PASSIVITY AND LINEAR SYSTEM STABILITY*

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Abstract. Using the network concept of passivity (or positive realness), new criteria for stability and instability of linear systems (with time-varying coefficients) are derived.

1. Introduction. Consider a system governed by the linear differential equation

$$p(D)y + \sum_{i=1}^{\eta} k_i(t)q_i(D)y = 0 \quad \text{on} \quad [t_0, \infty), \tag{1}$$

where

$$p(D) = D^n + p_{n-1}D^{n-1} + \cdots + p_0$$
$$q_i(D) = q_{i,m}D^m + q_{i,m-1}D^{m-1} + \cdots + q_{i,0}$$
$$q(D) = \sum_{i=1}^{\eta} q_i(D) = q_mD^m + q_{m-1}D^{m-1} + \cdots + q_0$$

are constant-coefficient differential operators with the order n of p(D) at least one higher than the order m of $q(D)$, and $k_i(t)$ are real not necessarily continuous but L-integrable on $[t_0, \infty)$. Let $G(s) = q(s)/p(s)$, $y = x_1$, $x_2 = dx_1/dt$, $\cdots$ $x_n = dx_{n-1}/dt$, and $\mathbf{x} =$ col $[x_1, x_2, \cdots, x_n]$. Then (1) can be written as the vector differential equation

$$d\mathbf{x}/dt = A(t)\mathbf{x} \tag{2}$$

where $A(t)$ is a $n \times n$ matrix.

Notation. $||\mathbf{x}||$ denotes the norm of $\mathbf{x}$ where $||\mathbf{x}||^2 = \mathbf{x}'\mathbf{x}$; $\mathbf{x}_0$ denotes $\mathbf{x}(t_0)$; $\mathbf{x}(t, t_0, \mathbf{x}_0)$ denotes the solution of (2) which takes the value $\mathbf{x}_0$ at $t = t_0$.

Definition 1. The null solution of (2) is said to be exponentially stable if there exist positive constants ϵ_1, ϵ_2 such that for $t \geq t_0$,

$$||\mathbf{x}(t, t_0, \mathbf{x}_0)|| \leq \epsilon_2 ||\mathbf{x}_0|| \exp(-\epsilon_1(t - t_0)). \tag{3}$$

Remark. If the constant ϵ_1 in (3) is zero, then the null solution of (2) is said to be stable.

Definition 2. The null solution of (2) is said to be completely unstable if there exist positive constants ϵ_1 and ϵ_2 such that for $t \geq t_0$

$$||\mathbf{x}(t, t_0, \mathbf{x}_0)|| \geq \epsilon_2 ||\mathbf{x}_0|| \exp(+\epsilon_1(t - t_0)). \tag{4}$$

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Definition 3. A real function of a complex variable $Z(s) = m(s)/n(s)$, where $m(s)$ and $n(s)$ are finite polynomials in s, is positive real if (i) $n(s) + m(s)$ has no zeros in the closed right halfplane (Re $s \geq 0$), and (ii) Re $Z(j\omega) \geq 0$ for all real ω.

Assumption 1. For some positive (known) constant K, and for some $\beta \geq 0$, $G(s - \beta) + (1/K)$ is positive real.

Problem 1. Based on assumption 1, find conditions for the null solution of (1) to be exponentially stable (stable) when $k_i(t)$ takes values in $(-\infty, \infty)$, $i = 1, 2, \cdots, \eta$.

Assumption 2. For no positive value of K is $G(s) + (1/K)$ positive real, but for a known value of $\beta > 0$ and a known $K > 0$, $G(s + \beta) + (1/K)$ is positive real.

Problem 2. Based on Assumption 2, find conditions for the null solution of (1) to be completely unstable when $k_i(t)$ takes values in $(-\infty, \infty)$, $i = 1, 2, \cdots, \eta$.

First we consider Problem 1, and in Sec. 3 we take up Problem 2.

A number of stability results concerning (1) are known. We now present the relevant few for comparison with the result of the present paper (for reference to the existing instability results, see Sec. 3 below).

