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STRONG ELLIPTICITY AND VAN HOVE'S LEMMA IN INHOMOGENEOUS MEDIA*

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1. Introduction. Let D be a bounded regular space domain, and let $\mathcal{C}$ be a fourth-order tensor-valued function on $\bar{D}$ with coefficients C_{ijkl} in $C^1(\bar{D})$. If the coefficients C_{ijkl} are regarded as elasticities, then a sufficient condition for uniqueness of solutions of the elastostatic displacement boundary-value problem in D is the existence of a positive constant c for which the inequality

$$\int_D C_{ijkl} v_{i,j} v_{k,l} \, dV \geq c \int_D |\nabla \mathbf{v}|^2 \, dV \tag{1}$$

holds for all $\mathbf{v} \in H_0^1(D)$. (In this expression, $v_{i,j}$ denotes the derivative of the ith component of the vector $\mathbf{v}$ with respect to x_j. Summation convention is used throughout this note.) Such an inequality is easily obtained by assuming that the coefficients C_{ijkl} are such that there exists a positive constant c_0 for which

$$C_{ijkl}(\mathbf{x}) \xi_{ij} \xi_{kl} \geq c_0 \xi_{ij} \xi_{ij} \tag{2}$$

for every tensor ξ_{ij} and every $\mathbf{x}$ in D (see, for example, Fichera [2] or Knops and Payne [3]).

A weaker assumption than the inequality (2) is the assumption that $\mathcal{C}$ is *uniformly strongly elliptic*, i.e., that there exists a constant c_1 for which

$$C_{ijkl}(\mathbf{x}) \alpha_i \beta_j \alpha_k \beta_l \geq c_1 |\boldsymbol{\alpha}|^2 |\boldsymbol{\beta}|^2 \tag{3}$$

for all vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ and all $\mathbf{x}$ in D. If the major symmetry condition $C_{ijkl} = C_{klij}$ is satisfied, then Wheeler [5] has shown that (3) implies uniqueness of solutions of the elastodynamic displacement boundary-value problem. However, Edelstein and Fosdick [1] have shown by example that uniform strong ellipticity alone is not sufficient in general to guarantee uniqueness for the elastostatic displacement boundary-value problem, although uniqueness can be regained in certain circumstances with a few additional assumptions [3].

Suppose that the elastic medium is homogeneous, i.e. that the elasticities C_{ijkl} are independent of $\mathbf{x}$ in D. Then one can establish without difficulty the following lemma [3].

VAN HOVE'S LEMMA: If $\mathcal{C}$ is uniformly strongly elliptic, then an inequality of the form (1) holds for all $\mathbf{v} \in H_0^1(D)$.

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It follows from this lemma that, in a homogeneous medium, uniform strong ellipticity implies uniqueness for the elastostatic displacement boundary-value problem.

It is apparent that Van Hove's Lemma is not valid as stated if the coefficients C_{ijkl} are allowed to vary arbitrarily in D. (Otherwise, uniform strong ellipticity would imply uniqueness for the elastostatic displacement boundary-value problem in an arbitrary inhomogeneous medium, contradicting the Edelstein-Fosdick example.) Out of both mathematical curiosity and a desire to shed light on questions of uniqueness for the elastostatic displacement boundary value problem, one is led to ask what becomes of Van Hove's Lemma in an arbitrary inhomogeneous medium. In the following, we first show that Van Hove's Lemma remains valid as stated, provided that the coefficients C_{ijkl} are "nearly constant" in D in a certain sense. We then outline the construction of a set of coefficients C_{ijkl} which are such that no inequality of the form (1) can hold for all $\mathbf{v} \in H_0^1(D)$. (Of course, such a set of coefficients can be recovered from the Edelstein-Fosdick example. However, we feel that the direct construction given here better illustrates why the algebraic condition of strong ellipticity fails in general to guarantee the analytic inequality (1).) Finally, we offer a generalization of Van Hove's Lemma which is valid in inhomogeneous media. Specifically, we show that if $\mathcal{C}$ is strongly elliptic in D, then it is possible to salvage an inequality of the form (1) for functions $\mathbf{v} \in H_0^1(D)$ which satisfy a finite set of orthogonality conditions, the number of which does not increase under small perturbations of C.

2. Van Hove's Lemma for tensors with "nearly constant" coefficients. In this section, we show that an inequality of the form (1) holds for all $\mathbf{v} \in H_0^1(D)$ provided $\mathcal{C}$ is uniformly strongly elliptic in D and the coefficients C_{ijkl} are "nearly constant" in a certain sense. We take the following approach to this objective: Letting $\mathcal{C}^0$ be a strongly elliptic tensor with constant coefficients, we observe that if $\mathcal{C}$ is sufficiently near $\mathcal{C}^0$ in the usual tensor norm, then not only is $\mathcal{C}$ strongly elliptic but also an inequality of the form (1) holds for $\mathcal{C}$ on $H_0^1(D)$.

