \) and equating to zero the coefficient of each power of \( \varepsilon \), we obtain

\[
h_s(\xi, 0) = -f_s(\xi) - \sum_{m=0}^{s-1} \frac{s!}{m! (s-m)!} h_{m,s-m}(\xi), \tag{3.5}
\]

where

\[
f_s(\xi) = \frac{1}{2} \partial_s \{[\xi^2 + (\beta(\xi, \varepsilon) - 1)^2] \ln[\xi^2 + (\beta(\xi, \varepsilon) - 1)^2] \} |_{\varepsilon=0}, \tag{3.6}
\]

\[
h_{s, \varepsilon}(\xi) = \partial_{\varepsilon} h_s(\xi, \beta(\xi, \varepsilon)) |_{\varepsilon=0}. \tag{3.7}
\]

Applying the same operation on (3.3), we get

\[
h_{s, \varepsilon}(\xi) = -f_{s, \varepsilon}(\xi) + \sum_{m=1}^{s} \frac{s! b_{m,s} \xi^m}{(s - m)! m!} f_{s, \xi}(s-m)(\xi)
\]

\[
+ \sum_{k=1}^{s} \left\{ \sum_{m=1}^{k} \frac{s! b_{m,s} \xi^m}{(s - k) (k! - m)! m!} h_{s-k, \xi}(k-m)(\xi) - \frac{s!}{k! (s-k)!} h_{s-k, \varepsilon}(\xi) \right\} \tag{3.8}
\]

where the notation

\[
P_{s, \varepsilon}(\xi) = \partial_{\varepsilon}^s [\partial_\varepsilon P(\xi, \eta) |_{\eta = \beta(\xi, \varepsilon)}] |_{s=0} \tag{3.9}
\]

\[
P_{s, \varepsilon}(\xi) = \partial_{\varepsilon}^s [\partial_\varepsilon P(\xi, \eta) |_{\eta = \beta(\xi, \varepsilon)}] |_{s=0} \tag{3.10}
\]

applies to \( f \) and \( h_m \). Eqs. (3.5) and (3.8) determine the values of \( h_s \) and \( h_{s, \varepsilon} \) on the \( \xi \)-axis, in terms of the \( h_m \) with \( m < s \). By determining regular biharmonic functions satisfying (3.5) and (3.8), starting with \( s = 0 \), the \( h_s \) can be found successively.

Setting \( s = 0 \) in (3.5) and (3.8), we get

\[
h_0(\xi, 0) = -\frac{1}{2}[\xi^2 + 1] \ln(\xi^2 + 1), \tag{3.11}
\]

\[
h_{0, \varepsilon}(\xi, 0) = \ln(\xi^2 + 1) + 1. \tag{3.12}
\]

An image analysis leads to the choice

\[
h_0(\xi, \eta) = -\frac{1}{2}[\xi^2 + (\eta - 1)^2] \ln[\xi^2 + (\eta - 1)^2] + 2\eta + A_0(\xi, \eta) \tag{3.13}
\]
where $A_0$ is an arbitrary biharmonic function satisfying the conditions $A_0 = A_0_{\infty} = 0$ on the $\xi$-axis. However, since the outer expansion is $O(\epsilon^3)$, the term $\epsilon^2 A_0$ cannot be matched with the outer expansion. Thus $A_0 = 0$. Then (3.1) becomes

$$\epsilon^{-2}G(\epsilon \xi, \epsilon \eta, \epsilon) \sim \frac{1}{2}[\xi^2 + (\eta - 1)^2] \ln \frac{\xi^2 + (\eta - 1)^2}{\xi^2 + (\eta + 1)^2} + 2\eta + O(\epsilon). \quad (3.14)$$

Aside from the term $O(\epsilon)$, this is the Green's function for the half plane $\eta \geq 0$, bounded by the tangent to $\partial D$ at $O$.

We need the solution of a fundamental problem for the determination of $h_n$ for $n \geq 1$. Let $\omega(\xi, \eta)$ be a biharmonic function defined on the half-plane $\eta > 0$. On the edge $\eta = 0$, $\omega$ satisfies the conditions

$$\omega(\xi, 0) = \omega_0(\xi), \quad \omega_{,\xi}(\xi, 0) = \omega_1(\xi) \quad (3.15)$$

where $\omega_0$ and $\omega_1$ are given functions satisfying the conditions

$$\lim_{\xi \to \infty} \frac{1}{\xi^3} \omega_0(\xi), \quad \frac{1}{\xi} \omega_1(\xi) = 0. \quad (3.16)$$

Using $G_*$ for $G$ in (1.4) and making appropriate substitutions, we get

$$\omega(\xi, \eta) = \frac{1}{\pi} \int_{-\infty}^{\infty} \left\{ \frac{2\eta^2 \omega_0(\xi')}{(\xi - \xi')^2 + \eta^2} - \frac{\eta^2 \omega_1(\xi')}{(\xi - \xi')^2 + \eta^2} \right\} d\xi'. \quad (3.17)$$

This is the solution of a semi-infinite cantilever plate subjected to edge displacement and rotation.