THEOREM 1. (Dini–Hukuhara). If $k_i(t)$, $0 \leq t < \infty$ are measurable functions and

$$\int_0^\infty |k_i(t) - c_i| \, dt < \infty, \qquad i = 1, \cdots, \eta;$$

if c_i are real numbers such that

$$p(D)y + \sum_{i=1}^{\eta} c_i q_i(D)y = 0 \tag{5}$$

has all solutions bounded on $[0, \infty)$, then the system (1) also has all solutions bounded.

THEOREM 2 (Cesari). If the real-valued continuous functions $k_i(t)$, $t_0 \leq t < \infty$ are of bounded variation in $[t_0, \infty)$ and $k_i(t) \to 0$ as $t \to \infty$; if c_i are real numbers such that Eq. (5) has all solutions bounded in $[t_0, \infty)$; if the roots $\varphi_i(t)$ (functions of t) of the algebraic equation

$$p(\varphi) + \sum_{i=1}^{\eta} k_i(t) q_i(\varphi) = 0 \tag{6}$$

have real parts Re $[\varphi_i(t)] \leq 0$ for all $t_0 \leq t \leq \infty$, then all solutions of Eqs. (1) are bounded on $[t_0, \infty)$.

Infante [1a] has obtained, using the second method of Lyapunov, stability criteria for an equation of the type

$$\frac{d^n y}{dt^n} + \beta_{n-1}(t) \frac{d^{n-1} y}{dt^{n-1}} + \cdots + \beta_1(t) \frac{dy}{dt} + \beta_0(t) y = 0$$

(where the $\beta_i(t)$ are real continuous functions). The criteria depend on n parameters which determine a family of elliptic paraboloids in the n-dimensional space $\beta_i(t)$.

Infante [1b] and Infante and Plaut [9] consider the system

$$d\mathbf{x}/dt = (A_0 + F(t))\mathbf{x} \tag{7}$$

where A_0 is a constant $n \times n$ stable matrix and $F(t)$ is a $n \times n$ time-varying matrix, and derive asymptotic stability conditions in terms of the maximum eigenvalue of $(A_0' + F' + BA_0 + FB^{-1})$ where B is some positive definite (constant) matrix. Infante [1b] presents the idea of finding the optimum form for B. Even for a second-order scalar differential equation (see Example 2 below), Infante concludes that the computation of an optimal matrix B is impossible and is not amenable to analysis in general.

Dickerson [8] also considers systems of the form (7) and derives criteria which, in essence, imply that the system (7) is stable if the system is asymptotically stable in the absence of $F(t)$, and either the time derivative of $F(t)$ or the time integral of $F(t)$, in some suitable sense, is small. But the idea of an optimal B is not employed.

We now apply Theorem 1 to an example and compare with Infante's [1a] criterion.

Example 1. The second-order system

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + k_0(t)y = 0 \tag{8}$$

is, according to Theorem 1, stable if

$$\int_0^\infty |k_0(t) - c_0|\, dt < \infty \tag{9}$$

for some constant $c_0 > 0$.

According to Infante [1a], the system (8) is asymptotically stable if for some $\epsilon > 0$

$$\epsilon \le k_0(t) \le 4 - \epsilon, \qquad t \ge 0. \tag{10}$$

Remarks. (i) A time-varying function $k_0(t)$ can be chosen such that it obeys inequality (10) but violates (9); (ii) Theorem 2 is not applicable if $k_0(t)$ assumes values in $(-\infty, \infty)$ and $k_0(t)$ does not to zero as $t \to \infty$; (iii) The inequality (10) is no different from that obtained by the so-called circle-criterion (see references in [2]). Brockett [3] has shown that (10) is a conservative bound on $k_0(t)$ and that the system (8) is stable if $0 \le k_0(t) \le 11.6$, $t \ge 0$.

