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A RESULT ON THE SINGULARITIES OF MATRIX FUNCTIONS*

BY

WILLIAM T. REID *(The University of Oklahoma)*

Abstract. For $F(t) = F(t_1, \cdots, t_p)$ an $n \times n$ complex-valued matrix function which is continuous on an open neighborhood of $t^0 = (t_\alpha{}^0)$ $(\alpha = 1, \cdots, p)$ and singular at t^0, there is presented a necessary and sufficient condition for $F(t)$ to be non-singular on a deleted neighborhood of t^0. If, in addition, $F(t)$ is differentiable at t^0 then a corollary to this criterion yields a differential condition that is sufficient for such isolation of a point of singularity. Applications of corollary are given, including in particular for $n = 1$ the correction of a result stated by George and Gunderson [1].

1. Introduction. Suppose that $F(t) = F(t_1, \cdots, t_p)$ is an $n \times n$ complex-valued matrix function which is continuous on an open neighborhood $\mathfrak{N}(t^0; \delta) = \{t : |t_\alpha - t_\alpha{}^0| < \delta, \alpha = 1, \cdots, p\}$ of t^0, and $F(t^0)$ is of rank $n - r$ $(0 < r \leq n)$; indeed, in our discussion it will be supposed that $r < n$, as the case $r = n$ is entirely similar with obvious simplifications due to the non-appearance of certain matrices and matrix functions. Then there exist $n \times r$ matrices G and H of rank r such that $F(t^0)G = 0$, $F^*(t^0)H = 0$, where in general M^* denotes the conjugate transpose of a matrix M. For W an $n \times (n - r)$ matrix such that the $n \times n$ matrix $[G\ W]$ is non-singular the matrix $F(t^0)W$ is of rank $n - r$ and the $n \times n$ matrix $[H\ F(t^0)W]$ is non-singular. Moreover, since $F(t)$ is continuous at t^0 there exists a δ_1 satisfying $0 < \delta_1 < \delta$ and such that the matrix $[H\ F(t)W]$ remains non-singular on $\mathcal{J} = \mathfrak{N}(t^0; \delta_1) = \{t : |t_\alpha - t_\alpha{}^0| \leq \delta_1, \alpha = 1, \cdots, n\}$, and a simple argument (see, for example, Mathis [2]) yields the existence of an $n \times r$ matrix function $H(t)$ which on $\mathcal{J}$ is continuous, of rank r, and satisfies

$$H(t^0) = H;\ H^*(t)F(t)W = 0 \text{ for } t \in \mathcal{J}. \tag{1.1}$$

The criterion mentioned in the abstract is as follows.

THEOREM 1.1. For $F(t)$, G, H, $H(t)$ and δ_1 as specified above, the matrix function $F(t)$ is non-singular on the deleted neighborhood $\mathcal{J}^d(t_0; \delta_1) = \{t : t \in \mathcal{J}, t \neq t^0\}$ if and only if the $r \times r$ matrix function

$$H^*(t)F(t)G = H^*(t)[F(t) - F(0)]G \tag{1.2}$$

is non-singular for $t \in \mathcal{J}^d(t^0; \delta_1)$.

As a ready consequence of this theorem one has the following result.

COROLLARY. If in addition to the conditions above stated the matrix function $F(t)$ is differentiable at t^0 and the $r \times r$ matrix

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$$H^*dF(t^0;\ \lambda)G = H^*\left[\sum_{\alpha=1}^{p} F_{t_\alpha}(t^0)\lambda_\alpha\right]G \tag{1.3}$$

is non-singular for

$$|\lambda| \equiv (|\lambda_1|^2 + \cdots + |\lambda_p|^2)^{1/2} = 1, \tag{1.4}$$

then there exists a δ_0, $0 < \delta_0 \leq \delta_1$ such that $F(t)$ is non-singular on the deleted neighborhood $\mathcal{G}^d(t^0;\ \delta_0) = \{t : |t_\alpha - t_\alpha{}^0| \leq \delta_0,\ t \neq t^0\}$. In particular, if $p = 1$ this condition may be written as the condition that $H^*F'(t^0)G$ is non-singular.

2. Proofs of Theorem 1.1 and its Corollary. It is to be remarked that in view of the choice of δ_1 so that the $n \times n$ matrix $[H\ F(t)W]$ is non-singular for $t \in \mathcal{G}$ it follows that the $(n - r) \times (n - r)$ matrix $W^*F^*(t)F(t)W$ is non-singular for such t, and consequently

$$H(t) = \{E - F(t)W[W^*F^*(t)F(t)W]^{-1}W^*F^*(t)\}H$$

is an $n \times r$ matrix function which on $\mathcal{G}$ is continuous, of rank r, and satisfies (1.1) for $t \in \mathcal{G}$. The result of Theorem 1.1 is then a direct consequence of the fact that in view of (1.1) the matrix product $[H(t)\quad F(t)W]^*F(t)[G\quad W]$ is of the form

$$\begin{bmatrix} H^*(t)F(t)G & 0 \\ W^*F^*(t)F(t)G & W^*F^*(t)F(t)W \end{bmatrix}.$$