Our desired result is a corollary of the following observation.

LEMMA: If C_{ijkl}^0 are constants for which an inequality (1) holds with constant c and if C_{ijkl} are functions on D, then, denoting by $\mathcal{C}$ and $\mathcal{C}^0$ the respective tensors defined by C_{ijkl} and C_{ijkl}^0,

$$\int_D C_{ijkl} v_{i,j} v_{k,l}\, dV \geq (c - \sup_D |\mathcal{C} - \mathcal{C}^0|) \int_D |\nabla \mathbf{v}|^2\, dV$$

for all $\mathbf{v} \in H_0^1(D)$.

Proof: The inequality of the lemma follows immediately from the inequality (1) and the expression

$$\int_D C_{ijkl} v_{i,j} v_{k,l}\, dV = \int_D [C_{ijkl} - C_{ijkl}^0] v_{i,j} v_{k,l}\, dV + \int_D C_{ijkl}^0 v_{i,j} v_{k,l}\, dV.$$

One sees that if the coefficients C_{ijkl} are sufficiently near the constant coefficients C_{ijkl}^0 uniformly in D, then $\sup_D |\mathcal{C} - \mathcal{C}^0| < c$, and the lemma implies that an inequality (1) holds for $\mathcal{C}$ with positive constant $(c - \sup_D |\mathcal{C} - \mathcal{C}^0|)$. It follows in turn from (1) that $\mathcal{C}$ is uniformly strongly elliptic in D. We close this section by remarking that, since both the constant c and the quantity $\sup_D |\mathcal{C} - \mathcal{C}^0|$ are calculable, one may determine quantitative limits within which the coefficients of $\mathcal{C}$ may vary while still preserving an inequality of the form (1).

3. A counterexample. If the coefficients C_{ijkl} are allowed to vary without restriction on D, then it may happen that no inequality of the form (1) can hold for all $\mathbf{v} \in H_0^1(D)$, even though $\mathcal{C}$ is uniformly strongly elliptic. Indeed, we now describe the construction of coefficients C_{ijkl} on $D \subseteq R^3$ and a function $\mathbf{v} \in H_0^1(D) \cap C^1(\bar{D})$ such that $\mathcal{C}$ is uniformly strongly elliptic and

$$\int_D C_{ijkl} v_{i,j} v_{k,l} \, dV < 0.$$

First, we define constant coefficients $C_{ijkl}{}^0$ as follows: Set

$$C_{1122}{}^0 = C_{2211}{}^0 = -2,$$
$$C_{1212}{}^0 = C_{1221}{}^0 = C_{2121}{}^0 = C_{2112}{}^0 = 2,$$
$$C_{1111}{}^0 = C_{1313}{}^0 = C_{1331}{}^0 = C_{2222}{}^0 = C_{3113}{}^0 = C_{2323}{}^0 = C_{2332}{}^0 = C_{3131}{}^0 = C_{3223}{}^0 = C_{3232}{}^0 = C_{3333}{}^0 = 1,$$

and take the remaining $C_{ijkl}{}^0$ to be 0. It is a straightforward matter to verify that there exists a positive constant c for which

$$\begin{aligned} C_{ijkl}{}^0 \alpha_i \beta_j \alpha_k \beta_l = {} & \alpha_1{}^2 \beta_1{}^2 + \alpha_2{}^2 \beta_2{}^2 + 2[\alpha_1{}^2 \beta_2{}^2 + \alpha_2{}^2 \beta_1{}^2] \\ & + (\alpha_1 \beta_3 + \alpha_3 \beta_1)^2 + (\alpha_2 \beta_3 + \alpha_3 \beta_2)^2 \\ & + \alpha_3{}^2 \beta_3{}^2 \geq c\,|\boldsymbol{\alpha}|^2\,|\boldsymbol{\beta}|^2 \end{aligned}$$

for all $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. Now for any $\mathbf{v} \in H_0^1(D)$, one calculates

$$\begin{aligned} C_{ijkl}{}^0 v_{i,j} v_{k,l} = {} & v_{1,1}{}^2 - 4 v_{1,1} v_{2,2} + v_{2,2}{}^2 + 2[v_{1,2} + v_{2,1}]^2 \\ & + [v_{1,3} + v_{3,1}]^2 + [v_{2,3} + v_{3,2}]^2 + v_{3,3}{}^2. \end{aligned}$$