In what follows, we obtain a more general stability condition for systems of the type (1). To this end, we exploit the assumption that $G(s) + (1/K)$ is positive real for a known (positive) value of K. The method adopted is the generation of a Lyapunov function candidate (due to Brockett) from $G(s) + (1/K)$ and then the use of Corduneanu's theorem [4]. When the system is described by an equation of the type (7) it is not known how to extend Brockett's technique to generate a Lyapunov function candidate. We show how this difficulty can be overcome by employing Anderson's [7] results on positive real matrices. It is believed that the present approach is superior to those of Infante and others in avoiding the construction of an optimum matrix B. As applied to Example 1, the criterion of the paper asserts that the system (8) is stable if

$$\int_{t_0}^\infty \left| \left\{ k_0(t)\left(1 - \frac{k_0(t)}{4}\right) \right\}^{-1} \right| dt < \infty \tag{11}$$

where the superscript $^-$ denotes negative lobes of the time function inside the curly brackets. Inequality (11) includes (8) as a special case and permits $k_0(t)$ to assume values in $(-\infty, \infty)$.

2. Stability criteria. In order to state the main stability results of the paper, many preliminaries are needed. To conserve space, only a summary is given; for details, see [2].

Based on the positive realness of $G(s - \beta) + (1/K)$ for some $\beta \ge 0$, and by suitably choosing a path for integration, we can generate the following positive definite quadratic function in $\mathbf{x}$:

$$V_1(\mathbf{x}, t) = \exp(-\beta t) \int_{t(0)}^{t(\mathbf{x})} \Big\{ \{p(D-\beta)[y \exp(\beta\tau)]\} \cdot \Big[\Big(q(D-\beta) + \frac{p(D-\beta)}{K}\Big)[y \exp(\beta\tau)]\Big] - \{r_1(D)[y \exp(\beta\tau)]\}^2 \Big\} \, d\tau \tag{12}$$

where $r_1(s)$ is the negative spectral factor of $G(s - \beta) + (1/K)$ (see [2] for explanation). Let $\zeta(t)$ be a nonnegative (integrable and bounded) function on $[t_0, \infty)$ and

$$h(t) \triangleq \exp\Big(-\int_{t_0}^{t} \zeta(\tau) \, d\tau\Big). \tag{13}$$

Assume that

$$\int_{t_0}^{t} \zeta(\tau) \, d\tau \leq M < \infty \text{ for all } t \text{ in } [t_0, \infty) \quad \text{and} \quad 0 < \epsilon \leq \lim_{t \to \infty} \int_{t_0}^{t} \zeta(\tau) \, d\tau \leq M < \infty.$$

Now let

$$V(\mathbf{x}, t) \triangleq h(t) V_1(\mathbf{x}, t) \tag{14}$$

and

$$W(\mathbf{x}, t) \triangleq -\sum_{i=1}^{n} (k_i(t) q(D) y)\Big(q(D)y - \frac{1}{K} \sum_{i=1}^{n} k_i(t) q_i(D) y\Big). \tag{15}$$

Based on a property of quadratic forms, it is possible to find a function of time $\lambda(t)$ as a solution to n algebraic inequalities such that

$$\lambda(t) V_1(\mathbf{x}, t) - W(\mathbf{x}, t) \geq 0, \qquad t \geq t_0. \tag{16}$$

Then

$$W(\mathbf{x}, t) \leq \sup_{k_i} \lambda(t) V_1(\mathbf{x}, t), \qquad t \geq t_0. \tag{17}$$

The proof of the following lemma is straightforward and is hence omitted (see, for instance, [2]).

Lemma 1. Let $V_1(\mathbf{x}, t)$, $h(t)$ and $V(\mathbf{x}, t)$ be as defined by (12), (13) and (14) respectively. Then for some positive constants γ_0 and γ_1, we have

$$\gamma_0 \, ||\mathbf{x}||^2 \leq V(\mathbf{x}, t) \leq \gamma_1 \, ||\mathbf{x}||^2.$$

