POLYCONICS 1. POLYELLIPSES AND OPTIMIZATION*

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A. Introduction. 1. Ellipses and hyperbolas may be defined as plane curves consisting of all points the sum, or the difference, of whose distances from two fixed points F_1 and F_2, the foci, is constant. This generalizes at once to a provisional definition of a polyconic with n foci $F_1, \cdots, F_n$ in the plane: a plane locus consisting of all points the sum of whose signed distances to $F_1, \cdots, F_n$ is constant, the signatures being $+1$ or -1. If all n signatures are $+1$ the polyconic is a polyellipse or, more specifically, an n-ellipse. Thus a circle in this terminology is a 1-ellipse and an ordinary ellipse becomes a 2-ellipse.

Why bother with polyconics? There is the possible intrinsic interest, novelty, and educational value. Then, certain properties of conics themselves appear to be easier and more natural in this generalized context; for instance, the radius of curvature of a polyellipse is easier to get from first principles than is that of an ellipse. Also, the numerical and graphical treatment of polyconics introduces early, simply and naturally certain important techniques of numerical analysis, such as e.g. the variable step-length procedures. However, the most important reasons for working with polyconics appear to us to be two: a) polyconics may help to revive interest in the neglected subject of geometry, b) polyconics, and especially polyellipses, arise naturally in the treatment of an important class of optimization problems.

2. We formulate now several of these optimization problems, starting with some very simple ones, and building up by means of various generalizations. Each problem is supplied with a codified description which is supposed to reflect something of its structure.

Problem 1—$P_n(F_1, \cdots, F_n)$—how may we produce the point(s) in the Euclidean plane which would minimize the sum of n distances to n given points $F_1, \cdots, F_n$ in the plane?

Next, we rephrase the problem and add a simple geometrical constraint.

Problem 2—$P_n(F_1, \cdots, F_n; R)$—let n points $F_1, \cdots, F_n$ in the plane represent n locations to be supplied from a central depot whose position X is a point in the plane of the F_is. For technical reasons X must be confined to a region R of the plane. Where must X lie in R in order to minimize the sum of the n distances, $\sum_1^n |XF_i|$?

The tentative answer here might be obtained as follows. Suppose that we know how to solve Problem 1. Let us determine first the point which minimizes the sum of the n distances unconditionally. If it lies in R we are finished. If not, we produce the polyellipse with the foci $F_1, \cdots, F_n$ which is externally tangent to R. Now the point of tangency is the desired optimal location.

Problem 3—$P_n(F_1, \cdots, F_n; w_1, \cdots, w_n; R)$—let the n points $F_1, \cdots, F_n$ be as in Problem 2, and let a positive weight w_i be associated with the ith location F_i, $i = 1, \cdots, n$.

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Where must X lie in R so as to minimize the weighted distance sum $\sum_1^n w_i|XF_i|$?

This is a generalization of the so-called generalized Weber problem in spatial economics, dealing with the optimal location of industries. For some references to the considerable literature on Weber's problem, especially in its economic ramifications, see [1, 2, 3]. The generalization from Problems 1 and 2 to 3 suggests that we extend our provisional definition of polyconics. The loci of constant sum of signed distances, which we have introduced before as polyconics, will be called simple polyconics. In addition, we shall have the loci of constant weighted sum of signed distances

$$\left\{X: \sum_1^n \pm w_i|XF_i| = c\right\};$$

these will be called general polyconics. The weights themselves shall be always taken as positive. Since several foci of a polyconic may coincide it ought to be observed that there is no real difference between simple polyconics and general ones, and we shall often leave out the specification. With the extension of simple polyellipses to general ones, our Problem 3 may be treated just as Problem 2 before.

For the formulation of our next problem the locations F_i, instead of being given in advance, are now to be selected from n disjoint plane regions $R_1, \cdots, R_n$. In other words, instead of connecting n points $F_1, \cdots, F_n$ by a minimal-length network as in Problem 1, we now wish to connect so the n set-terminals $R_1, \cdots, R_n$. This leads to

Problem 4—$P_n(R_1, \cdots, R_n)$—how may we produce the point minimizing the sum of n distances to n given sets $R_1, \cdots, R_n$ in the plane?

There are analogous extensions of the above to the case of minimizing with constraint as in Problem 2, and to the case of weighted sum as in Problem 3. We observe that our Problem 4 suggests a still further generalization of polyconics in which the n point-foci $F_1, \cdots, F_n$ are replaced by n disjoint plane sets $R_1, \cdots, R_n$. Note that parabolas are now included among simple polyconics: we take $n = 2$, the signatures are $+1$ and -1, and R_1 is a point while R_2 is a straight line.

The next problem arises by an iterated generalization; it might be called the problem of hierarchical network minimization. The idea is to replace n points, to be optimally connected, by n clumps of points, to be optimally connected in two stages: first over each clump separately, then all the local clump networks together.

Problem 5—$P_n^2(F_{ij}; w_{ij}; w_j)$—let n finite clumps of points, F_{ij}, be given in the plane. Here j is the clump index so that $j = 1, \cdots, n$, while $i = 1, \cdots, i(j)$ enumerates the $i(j)$ points of the jth clump. Together with the points F_{ij} go the positive weights w_{ij} indexed in the same way. As before, we suppose that the points of the jth clump represent locations to be supplied from a local depot X_j, for $j = 1, \cdots, n$. In addition, the local depots are themselves to be supplied from a central depot X, and we are given positive weights $w_1, \cdots, w_n$ corresponding to, say, the costs of connecting together the local depots $X_1, \cdots, X_n$. Under these conditions we have our hierarchical network minimization problem: how may we locate the n+1 points $X_1, \cdots, X_n, X$ in the plane so as to minimize the weighted sum of distances

$$\sum_{j=1}^n \left[w_j|XX_j| + \sum_{i=1}^{i(j)} w_{ij}|X_jF_{ij}| \right]?$$

Of course, we could now introduce a three-level problem P_n^3, we could extend to set-terminals and put constraints on local depots, etc. An interesting variant arises when we do not specify n in advance but just take the $N = \sum_{j=1}{}^n(j)$ points, and make some reasonable provision for the weights. The problem arises now of producing the optimal two-stage network without being given in advance the clumping specification. That is, we consider all the distinct set-partitions of the collection of N signed points $F_1, \cdots, F_N$ into the separate nonempty sets, and we wish to determine the set-partition leading to the minimum of our weighted sum of distances.

3. This is the first article of a projected series treating the elementary geometrical, computational, and graphical properties and uses of polyconics. The interest in this first part will be to develop sufficient elementary apparatus for a geometrical overview of polyellipses, to justify the relevance of polyellipses to our optimization problems, and to start producing practicable solutions of the simpler of those problems. It is expected that succeeding parts will be addressed to algorithmic details of the optimization, to the developing of methods for solving the hierarchical types of optimization, and to further general geometry of polyconics.

B. Distance-perturbation formula. 1. We obtain first a simple distance-perturbation formula which will be quite crucial in our work. Its use will amount to applying Taylor's theorem with terms up to and including the quadratic ones. The formula we need could actually be obtained by an appeal to Taylor's theorem for a function of two variables, but it will be simpler and faster to proceed directly.