If $\mathring{\mathbf{x}}$ is any point of D, then one can choose a particular $\mathbf{v} \in H_0^1(D) \cap C^1(\bar{D})$ which vanishes on ∂D and satisfies

$$v_{3,3} = v_{2,3} + v_{3,2} = v_{1,3} + v_{3,1} = v_{1,2} + v_{2,1} = 0$$
$$v_{1,1}{}^2 - 4 v_{1,1} v_{2,2} + v_{2,2}{}^2 < 0$$

at $\mathring{\mathbf{x}}$. It follows that $C_{ijkl}{}^0 v_{i,j} v_{k,l} < 0$ not only at $\mathring{\mathbf{x}}$ but, by continuity, in some neighborhood of positive radius δ about $\mathring{\mathbf{x}}$. For any $\epsilon < 0$, a continuous (scalar-valued) function φ_ϵ can be found such that

(i) $\varphi_\epsilon(\mathbf{x}) > 0$ for $\mathbf{x} \in \bar{D}$,
(ii) $\varphi_\epsilon(\mathbf{x}) = 1$ for $\mathbf{x} \in \bar{D}$ satisfying $|\mathbf{x} - \mathring{\mathbf{x}}| < \delta/2$,
(iii) $\varphi_\epsilon(\mathbf{x}) < \epsilon$ for $\mathbf{x} \in \bar{D}$ satisfying $|\mathbf{x} - \mathring{\mathbf{x}}| > \delta$.

If $\epsilon > 0$ is chosen sufficiently small and we define $C_{ijkl} = \varphi_\epsilon C_{ijkl}{}^0$, then, for our particular $\mathbf{v}$,

$$\int_D C_{ijkl} v_{i,j} v_{j,k} \, dV < 0.$$

4. A generalization of Van Hove's Lemma for inhomogeneous media. We complete this discussion by offering a generalization of Van Hove's Lemma which is valid in inhomogeneous media. Specifically, we show that if $\mathcal{C}$ is uniformly strongly elliptic in D

and if an arbitrary complete orthonomal set in $L^2(D)$ is given, then an inequality of the form (1) holds for all $\mathbf{v} \in H_0^1(D)$ which are orthogonal in $L^2(D)$ to a finite number of members of this set. In addition, we observe that if C is perturbed slightly, then an inequality of the form (1) continues to hold for the perturbed tensor and for all $\mathbf{v} \in H_0^1(D)$ which satisfy this same finite set of orthogonality conditions.

Let $\{\varphi_n\}_{n=1,2,\ldots}$ be any complete orthonormal set in $L^2(D)$. For $n = 1, 2, \cdots$, define S_n to be the span of $\{\varphi_1, \cdots, \varphi_n\}$ and let $S_n^\perp$ denote the orthogonal complement of S_n in $L^2(D)$. Let $\| \ \|_0$ and $\| \ \|_1$ denote the norms in $L^2(D)$ and $H_0^1(D)$, respectively. The following is our generalization of Van Hove's Lemma.

LEMMA: If $\mathcal{C}$ is uniformly strongly elliptic, then there exists a value of n for which an inequality of the form (1) holds for all $\mathbf{v} \in H_0^1(D) \cap S_n^\perp$.

Proof: It follows from the uniform strong ellipticity of $\mathcal{C}$ that there exist positive constants c' and c'' such that

$$c' \int_D |\nabla \mathbf{v}|^2 \, dV \leq c'' \|\mathbf{v}\|_0^2 + \int_D C_{ijkl} v_{i,j} v_{k,l} \, dV \tag{4}$$

for every $\mathbf{v} \in H_0^1(D)$. The inequality (4) is a special form of Gårding's Inequality. A derivation of (4) can be found in [5].

Now suppose that the lemma is false. Then for each positive integer n, one can find an element $\mathbf{v}^{(n)} \in H_0^1(D) \cap S_n^\perp$ such that $\|\mathbf{v}^{(n)}\|_0 = 1$ and

$$\int_D C_{ijkl} v_{i,j}^{(n)} v_{k,l}^{(n)} \, dV < \frac{1}{n} \int_D |\nabla \mathbf{v}^{(n)}|^2 \, dV$$

The inequality (4) yields

$$c' \int_D |\nabla \mathbf{v}^{(n)}|^2 \, dV \leq c'' + \frac{1}{n} \int_D |\nabla \mathbf{v}^{(n)}|^2 \, dV$$

and, for large n, one obtains

$$\|\mathbf{v}^{(n)}\|_1^2 \leq 1 + c'' \Big/ \left(c' - \frac{1}{n} \right).$$

Thus the norms $\|\mathbf{v}^{(n)}\|_1$ are bounded.