Further, the time derivative of $V(\mathbf{x}, t)$ along the trajectories of (1) satisfies the inequality

$$dV(\mathbf{x}, t)/dt \,|_{(1)} \leq [-2\beta + \sup_{k_i} \lambda(t) - \zeta(t)] V(\mathbf{x}, t) \tag{18}$$

which, on integrating and assuming that for some (small) constant $\epsilon > 0$ and some (arbitrarily large) positive constant M

$$\frac{1}{T} \int_{t_0}^{t_0+T} \sup_{k_i} \lambda(\tau) \, d\tau \leq 2\beta - \epsilon_0 + \frac{M}{T}, \qquad T > 0 \tag{19}$$

leads to the inequality

$$||\mathbf{x}(t)||^2 \leq \epsilon_1 \, ||\mathbf{x}_0||^2 \exp(-\epsilon_0 (t - t_0)), \qquad t \geq t_0$$

for some (positive) constant ϵ_1.

The main stability theorem and its proof are now given.

THEOREM 3. The system (1) is exponentially stable if (a) $G(s - \beta) + (1/K)$ is positive real for (known) values of $K > 0$, $\beta > 0$, and (b) $\sup_{k_i} \lambda(t)$ obtained from inequality (16) where $V(\mathbf{x}, t)$, $W(\mathbf{x}, t)$ are as defined in (12), (15) respectively, satisfies inequality (19).

Proof. As a Lyapunov–Corduneanu function candidate for (1) choose $V(\mathbf{x}, t)$ as defined in (12)–(14). Its time derivative along the trajectories of (1) satisfies inequality (18). Lemma 1 completes the proof of the theorem.

Remark 1. If there is no positive value of β for which hypothesis (a) of Theorem 3 is satisfied, then only stability of (1) can be guaranteed. In this case, inequality (19) reads:

$$\int_{t_0}^{\infty} \sup_{k_i} \lambda(\tau)\, d\tau < \infty. \tag{20}$$

Remark 2. The domain of the parameter space obtained from (19) is unbounded whereas the corresponding domain of Infante and others is finite.

Example 1 (See Introduction). $G(s) = 1/s(s + 2)$ and $G(s) + (1/4)$ is positive real for $\beta = 0$. Let $k(t) = k_0(t)\,(1 - (k_0(t)/4))$ and $k(t) = k^+(t) - k^-(t)$ where $k^+(t)$ and $(-k^-(t))$ are respectively the positive and nonpositive lobes of $k(t)$. It can be verified that $\lambda(t) = \gamma k^-(t)$, γ being a certain positive constant. Satisfaction of the integral inequality (20) guarantees stability of (8).

Example 2. Consider the system

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + k_1(t)\frac{dy}{dt} + k_0(t)y = 0. \tag{21}$$

Here $G(s) = (s + 1)/s(s + 2)$ which is positive real for $\beta = 0$. Note that $K = \infty$. Simple calculations yield $V(\mathbf{x}, t) = h(t)(x_1^2 + \frac{1}{2}x_2^2 + x_1x_2)$ and

$$W(\mathbf{x}, t) = -(k_1(t)x_2^2 + k_0(t)x_1^2 + (k_0 + k_1)(t)x_1x_2).$$

Inequality (16) becomes

$$[\lambda(t) + k_0(t)]x_1^2 + [\lambda(t) + k_0(t) + k_1(t)]x_2x_1 + \left(\frac{\lambda(t)}{2} + k_1(t)\right)x_2^2 \geq 0$$

from which

(i) $\lambda(t) \geq -k_0(t)$,

(ii) $\lambda^2(t) + 2k_1(t)\lambda(t) - (k_0(t) - k_1(t))^2 \geq 0$ (22)

We then have $\sup_{k_i} \lambda(t) = -k_0(t) + \mu(t)$ where $\mu(t) \geq 0$ is the solution of

$$\mu^2(t) + 2(k_1(t) - k_0(t))\mu(t) - k_1^2(t) \geq 0$$

obtained from (22). The system (21) is stable if

$$\int_{t_0}^{\infty} (\mu(t) - k_0(t))\, dt < \infty.$$

3. Instability criteria. If $G(s) + (1/K)$ is not positive real for any $K > 0$, but $G(s + \beta) + (1/K)$ becomes positive real for a (known) $K > 0$ and $\beta > 0$, then the system (1) may become unstable for certain $k_i(t)$s taking values in $(-\infty, \infty)$.