Let X be a point on a line L and F a point off L, let b be the distance $|XF|$ and θ the angle that XF makes with L. Perturb X to X_1 by giving it a small displacement ϵ along L and a small displacement η at right angles to L. A positive sense is defined on L and at right angles to L so that ϵ and η can be positive or negative, independently; this also removes the ambiguity in the determination of θ. If $b + \Delta b$ is the perturbed distance $|X_1F|$ then by Pythagoras' theorem

$$b + \Delta b = ((b\cos\theta - \epsilon)^2 + (b\sin\theta - \eta)^2)^{1/2} = b\left(1 - \frac{2\epsilon}{b}\cos\theta - \frac{2\eta}{b}\sin\theta + \frac{\epsilon^2 + \eta^2}{b^2}\right)^{1/2}.$$

On applying to the last radical the binomial theorem in the form

$$1 - x = 1 - \frac{1}{2}x + \frac{\frac{1}{2}(\frac{1}{2} - 1)}{1.2}x^2 + O(x^3) \qquad (x \text{ small})$$

we obtain

$$\Delta b = -\epsilon\cos\theta - \eta\sin\theta + \frac{\epsilon^2}{2}\frac{\sin^2\theta}{b} + \frac{\eta^2}{2}\frac{\cos^2\theta}{b} - \epsilon\eta\frac{\sin\theta\cos\theta}{b} + \cdots \tag{1}$$

Next, the same thing is done for n points $F_1, \cdots, F_n$, the angles $\theta_1, \cdots, \theta_n$ being determined in a consistent way, and the n distances from X to the F_i's being $b_1, \cdots, b_n$. Using (1) and summing over i we obtain the perturbation Δc in the sum of distances

$$c = \sum_{i=1}^{n} |XF_i| = b_1 + \cdots + b_n \tag{2}$$

resulting from the (ϵ, η) shift of X:

$$\Delta c = -\epsilon\sum_1^n \cos\theta_i - \eta\sum_1^n \sin\theta_i + \frac{\epsilon^2}{2}\sum_1^n \frac{\sin^2\theta_i}{b_i} + \frac{\eta^2}{2}\sum_1^2 \frac{\cos^2\theta_i}{b_i}$$
$$- \epsilon\eta\sum_1^n \frac{\cos\theta_i\sin\theta_i}{b_i} + \text{higher-order terms.} \tag{3}$$

This is the formula we need. Let P be the n-ellipse passing through the point X and having the foci $F_1, \cdots, F_n$; we apply our formula to finding the tangent T to P at X. We let $\eta = 0$ in (3) and we suppose that L is the tangent T. For best linear fit to P we must have $\Delta c = O(\epsilon^2)$ so that from (3)

$$\sum_1^n \cos \theta_i = 0. \tag{4}$$

This can be restated in vector terms: if X is not a focus F_i we have

$$\text{if } \bar{u} = \sum_1^n \frac{XF_i}{|XF_i|} \text{ then } T \perp \bar{u}. \tag{5}$$

This simple geometrical construction produces the tangent to P at any point X other than a focus F_i. The discussion of what happens when X is a focus F_i will be deferred for the moment, but at all other points a polyellipse P is smooth, i.e. has a unique tangent. An easy additional argument could be supplied to show that in fact P is then real-analytic, but we shall not use this. It is noted that (5) generalizes the standard properties of circles and ellipses: the tangent to a circle is at right angles to the radius through the tangency point, the tangent to an ellipse makes equal angles with the radii joining the tangency point to the foci.

The preceding exploits the first-order osculation and the best *linear* fit; the second-order osculation and the best *circular* fit are considered next. For this purpose ϵ and η are arranged so that the perturbation of X to X_1 is along a circle C tangent to both P and T at X, and of radius R, say. Therefore

$$\eta = \frac{\epsilon^2}{2R} + O(\epsilon^3)$$

so that (3) becomes

$$\Delta c = -\epsilon \sum_1^n \cos \theta_i + \frac{\epsilon^2}{2}\left(\sum_1^n \frac{\sin^2 \theta_i}{b_i} - \frac{1}{R}\sum_1^n \sin \theta_i\right) + O(\epsilon^3).$$

If C is to be the circle of best fit to P, so that R is the radius of curvature of P at X, then we must have $\Delta c = O(\epsilon^3)$. Hence

$$R = \sum_1^n \sin \theta_i \Big/ \sum_1^n \frac{\sin^2 \theta_i}{b_i}. \tag{6}$$

Accordingly, R is well defined at all points of a polyellipse P except where this P happens to pass through one of its foci. For in that case one of the focal distances b_i is 0 so R is, formally, undefined through an easy argument shows that R is then 0. We may therefore expect that the focus crossed by P is a cusp point for P. Only quantities $\sin \theta_i$ enter into R in (6), obviating the need for distinguishing between θ_i and $\pi - \theta_i$. Generally, (6) shows that a polyellipse turns slowly at its points which are far away from a focus, and fast when it passes close to a focus. This can be seen on the elementary example of the family of confocal ellipses with the foci F_1 and F_2; we recall that this family includes the degenerate ellipse consisting of the segment F_1F_2 described twice; here F_1 and F_2 are the cusps, with cusp angle 0.

2. The preceding section applies to simple polyellipses, but there is no difficulty whatever in extending the results on tangents and curvature to general polyellipses. We replace accordingly (2) by

$$c = \sum_1^n w_i |XF_i| \tag{7}$$

and we trace the corresponding changes in (3). These result in replacing (4) by

$$\sum_1^n w_i \cos \theta_i = 0 \tag{8}$$

and (5) by

$$\text{if } \bar{u} = \sum_1^n w_i \frac{XF_i}{|XF_i|} \quad \text{then} \quad T \perp \bar{u}. \tag{9}$$

Similarly, the radius of curvature R of a general polyellipse is given by

$$R = \sum_1^n w_i \sin \theta_i \Big/ \sum_1^n w_i \frac{\sin^2 \theta_i}{b_i}. \tag{10}$$

On prefacing the weights w_i in (9) and (10) by suitable plus and minus signs we get formulas valid for any polyconic.

C. Confocal polyellipses. 1. We consider now the elementary geometry of the family of all confocal n-ellipses with n fixed foci $F_1, \cdots, F_n$. The curves are the loci $P(c)$ given by

$$\left\{X : \sum_1^n |XF_i| = c\right\} \quad \text{or} \quad \left\{(x, y) : \sum_1^n ((x - x_i)^2 + (y - y_i)^2)^{1/2} = c\right\} \tag{11}$$

where c varies over its entire allowed range: $c_0 \leq c$. There is the obvious modification—introduction of weights—for generalized polyellipses. As a guide and for comparison we may take the family of all concentric circles ($n = 1$) or of all confocal ellipses ($n = 2$). In either case there is one degenerate locus associated with the minimum sum c_0 of the n distances: for $n = 1$ $c_0 = 0$ and the 0-radius circle F_1, for $n = 2$, $c_0 = |F_1F_2|$ and the collapsed ellipse F_1F_2.

In the general case of arbitrary n there is also one degenerate locus $D = P(c_0)$ corresponding the the minimum value c_0 of c in (11). An attempt to find c_0, that is to say to solve our Problem 1, by calculus leads to the equations

$$\sum_1^n \frac{x - x_i}{((x - x_i)^2 + (y - y_i)^2)^{1/2}} = 0, \sum_1^n \frac{y - y_i}{((x - x_i)^2 + (y - y_i)^2)^{1/2}} = 0 \tag{12}$$

or in vector form

$$\sum_1^n \frac{\overline{XF_i}}{|XF_i|} = 0.$$

That is, the sum of n distances to the foci $F_1 \cdots, F_n$ is minimized at the point D where the n unit vectors toward the F_is add up to 0. This allows us to find D approximately by mechanical analogues used on balancing the forces in strings with attached equal weights (or, for general polyellipses, with unequal weights). However, the straightforward attempt to find D by solving the above Eqs. (12) bogs down on account of boundary extrema and discontinuity of derivatives even more than through the formal complexity of system (12) for n of any size.