For $s = 1$, (3.5) and (3.8) yield

$$h_1(\xi, 0) = -f_1(\xi) - h_{01}(\xi) = 0, \quad (3.18)$$

$$h_{1,1}(\xi, 0) = -(f_{,\xi} + h_{0,\xi}) + (f_{,\xi} + h_{0,\xi})b_2 \xi$$

$$= -\frac{b_2 4 \xi^2}{\xi^2 + 1} \quad (3.19)$$

where $b_2 = \pm 1$. Using (3.17) we obtain

$$h_1(\xi, \eta) = -4b_2 \frac{\eta^2 + \eta(\xi^2 + \eta^2)}{\xi^2 + (\eta + 1)^2} + A_1(\xi, \eta) \quad (3.20)$$

where $A_1$ has the same property as $A_0$ and must again vanish because $\epsilon^3 A_1(\xi, \eta)$ cannot be matched with the outer expansion. Now (3.1) becomes

$$\epsilon^{-2}G(\epsilon \xi, \epsilon \eta, \epsilon) \sim \frac{1}{2}[\xi^2 + (\eta - 1)^2] \ln \frac{\xi^2 + (\eta - 1)^2}{\xi^2 + (\eta + 1)^2} + 2\eta$$

$$- \epsilon^2 b_2 \frac{\eta^2 + \eta(\xi^2 + \eta^2)}{\xi^2 + (\eta + 1)^2} + O(\epsilon^2). \quad (3.21)$$

If $b_2 = + 1$, this agrees to $O(\epsilon)$ with the expansion $G_+^*$ given by (1.5). If $b_2 = - 1$, it agrees to $O(\epsilon)$ with the expansion of $G_-^*$ for the exterior of the circle $x^2 + (y + 1)^2 = 1$.

For $s > 1$, $h_s$ is just

$$h_s(\xi, \eta) = \frac{1}{\pi} \int_{-\infty}^{\infty} \left\{ \frac{2\eta^2 h_{s,1}(\xi', 0)}{[(\xi - \xi')^2 + \eta^2]^2} + \frac{\eta^2 h_{s,1}(\xi', 0)}{(\xi - \xi')^2 + \eta^2} \right\} d\xi' + A_s(\xi, \eta) \quad (3.22)$$
where \( f \) indicates that the integral is to be integrated as a distribution because the
integrand may be unbounded at \(| \xi' | = \infty \). The function \( A_* \) again is regular biharmonic
and satisfies the conditions

\[
A_*(\xi, 0) = A_{*,*}(\xi, 0) = 0.
\]

It must also satisfy the condition

\[
A_*(\xi, \eta) = O(\rho^{s-1}) \quad \text{as} \quad \rho \to \infty
\]

so that \( \epsilon^{s+2} A_*(\xi, \eta) \) can be matched with the outer expansion. It follows from (3.23)
and (3.24) that

\[
A_*(\xi, \eta) = \sum_{m=1}^{s-1} [C_{m}C_{m}(\theta) + S_{m}S_{m}(\theta)] \rho^m,
\]

where \( C_{m} \), \( S_{m} \) are constants and \( S_{s_2} = 0 \). The sum in (3.25) is taken from \( m = 2 \)
because no biharmonic function of the form \( f(\theta) \rho \) can be found to satisfy the required
conditions. An immediate consequence is that \( A_2 = 0 \). This suggests that \( h_2 \) can be
calculated explicitly. The calculation, however, is not given here.

The constants \( C_{m} \) and \( S_{m} \) are so far undertermined. It is the purpose of matching
to determine them. For this purpose we need to know the outer expansion of the inner
expansion. To find the behavior of the inner expansion for large \( \rho \), we use (3.22) together
with \( h_0 \) and \( h_1 \) given by (3.13) and (3.20). Then we can prove inductively that

\[
h_*(\xi, \eta) = O(\rho^{s-1}), \quad \rho \to \infty, \quad s \geq 1,
\]

and

\[
h_*(\xi, \eta) \sim \sum_{m=-\infty}^{s-1} I_{s_m}(\theta) \rho^m, \quad \rho \to \infty, \quad s \geq 1.
\]

Furthermore

\[
\frac{1}{2} [\xi^2 + (\eta - 1)^2] \ln [\xi^2 + (\eta - 1)^2] + h_0(\xi, \eta) \sim \sum_{m=-\infty}^{s-1} I_{s_m}(\theta) \rho^m.
\]

The fact that \( h_* \) is harmonic together with (3.25) implies that

\[
I_{s_m}(\theta) = (A_{s_m} + S_{s_m})S_{s_m}(\theta) + (B_{s_m} + C_{s_m})C_{s_m}(\theta) + \tilde{A}_{s_m}\tilde{S}_{s_m}(\theta) + \tilde{B}_{s_m}\tilde{C}_{s_m}(\theta)
\]

where \( S_{s_2} = 0 \), and \( S_{s_m} = C_{s_m} = 0 \) unless \( s \geq 2 \) and \( 2 \leq m \leq s - 1 \). The constants
\( A_{s_m} \), \( B_{s_m} \), \( \tilde{A}_{s_m} \) and \( \tilde{B}_{s_m} \) are determined by the integral in (3.22).