Below we shall derive conditions on the $k_i(t)$'s for a lower bound on $\mathbf{x}(t)$ of the form

$$||\mathbf{x}(t)|| \geq \epsilon_0 \, ||\mathbf{x}_0|| \exp (\epsilon_1(t - t_0)), \qquad t \geq t_0 \cdots \tag{23}$$

where ϵ_0 and ϵ_1 are constants and $\epsilon_0 > 0$. The solutions of (1) are then said to have Property C. If $\epsilon_1 > 0$, we have complete instability; $\epsilon_1 = 0$ implies that the system (1) is unstable in the sense of Lyapunov.

Not many results seem to be available for the instability of (1). The instability criterion of Bickart [5] is based on the assumption that the coefficients are periodic (with the same period), and is expressed in terms of bounds on $k_i(t)$ much as in [1] for stability. In essence, this instability criterion is the counterpart of the circle criterion for stability and like the latter is a sufficient condition.

For "slowly" varying systems, Skoog and Lau [6] have derived an instability criterion using the Lyapunov–Chetaev instability theorem. Their criterion reads as follows: suppose the matrix $A(t)$ in (2) has some eigenvalues in the right-half plane and all eigenvalues are bounded away from the imaginary axis; then if $\sup_{t \geq 0} ||dA/dt||$ is sufficiently small, the system (2) has unbounded solutions. Here *we do not make Skoog-and-Lau type assumptions* on the eigenvalues of $A(t)$.

Assumption. $G(s) + (1/K)$ is not positive real for any $K > 0$ but $G(s + \beta) + (1/K)$ becomes positive real for (a known) $K > 0$ and some (known) $\beta > 0$.

Problem. Find conditions for the solutions of (1) to have Property C. The method adopted to solve the above problem is an application, perhaps for the first time, of the Corduneanu theorem on instability [4].

Preliminaries. Based on the positive realness of $G(s + \beta) + (1/K)$, choosing a suitable path of integration, we can define the following positive definite function quadratic in $\mathbf{x}$:

$$J_1(\mathbf{x}, t) = \exp (2\beta t) \int_{t(o)}^{t(\mathbf{x})} \Big\{ \{p(D + \beta)[y \exp (-\beta\tau)]\} \cdot \Big[\Big(q(D + \beta) + \frac{p(D + \beta)}{K}\Big)[y \exp (-\beta\tau)]\Big] - \{r_1(D)[y \exp (-\beta\tau)]\}^2 \Big\} \, d\tau \tag{24}$$

where $r_1(s)$ is the negative spectral factor of $G(s + \beta) + (1/K)$. Now let

$$J(\mathbf{x}, t) \triangleq (h(t))^{-1} J_1(\mathbf{x}, t) \tag{25}$$

where $h(t)$ is as defined in (13), and

$$U(\mathbf{x}, t) \triangleq \Big(-\sum_{i=1}^{\eta} k_i(t) q_i(D) y\Big)\Big(q(D)y - \frac{1}{K} \sum_{i=1}^{\eta} k_i(t) q_i(D) y\Big). \tag{26}$$

Based on a property of quadratic forms, it is possible to find a function of time $\xi(t)$ such that

$$U(\mathbf{x}, t) + \xi(t) J_1(\mathbf{x}, t) \geq 0, \qquad t \geq t_0 . \tag{27}$$

Then

$$U(\mathbf{x}, t) \geq -\sup_{k_i} \xi(t) J_1(\mathbf{x}, t), \qquad t \geq t_0 . \tag{28}$$

Further, let

$$\max_{x} \{[r_1(D - \beta)y]^2 / J_1(\mathbf{x}, t)\} = \delta_m . \tag{29}$$

The following lemma (similar to Lemma 1) can be established:

LEMMA 2. Let $J_1(x, t)$, $h(t)$ and $J(x, t)$ be as defined by (24), (13) and (25) respectively. Then for some positive constants γ_0 and γ_1, we have

$$\gamma_0 \|\mathbf{x}\|^2 \leq J(\mathbf{x}, t) \leq \gamma_1 \|\mathbf{x}\|^2.$$

The time derivative of $J(\mathbf{x}, t)$ along the trajectories of (1) satisfies the inequality

$$dJ(\mathbf{x}, t)/dt \,|_{(1)} \geq [2\beta - \delta_m - \sup_{k_i} \xi(t) + \zeta(t)]J(\mathbf{x}, t) \tag{30}$$

which on integrating and assuming that for some constant ϵ_0 and some (arbitrarily large) positive constant M,

$$\frac{1}{T}\int_{t_0}^{t_0+T} \sup_{k_i} \xi(t)\, dt \leq 2\beta - \delta_m - \epsilon_0 + \frac{M}{T}, \tag{31}$$

leads to the inequality

$$\|\mathbf{x}(t)\|^2 \geq \epsilon_1 \|\mathbf{x}_0\|^2 \exp\,(\epsilon_0(t - t_0)), \qquad t \geq t_0$$

for some (positive) constant ϵ_1.

We now give the main instability theorem and its proof:

THEOREM 4. The solutions of (1) have Property C if

(a) $G(s + \beta) + (1/K)$ is positive real for known values of $K > 0$, $\beta > 0$, and

(b) $\sup_{k_i} \xi(t)$ as obtained from (27), where $J_1(\mathbf{x}, t)$ and $U(\mathbf{x}, t)$ are defined in (24) and (26) respectively, satisfies inequality (31) with δ_m given by (29).

Proof. As a Lyapunov–Corduneanu function candidate for (1) choose $J(\mathbf{x}, t)$ defined by (25). Its time derivative along the trajectories of (1) satisfies inequality (30). Invoke Lemma 2 to complete the proof of the theorem.

Remark. The parameter space guaranteeing Property C of the system (1) is unbounded and there is no restriction on the rate of variation of the $k_i(t)$s unlike the criterion of Skoog and Lau [6].

Example 3. Consider the system

$$\left(\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 5y\right) + k(t)\left(3\frac{dy}{dt} - 5y\right) = 0 \tag{32}$$

where $G(s) = (3s - 5)/(s^2 - 4s + 5)$ and $G(s + 3) = (3s + 4)/(s^2 + 2s + 2)$ is positive real.

$$J_1(\mathbf{x}, t) = \tfrac{3}{2}(-3x_1 + x_2)^2 + 11x_1^2 + 4x_1(x_2 - 3x_1).$$

$$[r_1(D - 3)y]^2 = 2(5x_1 - x_2)^2,$$

$$[r_1(D - 3)y]^2 \leq 4J_1(\mathbf{x}, t).$$

Further

$$(3x_2 - 5x_1)^2 \leq 6J_1(\mathbf{x}, t).$$

Hence

$$dJ(\mathbf{x}, t)/dt \,|_{(32)} \geq (2 + \xi(t) - 6k^+(t))J(\mathbf{x}, t)$$

where $k^+(t)$ denotes the positive lobes of $k(t)$. We conclude that the system (32) is unstable if

$$\frac{1}{T}\int_{t_0}^{t_0+T} k^+(t)\,dt \leq \frac{1}{3} - \epsilon_0 + \frac{M}{T}, \qquad T > 0,$$

for some small constant $\epsilon_0 > 0$, and some (arbitrarily large) constant $M > 0$.

4. More general systems. Consider the system of differential equations

$$d\mathbf{x}/dt = A\mathbf{x} - B\mathcal{K}(t)\boldsymbol{\sigma} \tag{33}$$
$$\boldsymbol{\sigma} = C'\mathbf{x},$$

where $\mathcal{K}(t)$ is a $n \times n$ matrix of time-varying functions $k_{ij}(t)$ assumed to be L-integrable on $[t_0, \infty)$. Eq. (33) describes a negative feedback system with $\mathcal{K}(t)$ as time varying matrix gain. The transfer matrix function of the linear time invariant part is given by $G(s) = C(sI - A)^{-1}B$. For lack of space, we deal only with the stability problem. An analysis similar to that in Sec. 3 holds for instability.