In general, we have

PROPOSITION 1. The degenerate locus D is a point if not all F_is are collinear.

If the n foci F_i are distinct points on a line and n is odd then D is again a point: the middle one of the foci. If n is even then D is the whole central segment F_iF_j. The latter case corresponds to and generalizes the collapsed ellipse F_1F_2 for $n = 2$. There are obvious generalizations of the foregoing for the general case when we have weights $w_1, \cdots, w_n$. Further,

PROPOSITION 2. For $c > c_0$ the polyellipse $P(c)$ is a closed strictly convex curve. If $c > c_1 \geq c_0$ then $P(c)$ contains $P(c_1)$ in its interior.

The proofs of Propositions 1 and 2 follow from some simple convexity considerations. Take a line L and let X be a moving point on L, parametrized by its distance s from an arbitrary origin on L. Let F_i be the ith focus of the polyellipse and let $b_i = b_i(s)$ be the distance $|XF_i|$. If F_i is on L then $b_i(s)$ is a piecewise linear convex function of s. If $F_i$is off L then $b_i(s)$ is a strictly convex function of s. Here we define a function to be convex if its second derivative is always ≥ 0, and strictly convex if it is always >0. The second derivative is allowed to be undefined at a finite number of points. Since the second derivative of a sum is the sum of second derivatives it follows that

$$b(s) = \sum_1^n b_i(s) = \sum_1^n |XF_i|$$

is a strictly convex function of s if not all foci F_i are collinear. Now the proofs of Propositions 1 and 2 are immediate. We add as a formal statement the obviously true

PROPOSITION 3. Through any point of the plane there passes exactly one polyellipse $P(c)$.

Propositions 1–3 begin to give us some idea of the general appearance of our family of confocal polyellipses. It is to be noted that these propositions hold for general polyellipses as well as for the simple ones.

2. We continue with the question of finding the degenerate locus $D = P(c_0)$. For $n = 3$ there are two cases. If the triangle $F_1F_2F_3$ has an angle $\geq 120°$ then D is the vertex F_i of that angle and we have a boundary extremum. Otherwise, D is the unique Steiner point of our triangle at which all three sides subtend the angle $120°$. This will be shown at once by exploiting external tangencies of circles and ellipses; the same technique will be applied later in our optimization problems: external tangency of polyellipses. In our simple case of three foci it is enough to observe that the 2-ellipse through D with the foci F_i, F_j must be externally tangent at D to the 1-ellipse through D with the focus F_k, for any one of the three choices of F_k.

An easy geometrical construction will yield us the point D [4]: if equilateral triangles are built outward on the sides of the triangle $F_1F_2F_3$ and their third vertices are F_{12}, F_{13}, F_{23} (in obvious enumeration), then the straight segments $F_{12}F_3$, $F_{13}F_2$, $F_{23}F_1$ are concurrent in D. Moreover, these three segments are of equal length:

$$d = F_1D + F_2D + F_3D.$$

For $n = 4$ there are also two cases. If F_1, F_2, F_3, F_4 are the vertices of a convex quadrilateral then D is the intersection of the diagonals, say F_1F_3 and F_2F_4. This is clear since for any point X other than D

$$|F_1X| + |XF_3| \geq |F_1F_3|, |F_2X| + |XF_4| \geq |F_2F_4|$$

with at least one strict inequality so that by adding

$$\sum_1^4 |F_iX| > \sum_1^4 |F_iD| = |F_1F_3| + |F_2F_4|.$$

If one of the F_is lies inside, or on the periphery of, the triangle formed by the other three, then it itself is D. We observe the consistency of the 'intersection of the diagonals' with the 'middle segment' case of four collinear points mentioned earlier. This consistency is seen by collapsing the convex quadrilateral onto a line.

We have gone over these simple matters in some detail in order to emphasize the essential difference between the cases $n \leq 4$ and $n > 4$. This has considerable analogy to the classic result of Ruffini-Abel-Galois on the solvability of general polynomial equations with integer coefficients: we can solve such equations by standard algebraic processes, i.e. by radicals, if and only if their degree is ≤ 4. It turns out that for $n \geq 5$ there is in general no Euclidean geometrical construction to yield the point D minimizing the sum of distances from $F_1, \cdots, F_n$. This is proved in [5] as an exercise in computing certain Galois groups. Specifically, it is shown there that no Euclidean construction will produce the point $D(x, 0)$ minimizing the sum of distances to $(0, 1), (0, 0), (0, -1), (3, 3), (3, -3)$. This difference between the cases $n \leq 4$ and $n > 4$ is yet another reason, a technical and not an a priori obvious one, for considering polyellipses in connexion with distance minimization problems.

3. We take up now the matter of cusps of polyellipses. Let $P(c)$ be the n-ellipse through a point X, with the foci $F_1, \cdots, F_n$. We suppose that the F_i's are not collinear, in particular $n \geq 3$. Using the distance-perturbation formula (3) with $\eta = 0$ and demanding that $\Delta c = 0(\epsilon^2)$, we have found for the tangent T to $P(c)$ at X the formula (5) for simple polyellipses and (9) for general ones. Suppose, however, that X itself is a focus, say F_i; this means that

$$c = \sum_{\substack{k=1 \\ k \neq i}}^{n} |F_1F_k|.$$

Now (3) no longer applies and we have instead

$$\Delta c = \pm\epsilon - \epsilon \sum_{\substack{k=1 \\ k \neq i}}^{n} \cos\theta_k + 0(\epsilon^2), \quad \epsilon \text{ small}, \quad \eta = 0. \tag{13}$$

What happens to $P(c)$ at and near X? The polyellipse $P(c)$ is still a strictly convex curve, and so instead of one tangent we have two one-sided tangents to $P(c)$ at X, which itself is a cusp. The two signs in (13) correspond precisely to those two one-sided tangents. To find them we put $\Delta c = O(\epsilon^2)$ in (13), getting two equations

$$\sum_{\substack{k=1 \\ k \neq i}}^{n} \cos\theta_k = 1, \ \sum_{\substack{k=1 \\ k \neq i}}^{n} \cos\theta_k = -1 \tag{14}$$

which replace (4). Equivalently, proceeding in vector terms, we get the following modification of the recipe (5): from F_i as origin draw the vector

$$\overline{u_i} = \sum_{\substack{k=1 \\ k \neq i}}^{n} \overline{F_iF_k}/|F_iF_k|, \tag{15}$$

from the tip of $\bar{u}_i$ draw the two tangents to the unit circle about F_i, touching that circle at A and B, say; then the directions of the two one-sided cusp tangents to $P(c)$ at F_i are F_iA and

F_iB. If $|\bar{u}_i| \leq 1$ then we have the case of a boundary minimum and F_i is the singular locus $D = P(c_0)$.