We now use (3.27) and (3.28) to obtain the expansion of \( \epsilon^{-2} G(\xi, \eta, \epsilon) \) for \( \rho \) large.
It is

\[
\epsilon^{-2} G(\xi, \eta, \epsilon) \sim \sum_{s=0}^{\infty} \frac{\xi^s}{s!} \sum_{m=-\infty}^{s-1} I_{s_m}(\theta) \rho^m.
\]

Next we set \( (\xi, \eta, \rho) = \epsilon^{-1/2}(\sigma, \nu, \lambda) \) in (3.30). Then upon rearranging the series, we get

\[
\epsilon^{-2} G(\epsilon^{1/2} \sigma, \epsilon^{1/2} \nu, \epsilon) \sim \sum_{k=1}^{\infty} \epsilon^{k/2} \sum_{n=0}^{k-1} \frac{1}{n!} [A_{n, 2n-k} + S_{n, 2n-k}] S_{2n-k}(\theta)
\]

\[
+ (B_{n, 2n-k} + C_{n, 2n-k})C_{2n-k}(\theta) + \tilde{A}_{n, 2n-k}\tilde{S}_{2n-k}(\theta) + \tilde{B}_{n, 2n-k}\tilde{C}_{2n-k}(\theta)] \lambda^{2n-k}.
\]

We note that (3.31) satisfies asymptotically the boundary condition on \( \nu = \beta(\sigma, \epsilon^{1/2}) \)
for all values of \( C_{mn} \) and \( S_{mn} \), which are still undetermined.
4. Matching. To match the inner and outer expansions, we note that the left sides of (2.12) and (3.31) are the same. Therefore the right sides must be asymptotically equal. This yields
\[
\frac{1}{(p - q)!} \{ \tilde{A}_{p-q,q} \text{ or } \tilde{B}_{p-q,q} \} = \frac{1}{p!} \{ A_{pq} \text{ or } B_{pq} \}, \quad p \geq 0, \; q \leq p - 1, \quad (4.1)
\]
\[
\frac{1}{(p - q)!} \{ A_{p-q,q} \text{ or } B_{p-q,q} \} = \frac{1}{p!} \{ A_{pq} \text{ or } B_{pq} \}, \quad p \geq 0, \; q \leq 1, \quad (4.2)
\]
\[
\frac{1}{(p - q)!} A_{p-q,q} = \frac{1}{p!} (A_{pq} + S_{pq}), \quad p > 2, \; 2 \leq q \leq p - 1, \quad (4.3)
\]
\[
\frac{1}{(p - q)!} B_{p-q,q} = \frac{1}{p!} (B_{pq} + C_{pq}), \quad p > 2, \; 2 \leq q \leq p - 1. \quad (4.4)
\]
The constants \( A, B, \tilde{A} \text{ and } \tilde{B} \) are known, and hence (4.1) and (4.2) yield the corresponding \( A', B', \tilde{A}' \text{ and } \tilde{B}' \). It remains to be shown that the constants \( A' \text{ and } B' \) associated with (4.3) and (4.4) can be determined from the outer expansion. Then \( C_{pq} \text{ and } S_{pq} \) can be found from (4.3) and (4.4) to complete the inner expansion.

The function \( G_n \) defined by (2.3) can be written as
\[
G_n(x, y) = K_n(x, y) + H_n(x, y) \quad (4.5)
\]
where
\[
K_n(x, y) = \sum_{m=-n}^{-1} J_{nm}(\theta) r^m, \quad H_n = \sum_{m=0}^{\infty} J_{nm}(\theta) r^m \quad (4.6)
\]
and \( J_{nm} \) is defined by (2.4). We note that the coefficients involved in \( K_n \) are completely determined by (4.1) and (4.2).

Let \( R_n(x, y) \) be a regular biharmonic function defined by
\[
\nabla^4 R_n = 0, \quad (x, y) \in D, \quad (4.7)
\]
\[
R_n = -K_n, \quad \partial_n R_n = -\partial_n K_n, \quad (x, y) \in \partial D, \quad r \neq 0. \quad (4.8)
\]
The functions \( R_n, n \geq 1, \) are well defined and can be determined by solving the series of boundary-value problems defined by (4.7) and (4.8). The following lemma enables us to determine the unknown coefficients involved in \( H_n \) in terms of the coefficients in the expansions of \( R_n \).

**Lemma.** \( R_n(x, y) = H_n(x, y) \).

**Proof.** This follows by setting \( G_n = K_n + R_n \) and noting that \( G_n \), together with its normal derivative, vanishes on \( \partial D \).

With this lemma, the left-hand sides of (4.3) and (4.4) are completely determined. The coefficients \( C_{pq} \text{ and } S_{pq} \) can now be found.

**References**