Assumption. $G(s - \beta)$ is positive real for some (known) $\beta > 0$ (see Anderson [7] for a definition of positive real matrices and other results).

Then, according to Anderson [7], there exists a positive definite matrix P and a matrix L such that

$$\begin{aligned} AP + PA &= -\beta P - LL' \\ PB &= C. \end{aligned} \tag{34}$$

For a calculation of P from (34) see Anderson [7] and Potter [10]. Let $\mathcal{K}_s(t)$ be the symmetric part of $\mathcal{K}(t)$; $\mathcal{K}_+(t)$, $-\mathcal{K}_-(t)$ denote respectively the positive definite and nonpositive definite parts of $\mathcal{K}_s(t)$. Find $\sup_{k_{ij}} \lambda(t)$:

$$\lambda(t)\mathbf{x}'P\mathbf{x} - \mathbf{x}'C\mathcal{K}_-(t)C\mathbf{x} \geq 0, \qquad t \geq t_0 .$$

Then by choosing $\mathbf{x}'P\mathbf{x}$ as a Lyapunov–Corduneanu function candidate for the system (33), it can be shown that the system (33) is exponentially stable if

$$\frac{1}{T}\int_{t_0}^{t_0+T} \sup_{k_{ij}} \lambda(\tau)\,d\tau \leq 2\beta - \epsilon_0 + \frac{M}{T}, \qquad T > 0$$

for some (small) constant $\epsilon_0 > 0$, and (arbitrarily large) positive M.

Remark 1. Brockett somewhere comments that the method of path-independent integrals for generation of a Lyapunov function candidate has not been extended to matrix positive real functions. An interpretation of Anderson's result [7] avoids the use of such integrals. Such an interpretation is believed to be of independent interest in the stability analysis of systems governed by partial differential equations.

Remark 2. If $G(s - \beta) + \mathcal{K}_0^{-1}$ is positive real for known values of $\beta > 0$ and constant matrix $\mathcal{K}_0$, then in the stability analysis of (33) it is sufficient to replace $\mathcal{K}(t)$ by $\mathcal{K}(t)(I - \mathcal{K}_0^{-1}\mathcal{K}(t))^{-1}$.

References

[1] E. F. Infante, (a) *Stability criteria for nth order homogeneous linear differential equations*, in *Differential equations and dynamical systems* (ed. J. P. LaSalle), Academic Press, New York, 1967, pp. 309–321; (b) *On the stability of some linear nonautonomous systems*, J. Appl. Mech. **35**, 7–12 (1968)

[2] Y. V. Ventakesh, *Global variation criteria for the stability of linear time varying systems*, SIAM J. Control **9**, 431–440 (1971)

[3] R. W. Brockett, *Optimisation theory and the converse of the Circle Criterion*, Proc. 1965 NEC, pp. 697–701

[4] C. Corduneanu, *The application of differential inequalities to the theory of stability*, An. Sti. Uni. Al. I. Cuza Iasi Sect. I (N.S.) **6**, 47–58 (1960)

[5] T. A. Bickart, *Periodically time varying system: an instability criterion*, Proc. IEEE **55**, no. 12, 2057 (1967)

[6] R. A. Skoog and C. C. G. Lau, *Instability of slowly varying systems*, IEEE Trans. Automatic Control **AC-17**, 86–92 (1972)

[7] B. D. O. Anderson, *A system theory criterion for positive real matrices*, SIAM J. Control **5**, 171–182 (1967)

[8] J. R. Dickerson, *Stability of systems with parametric excitation*, J. Appl. Mech. **37**, 228–230 (1970)

[9] E. F. Infante and R. Plaut, *Stability of a column subjected to a timc dependent axial load*, AIAA J. **7**, 766–768 (1969)

[10] J. F. Potter, *Matrix quadratic solutions*, SIAM J. Appl. Math. **14**, 496–501 (1966)