The case of $|\bar{u}_i| = 1$ means that either F_i is the singular locus D or else it is a 0-angle cusp of the polyellipse $P(c)$. The latter case is excluded since $P(c)$ is strictly convex, and the F_i's are not collinear. Since there is exactly one singular locus D it follows that at most one focus F_i can be D. We have therefore proved the following proposition which is independent of the context of polyellipses:

PROPOSITION 4. Let $F_1, \cdots, F_n$ be any n noncollinear points in the plane. At each F_i let $\overline{u_i}$ be the sum of the $n - 1$ unit vectors toward the other Fs. Then of the n vectors $\overline{u_1}, \cdots, \overline{u_n}$ at most one is of the length ≤ 1.

From the point of view of solving the simple optimization problem $1 - P_n(F_1, \cdots, F_n)$—this has the important consequence of enabling us to decide, by a simple test and in advance, when there will be a boundary minimum, and if there is one, giving us that minimum.

We observe that introducing weight w_k produces formulas valid for general polyellipses. Eq. (4) is replaced by

$$\sum_{\substack{k=1\\k\neq i}}^{n} w_k \cos\theta_k = w_i, \quad \sum_{\substack{k=1\\k\neq i}}^{n} w_k \cos\theta_k = -w_i \tag{16}$$

and the vectors $\bar{u}_i$ in (15) are replaced by

$$\overline{u_i} = \frac{1}{w_i}\sum_{\substack{k=1\\k\neq i}}^{n} w_k \frac{\overline{F_iF_k}}{|F_iF_k|}, \quad i = 1, \cdots, n. \tag{17}$$

In place of Proposition 4 we have its generalization:

PROPOSITION 5. Let $F_1, \cdots, F_n$ be any n noncollinear points in the plane, and let positive weights $w_1, \cdots, w_n$ be associated with the F_i's. Let $\overline{u_i}$ be as in (17); then of the n vectors $\overline{u_1}, \cdots, \overline{u_n}$ at most one is of length ≤ 1.

Computation with polyellipses showed that we may want to know in advance when a polyellipse for n given foci $F_1, \cdots, F_n$ may have several cusps. There is a simple test based on examining the $n(n-1)/2$ interfocal distances. Let

$$D_j = \sum_{\substack{i=1\\i\neq j}}^{n} |F_iF_j| \quad j = 1, \cdots, n;$$

if the n numbers D_j are pairwise different, then all polyellipses $P(c)$ are smooth except only when c equals one of the D_j. The polyellipse $P(D_j)$ has exactly one cusp, at F_j, and is otherwise smooth. If $D_1, \cdots, D_n$ are not all distinct, let the distinct ones among them be $d_1, \cdots, d_k$ where d_s is taken up with the multiplicity m_s (so that $\Sigma m_s = n$). Now the only polyellipses with cusps are $P(d_1), \cdots, P(d_k)$, and $P(d_s)$ has exactly m_s cusps. These occur at the foci F_i for the m_s indices i such that $D_i = d_s$.

The cusp-angles of an n-ellipse are easily computed for the special case $n = 3$. Let A, B, C be the three foci of a 3-ellipse and let A denote the angle of the triangle ABC as well as the vertex. Let α be the cusp-angle of the 3-ellipse passing through A. Then an application of (14) shows that

$$\cos\frac{A}{2}\cos\frac{\alpha}{2} = \frac{1}{2}.$$

In particular, α has a real value if and only if $A \leq 120°$. This is in complete accord with the previously mentioned problem of minimizing the sum $|AX| + |BX| + |CX|$: if an angle of the triangle ABC is $\geq 120°$ then its vertex is the minimizing X and we have a boundary minimum. Our Proposition 4 gives us a simple test for such a boundary minimum for general value n. This enabled us to run some Monte-Carlo tests for the probability of occurrence of such a boundary minimum, when n points are taken independently and uniformly at random in the square. Such tests were run for 5000 sets of n points; the frequencies f_n of occurrence of the minimum at one of the n points were found to be

n =	3	4	5	6	10	20	40
f_n =	.4012	.3036	.2214	.1848	.1042	.0568	.0258.

This suggests strongly that for large n $f_n \to 0$ and (much less strongly) that $f_n \sim 1/n$ approximately.

4. As we might expect, for c large enough any polyellipse $P(c)$ is approximately circular. More precisely, we have

PROPOSITION 6. Let $F_1, \cdots, F_n$ be the foci and $w_1, \cdots, w_n$ the weights of the general polyellipse $P(c)$. If c is large enough then $P(c)$ is contained between two concentric circles whose radii differ by an arbitrarily small amount.

To prove this we introduce polar coordinates with an origin to be specified later, and an arbitrary initial direction. Let α_i, r_i be the polar coordinates of F_i, $i = 1, \cdots, n$, and let $X = (\theta, r)$ be a point on the polyellipse $P(c)$, with r large. Then

$$|F_iX| = (r^2 - 2rr_i \cos(\theta - \alpha_1) + r_i^2)^{1/2}$$

$$= r - [r_i \cos \alpha_i \cos \theta + r_i \sin \alpha_i \sin \theta] + \frac{r_i^2}{2r} \sin^2(\theta - \alpha_i) + O\left(\frac{1}{r^2}\right). \qquad (18)$$

The polyellipse $P(c)$ is given by

$$\Sigma\, w_i\, |F_iX| = c;$$

using (18) we rewrite this as

$$r \sum w_i - \left[\cos \theta \sum w_i r_i \cos \alpha_i + \sin \theta \sum w_i r_i \sin \alpha_i\right]$$

$$+ \frac{1}{2r} \sum w_i r_i^2 \sin^2(\theta - \alpha_i) + O\left(\frac{1}{r^2}\right) = c. \qquad (19)$$

We now choose our origin so as to make vanish the two sums in the square brackets; this means that the origin is the weighted centre of mass of the foci:

$$\sum_1^n w_i\, \overline{XF}_i \Big/ \sum_1^n w_i\,.$$

As a consequence, (19) becomes

$$Wr + \frac{K}{r} + O(1/r^2) = c$$

where K is a positive constant and $W = \Sigma w_i$. Therefore there exists another positive constant K_1 such that

$$\frac{c}{W} - \frac{K_1}{c} < R < \frac{c}{W}. \qquad \text{q.e.d.}$$

What happens to the polyellipses $P(c)$ at the other extreme, when c is close to its minimum value c_0, can be similarly determined. We use now the Cartesian coordinates $F_i(x_i, y_i)$, and let (x_0, y_0) be the singular locus D corresponding to the minimum sum c^1 of the distances to the F_is. Let c Æ c^1 where $c - c^1$ is small; to find the approximate shape of the polyellipse $P(c)$