In case $F(t)$ is also differentiable at t^0, and the matrix (1.3) is non-singular for all λ satisfying (1.4), there are positive constants k_1, k_2 such that

$$0 < k_1 \leq |det[H^*dF(t^0;\ \lambda)G]| \leq k_2$$

for λ satisfying (1.4), and the non-singularity of $H^*(t)F(t)G = H^*(t)[F(t) - F(t^0)]G$ on a deleted neighborhood of t^0 follows from the fact that for $t \in \mathcal{G}^d(t^0;\ \delta_1)$ and $\alpha, \beta = 1, \cdots, n$, $\gamma = 1, \cdots, r$ we have

$$|t - t^0|^{-1}[F_{\alpha\beta}(t) - F_{\alpha\beta}(t^0)] = dF_{\alpha\beta}(t^0; |t - t^0|^{-1}[t - t^0]) + o\{1\},\ H_{\alpha\gamma}(t) = H_{\alpha\gamma} + o\{1\}$$

as $t \to t^0$, and hence

$$|t - t^0|^{-r}\ det[H^*(t)F(t)G] = det[H^*dF(t^0;\ |t - t^0|^{-1}(t - t^0))G] + o\{1\}$$

as $t \to t^0$.

3. Applications of Corollary. For a first application of the result of the above Corollary in the case of $p = 1$, consider a two-point boundary problem for a vector ordinary differential equation of the form

$$y'(t) = A(t)y(t),\ B_1y(a) + B_2y(b) = 0 \tag{3.1}$$

wherein $A(t)$ is a continuous $n \times n$ matrix function on an interval $J : a \leq t < b_0$, B_1 and B_2 are $n \times n$ matrices such that the $n \times 2n$ matrix $[B_1\ B_2]$ is of rank n, and $a \leq b < b_0$. If $Y(t)$ is a fundamental matrix solution of $Y'(t) = A(t)Y(t)$ satisfying $Y(a) = E$, and $F(t) = B_1 + B_2Y(t)$ for $a \leq t < b_0$, then (3.1) has a non-trivial solution if and only if $F(b)$ is singular. Moreover, if $F(b)$ is non-singular then for arbitrary n-dimensional vectors η and continuous n-dimensional vector functions $f(t)$ the non-homogeneous system

$$y'(t) = A(t)y(t) + f(t), \; B_1y(a) + B_2y(b) = \eta, \tag{3.2}$$

has a unique solution. Now suppose that $a \leq t^0 < b_0$, with $F(t^0)$ of rank $n - r$, $0 < r \leq n$, and G, H are $n \times r$ matrices of rank r satisfying $F(t^0)G = 0$, $F^*(t^0)H = 0$. Application of the above Corollary then yields the result that if the $r \times r$ matrix $H^*B_2A(t^0)Y(t^0)G$ is non-singular then there exists a $\delta_1 > 0$ such that if $0 < |b - t^0| \leq \delta_1$ and $b \geq a$, then (3.1) has only the identically vanishing solution, and (3.2) has a unique solution for arbitrary η and continuous $f(t)$. In the special case of $t^0 = a$ we have the following result. If $B_1 + B_2$ is of rank $n - r$ and G, H are $n \times r$ matrices of rank r satisfying $[B_1 + B_2]G = 0$, $[B_1^* + B_2^*]H = 0$, then in case the $r \times r$ matrix $H^*B_2A(a)G$ is non-singular there exists a $\delta_1 > 0$ such that if $a < b \leq a + \delta_1 < b_0$ then (3.1) has only the identically vanishing solution and (3.2) has a unique solution for arbitrary η and continuous $f(t)$. This last statement is the correct answer to a problem considered by George and Gunderson [1], and for which their stated result is incorrect. As presented in the Corollary to the Theorem of [1], if $n = 2k$ and

$$A(a) = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B_1 = \begin{bmatrix} B_{11} & 0 \\ 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 0 \\ B_{21} & 0 \end{bmatrix}, \tag{3.3}$$

where A_{ij}, $(i, j = 1, 2)$, and B_{11}, B_{21} are $k \times k$ matrices with B_{11} and B_{21} non-singular, the criterion of [1] yields the statement that for arbitrary η and continuous $f(t)$ the system (3.2) has a unique solution for $b > a$ and $b - a$ sufficiently small whenever $B_{21}A_{12}$ is a non-zero matrix. The argument presented is invalid and the criterion given is false, however, as is demonstrated by the particular example of $k = 2$, $n = 4$, $A_{11} = A_{21} = A_{22}$ the 2×2 zero matrix,

$$A_{12} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

while $B_{11} = B_{21}$ is the 2×2 identity matrix. For (3.1) the system $n = 2k$ with $A(a)$, B_1, B_2 as in (3.3), the criterion of the above Corollary is that the matrix A_{12} be non-singular.