It follows from the Rellich compactness theorem* that there exists a subsequence $\{\mathbf{v}^{(n_j)}\}_{j=1,2,\ldots}$ of $\{\mathbf{v}^{(n)}\}_{n=1,2,\ldots}$ which converges in $L^2(D)$ to an element $\mathbf{v}^{(o)} \in L^2(D)$. Now $\mathbf{v}^{(o)}$ is the limit of a sequence which is eventually in $S_n^\perp$ for every n; hence, $\mathbf{v}^{(o)} \in S_n^\perp$ for every n. Since $\{\varphi_n\}_{n=1,2,\ldots}$ is complete, this implies $\mathbf{v}^{(o)} = 0$. But this is a contradiction since $\|\mathbf{v}^{(o)}\|_0 = \lim_{j\to\infty} \|\mathbf{v}^{(n_j)}\|_0 = 1$, and the lemma is proved.

We conclude with the observation that, if an inequality of the form (1) holds for a tensor $\mathcal{C}$ and all $\mathbf{v} \in H_0^1(D) \cap S_n^\perp$, then for any other tensor $\mathcal{C}'$, one has

$$\int_D C_{ijkl}' v_{i,j} v_{k,l} \, dV \geq (c - \sup_D |\mathcal{C} - \mathcal{C}'|) \int_D |\nabla \mathbf{v}|^2 \, dV$$

for all $\mathbf{v} \in H_0^1(D) \cap S_n^\perp$. The implication is that, if (1) holds on $H_0^1(D) \cap S_n^\perp$ for a tensor

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* This theorem states that bounded subsets of $H_0^1(D)$ are relatively compact in $L^2(D)$. For a proof of a general version of this theorem which uses Fourier transforms, see Lair [4].

$\mathcal{C}$, then a similar inequality holds on $H_0^1(D) \cap S_n^\perp$ for all $\mathcal{C}'$ which are sufficiently near $\mathcal{C}$ that

$$\sup_D |\mathcal{C} - \mathcal{C}'| < c.$$

In other words, the set of orthogonality conditions sufficient to guarantee an inequality of the form (1) does not suddenly increase for small perturbations of $\mathcal{C}$. Of course, this result, together with Van Hove's Lemma, implies the result of Sec. 2 for tensors with "nearly constant" coefficients. However, it should be noted that the result of Sec. 2 is given in terms of calculable quantities, while this result is given in terms of "soft" constants and, therefore, must be regarded as only qualitative in nature.

Addendum. We are grateful to the referee of this paper for observing that one can obtain a "hard" version of the lemma of Sec. 4 for a particular complete orthonormal set in $L^2(D)$ via the variational characterization of the clamped membrane eigenvalues. We reproduce his comments below.

The clamped membrane eigenvalues are the successive minima of the Rayleigh quotient

$$R(\varphi) = \int_D |\nabla\varphi|^2 \, dV / \|\varphi\|_0^2,$$

defined for non-zero $\varphi \in H_0^1(D)$. Specifically, a monotone sequence $\{\lambda_n\}_{n=1,2,\cdots}$ of these eigenvalues and the sequence of corresponding eigenfunctions $\{\mathbf{u}_n\}_{n=1,2,\ldots}$ can be found as follows: Setting $\mathbf{u}_0 = 0$ and $S_0 = \{0\}$ for convenience, define inductively, for $n = 1, 2, \cdots$,

$\lambda_n = \inf_{\varphi \in H_0^1(D) \cap S_{n-1}^\perp} R(\varphi)$,
$\mathbf{u}_n$ = any minimum of $R(\varphi)$ in $H_0^1(D) \cap S_{n-1}^\perp$ having norm 1 in $L^2(D)$,
S_n = span of $\{\mathbf{u}_i\}_{i=0,\cdots,n}$.

Clearly, $\lambda_1 \leq \lambda_2 \leq \cdots$. In fact, it is known that all λ_n are positive, that $\lim_{n\to\infty} \lambda_n = \infty$, and that $\{\mathbf{u}_n\}_{n=1,1,\ldots}$ is a complete orthonormal set in $L^2(D)$.

For $\mathbf{v} \in H_0^1(D) \cap S_n^\perp$, one has

$$\|\mathbf{v}\|_0^2 \leq \frac{1}{\lambda_{n+1}} \int_D |\nabla \mathbf{v}|^2 \, dV$$

and thus, from the inequality (4),

$$\int_D C_{ijkl} v_{i,j} v_{k,l} \, dV \geq \left[c' - \frac{c''}{\lambda_{n+1}}\right] \int_D |\nabla \mathbf{v}|^2 \, dV.$$

Since suitable constants c' and c'' can be found by direct calculation, it follows that, whenever n is sufficiently large, an inequality of the form (1) holds with a positive, calculable constant c for all $\mathbf{v} \in H_0^1(D) \cap S_n^\perp$.

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