$$\Sigma|F_iX| = c$$

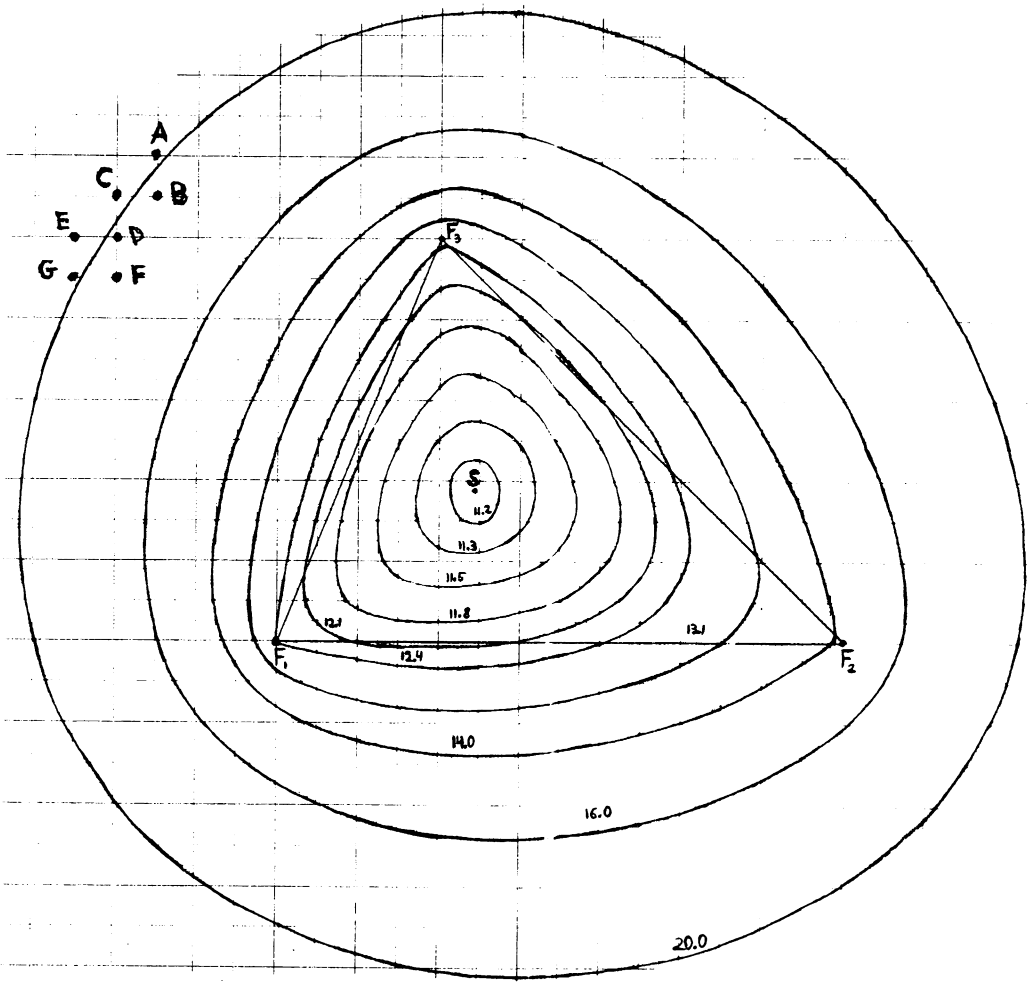
we let $X = (x, y)$ be a point on $P(c)$. Hence

$$x = x_0 + \xi, y = y_0 + \eta \qquad \xi \text{ and } \eta \text{ small;}$$

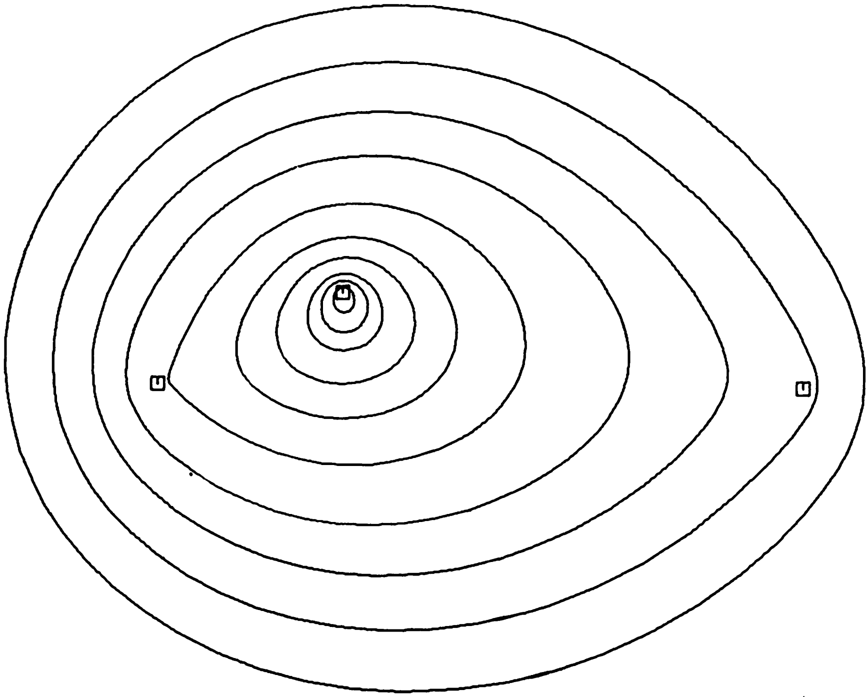
the polyellipse in Cartesian coordinates is

$$(x_0 + \xi - x_i)^2 + (y_0 + \eta - y_i)^2 = c.$$

Using some elementary expansions and recalling that



a)



b)

FIG. 1. 3-ellipses.

$$\sum \frac{x_0 - x_i}{((x_0 - x_i)^2 + (y_0 - y_i)^2)^{1/2}} = \sum \frac{y_0 - y_i}{((x_0 - x_i)^2 + (y_0 - y_i)^2)^{1/2}} = 0$$

by the definition of c_0, we find eventually that the approximate equation of $P(c)$, in terms of the ξ, η coordinates, is

$$A\xi^2 - 2B\xi\eta + C\eta^2 = D$$

where

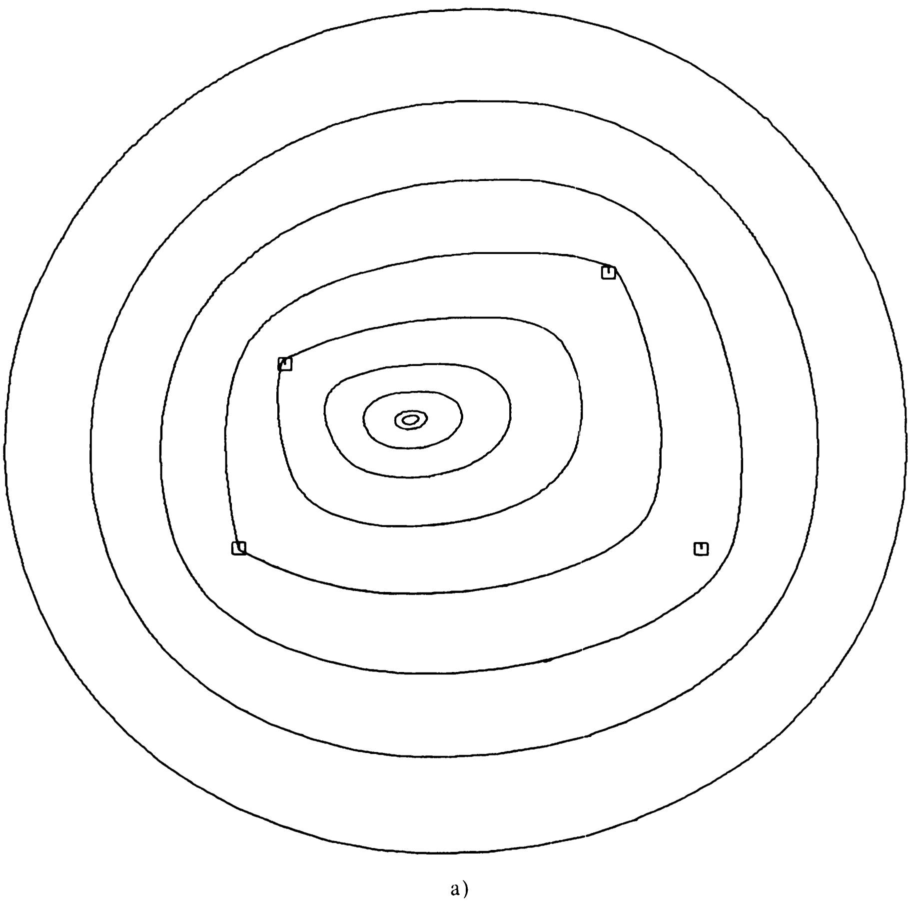
$$\begin{aligned} A &= \Sigma(y_0 - y_i)^2/[(x_0 - x_i)^2 + (y_0 - y_i)^2]^{3/2}, \\ B &= \Sigma(x_0 - x_i)(y_0 - y_i)/[(x_0 - x_i)^2 + (y_0 - y_i)^2]^{3/2}, \\ C &= \Sigma(x_0 - x_i)^2/[(x_0 - x_i)_0 + (y_0 - y_i)^2]^{3/2}, \\ D &= 2(c - c_0). \end{aligned} \tag{20}$$