As another application of the result of the Corollary with $p = 1$, consider a Hamiltonian differential system

$$-v'(t) + C(t)u(t) - A^*(t)v(t) = 0, \; u'(t) - A(t)u(t) - B(t)v(t) = 0, \tag{3.4}$$

with $n \times n$ coefficient matrix functions satisfying on a non-degenerate interval $\mathcal{J}$ of the real line the following hypothesis.

> $A(t)$, $B(t)$, $C(t)$ are continuous with $B(t)$ and $C(t)$ Hermitian and $B(t)$ non-negative definite for $t \in \mathcal{J}$. Moreover, (3.4) is identically normal on $\mathcal{J}$; that is, if $u(t) \equiv 0$, $v(t)$ is a solution of (3.4) on a non-degenerate subinterval $\mathcal{J}_0$ of $\mathcal{J}$, then $u(t) \equiv 0$, $v(t) \equiv 0$ on $\mathcal{J}$. (H)

For a discussion of such systems and a detailed treatment of the concepts utilized here, the reader is referred, in particular, to Chapter VII of [3]. Let $(U(t); V(t))$ be a conjoined basis for (3.4); that is, the matrix functions $U(t)$, $V(t)$ are $n \times n$, with the column vector functions of the $2n \times n$ matrix function

$$Y(t) = (U(t); V(t)) = \begin{bmatrix} U(t) \\ V(t) \end{bmatrix}$$

linearly independent solutions of (3.4), while the necessarily constant matrix function $V^*(t)U(t) - U^*(t)V(t)$ is the zero matrix. A value $t^0 \in \mathcal{J}$ is called a *focal point* of the conjoined basis $Y(t)$ of order r if $U(t^0)$ is of rank $n - r$. If G is an $n \times r$ matrix of rank r

such that $U(t^0)G = 0$, then $0 = V^*(t^0)U(t^0)G = U^*(t^0)V(t^0)G$, so that $H = V(t^0)G$ is an $n \times r$ matrix satisfying $U^*(t^0)H = 0$, and H is of rank r since $Y(t^0)$ is of rank n. Since

$$H^*U'(t^0)G = H^*[A(t^0)U(t^0) + B(t^0)V(t^0)]G = H^*B(t^0)H,$$

with the aid of the above Corollary it then follows that if in addition to the conditions of hypothesis (H) the matrix $B(t)$ is positive definite for $t = t^0$, then $t = t^0$ is an isolated focal point of $Y(t)$. In particular, whenever $B(t)$ is positive definite for all $t \in \mathcal{J}$ then all focal points of a conjoined basis are isolated. For an alternate proof of this latter result the reader is referred to Lemma 7.1 in Chapter VII of [3].

A related, but not equivalent, criterion for the isolation of focal points is attainable in the generalization of the polar coordinate method for the extension of the Sturmian theory emanating from the work of Lidskiĭ, as discussed in Chapter X of Atkinson [4]. For a Hamiltonian system (3.4) with coefficient matrices satisfying hypothesis (H), consider again a conjoined basis $Y(t) = (U(t); V(t))$. Then the $n \times n$ matrix functions $\hat{U}(t) = V(t) - iU(t)$, $\hat{V}(t) = V(t) + iU(t)$ are non-singular on $\mathcal{J}$, and on this interval the matrix function $\Theta(t) = \Theta(t \mid Y) = \hat{V}(t)\hat{U}^{-1}(t)$ is unitary and satisfies the matrix differential equation

$$\Theta'(t) = i\Theta(t)N(t \mid Y) \tag{3.5}$$

where $N(t \mid Y)$ is the Hermitian matrix function $-2\hat{U}^{*-1}(t)Y^*(t)\mathcal{A}(t)Y(t)\hat{U}^{-1}(t)$, and $\mathcal{A}(t)$ is the $2n \times 2n$ hermitian matrix function

$$\mathcal{A}(t) = \begin{bmatrix} C(t) & -A^*(t) \\ -A(t) & -B(t) \end{bmatrix}.$$

As $\Theta(t)$ is unitary, for $t \in \mathcal{J}$ all of its eigenvalues are on the unit circle $|\lambda| = 1$ in the complex λ-plane, and under hypothesis (H) the focal points of $Y(t)$ are those values of t for which $\lambda = 1$ is an eigenvalue of $\Theta(t)$. If $\lambda = 1$ is an eigenvalue of $\Theta(t^0)$ of index r and G is an $n \times r$ matrix of rank r such that $[\Theta(t^0) - E]G = 0$, then $[\Theta(t^0)G]^*[\Theta(t^0) - E] = G^*[E - \Theta^*(t^0)] = -([\Theta(t^0) - E]G)^* = 0$, so that $H = \Theta(t^0)G$ is of rank r and satisfies $[\Theta(t^0) - E]^*H = 0$. Consequently, in view of the Corollary there is a deleted neighborhood of t^0 on which $\lambda = 1$ is not an eigenvalue of $\Theta(t)$, and t is not a focal point of $Y(t)$, whenever $-iG^*\Theta^*(t^0)\Theta'(t^0)G = G^*N(t^0 \mid Y)G$ is non-singular.

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