It follows that for c close to c_0 the polyellipse $P(c)$ is approximately an ellipse whose eccentricity, size, and position are easily computed. The only change for a general polyellipse is that weights w_i appear in the first three sums in (20).

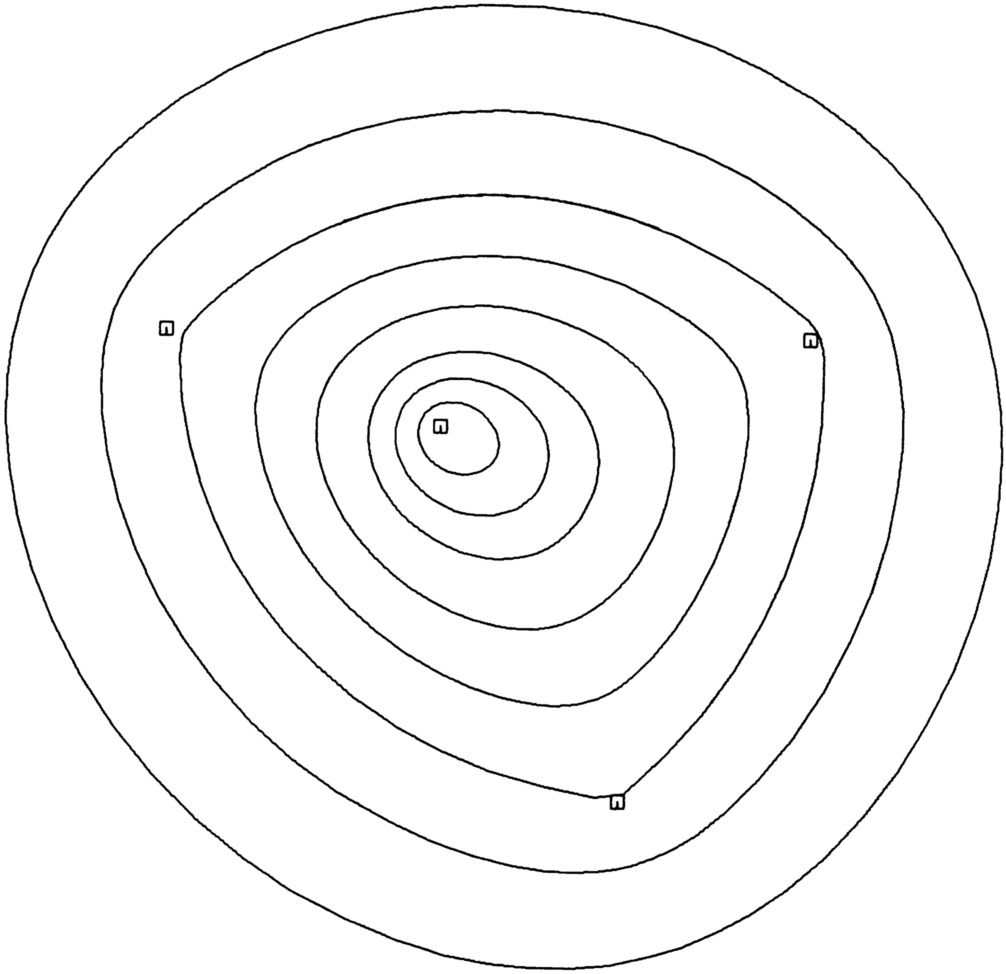
D. Numerical and graphical work. 1. The formulas (9) and (10) gave us the tangent and the radius of curvature, at a point X, of a general polyellipse $P(c)$ with known foci and weights. These formulas lead to a simple and practical method of drawing the whole of

$P(c)$. The idea is to replace locally $P(c)$ by an arc of its circle of curvature at X, go along that arc so as to turn through a fixed angle ϕ_0, then get back onto $P(c)$ and repeat the procedure till the return to X. Thus $P(c)$ is approximated by a sequence of $360°/\phi_0$ circular arcs and we adjust the angle ϕ_0 so as to make the approximation adequate. In practice it was found that $\phi_0 = 5°$ makes the gap between the approximating arcs sufficiently much thinner than the thickness of the drawing line.

There is no difficulty in starting the procedure at X; we compute first the weighted sum of distances to the foci. Then, by means of the formulas (9) and (10) we find the center of curvature of $P(c)$ at X and we produce the circle of curvature. Let XX_0 be an arc of it, turning through the angle ϕ_0. Then XX_0 and $P(c)$ have a contact of order ≥ 1 at X but X_0 will in general lie off $P(c)$. That is, the weighted sum of distances from X_0 to the foci (which we compute) is not equal to c. We therefore determine a new point X_1, possibly close to X, at which the weighted sum of distances to the foci has the same value as at X. This X_1 can be found in a variety of ways, all based on interpolating from the data which are the sums of distances computed at several points; one of these is, of course, the point X_0 itself. Now we repeat the procedure at X_1 and we continue till our return to X after $360°/\phi_0$ steps. The distance between the starting point X and the finishing point, which ideally should be 0, may serve as an overall check on the procedure.



a)



b)
FIG. 2. 4-ellipses.

One considerable advantage of our method will be clear at once: it is a variable step-length method and the length of the step is automatically self-adjusting. That is, each of our approximating circular arcs turns through the same angle ϕ_0; hence, where the polyellipse curves strongly the approximating arc is short, and where it turns slowly the arc is long, which is just as it should be. In practice it has been found that computing the coordinates and preparing for mechanical plotting of a family of 6–9 confocal general ellipses with given weights and 2–6 foci takes several seconds (2–10) on a reasonably large and fast modern computer.

2. We describe briefly another method of plotting polyellipses, adapted to manual rather than machine computing. We start with a square grid, for instance the centimeter grid of Fig. 1a. The three foci F_1, F_2, F_3 are as shown in the figure. Suppose that the simple 3-ellipse $P(20)$ is to be drawn. For any point X let $f(X)$ stand for the sum of its distances to F_1, F_2, F_3; since $f(X)$ will be needed only for grid points X, it can be obtained

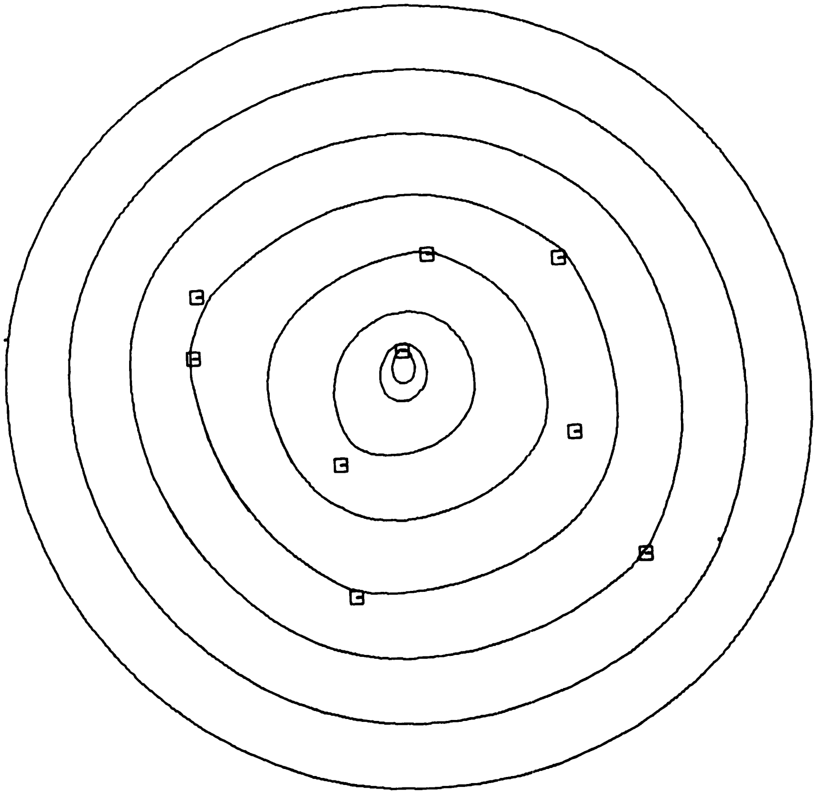


FIG. 3. A 9-ellipse family.

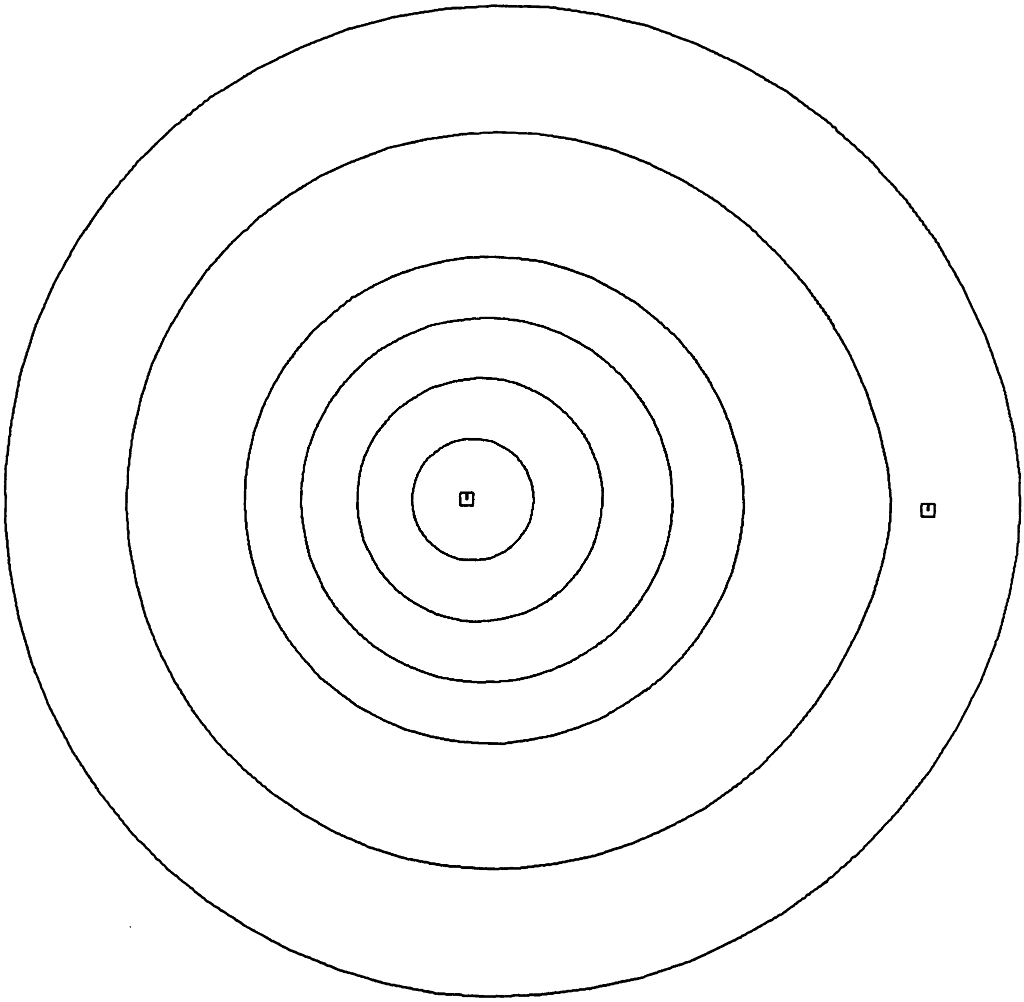
as sum of three square roots of sums of squares of integers, and is therefore simply calculated by a reference to a table of square roots. We determine, by trial and error, two neighbouring points of our grid, A and B, such that

$$f(A) > 20 > f(B).$$

Then, exploiting of course the convexity of our polyellipses, we determine a chain of grid-neighboring points along which the function $f(X) - 20$ changes signs: $A, B, C, D, E, F, G, \cdots$. Now, by a straightforward linear interpolation along $AB, BC, CD, DE, FG, \cdots$. we determine approximately the points of intersection of those segments and the poly-ellipse $P(20)$, and we joint them. The 10 confocal 3-ellipses of Fig. 1a were all obtained in this way; S is the Steiner point, i.e. the singular locus.

All the other graphs shown were obtained by the first method and were drawn by the machine. Fig. 1b shows a similar family of confocal simple 3-ellipses but drawn for the boundary-extermum case, when one of the angles of the triangle with the foci as vertices exceeds 120°. The next figure shows the two possibilities for simple 4-ellipses: in Fig. 2a the four foci are the vertices of a convex quadrilaterial, in Fig. 2b one of the four lies inside the triangle given by the other three. A family of confocal simple 9-ellipses with nine randomly determined points for foci is shown in Fig. 3.

The next figure shows the effects of different weights. Figs. 4*a, b, c, d* show the families of confocal 2-ellipses with weights 9:1, 8:2, 7:3, 6:4.



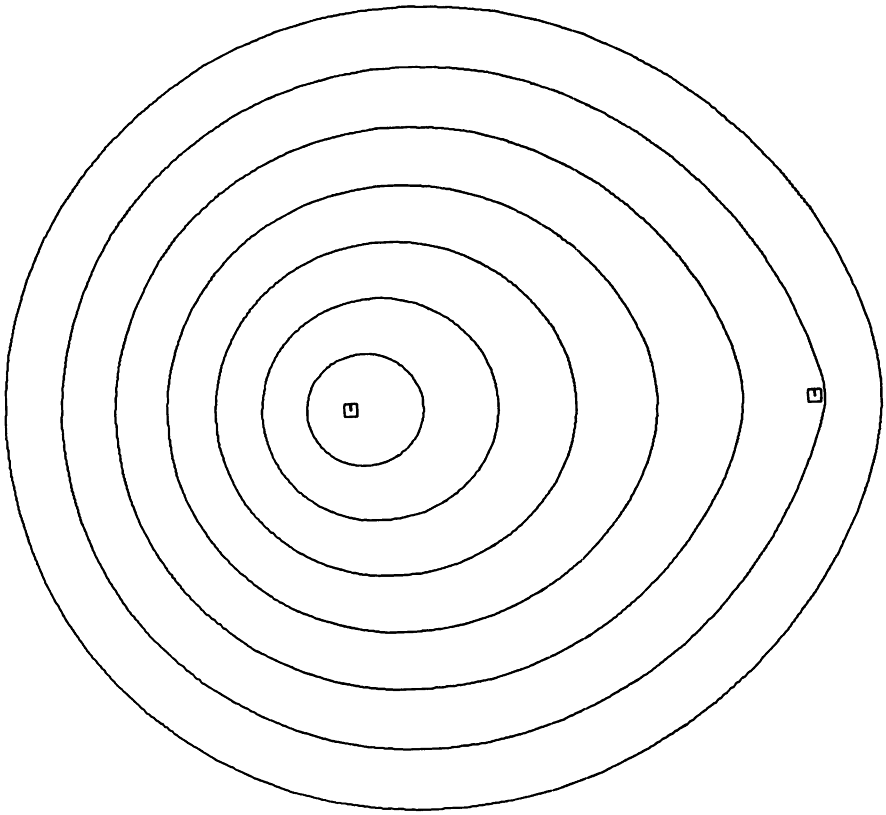
a)

3. We finish this installment with some preliminary remarks on the graph-minimization problems. Suppose that n points $F_1, \cdots F_n$ in the plane are given, together with corresponding positive weights, and the weighted sum of distances

$$\sum_1^n w_i |F_i X|$$

is to be minimized. This is, in effect, the generalized Weber problem. We determine first an initial approximation X_0 to the minimum $\bar{X}$, for instance as follows. To guard against the possibility of a boundary minimum, i.e. of $\bar{X}$ coinciding with an F_i, we apply the criterion of Proposition 5 to check whether a vector u_i exists, of length ≤ 1. If it does we are finished: there is a boundary minimum F_i and we have found it. If not, the vector u_i of shortest length determines the focus F_i which we use for X_0.

Next, since we know how to draw general polyellipses, we draw the polyellipse P_0



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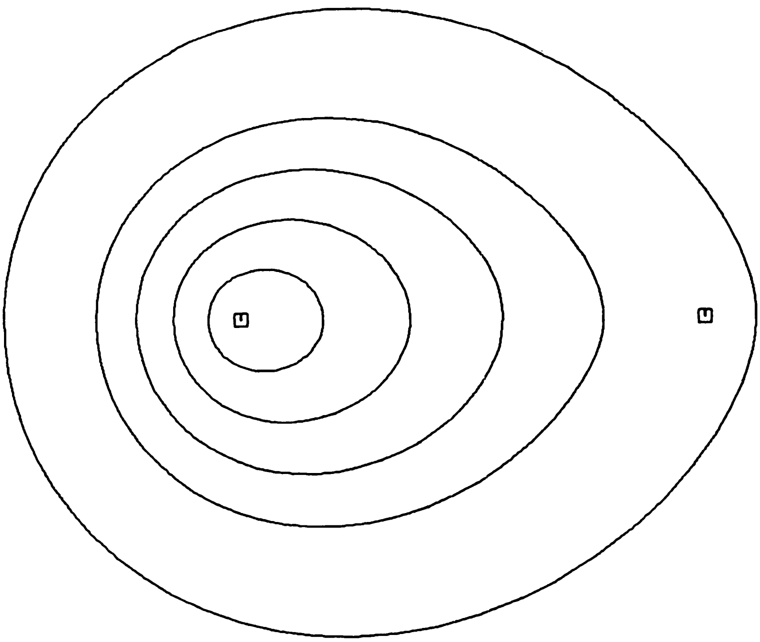

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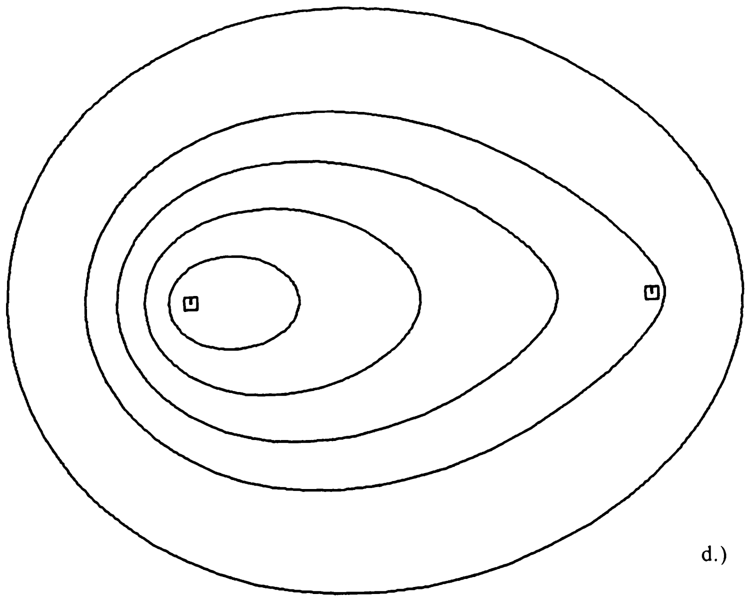

FIG. 4. General 2-ellipses with different weights.

corresponding to our foci and weights and passing through X_0. The true minimizing point X, i.e. the singular locus, lies inside P_0. We now produce the next approximation stage and take X_1 to be the 'center' of P_0; several possibilities exist, and are currently considered, for such a 'center'. Now the procedure is repeated on X_1 as the starting point. Two fundamental questions arise:

A) does our procedure converge?
B) if so, how fast?

With respect to (A), let us suppose that for any polyellipse P our method of 'centering' gives us a 'center' which is a point of a subset Q of P, such that

$$\operatorname{diam} Q / \operatorname{diam} P \leq \lambda < 1.$$

Then an application of the contraction mapping principle shows that our procedure indeed converges: $X_0, X_1, X_2, \cdots$ tend to the unique limit X. With respect to (B), it would therefore appear that the speed of convergence is exponential:

$$|X_n X| \sim K\lambda^n \qquad n \text{ large}.$$

However, suppose that our 'centering' method is nonuniform: the smaller P and the closer it is to being an ellipse (as suggested by C.4), the smaller the region Q, relative to P. Now the speed of convergence will be faster than exponential, perhaps something like Newton's method for roots of equations. It is proposed to consider these matters in a future paper.

BIBLIOGRAPHY

[1] H. W. Kuhn and R. E. Kuenne, Regional Sci. **4,** 21–33 (1962)
[2] A. Weber, *Theory of the location of industries,* Chicago, 1929 (transl. from German by C. J. Friedrich)
[3] K. Eisemann, SIAM Rev. 4, 394–395 (1962)
[4] Z. A. Melzak, *Companion to concrete mathematics,* J. Wiley & Sons, 1973
[5] E. J. Cockayne and Z. A. Melzak, Math. Magazine **42,** 206 (1969)