OPTIMIZATION OF ELASTOHYDRODYNAMIC CONTACTS*

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Introduction. Hydrodynamic lubrication is concerned with a particular form of creeping flow between surfaces in relative motion. The momentum and continuity equations for this situation can be combined into a single equation—the Reynolds equation—derived by O. Reynolds near the end of the last century [1]. In this note a class of optimization problems associated with these flows will be discussed.

Analysis. The one-dimensional Reynolds equation governing the lubrication problem is given by the two-point elliptic boundary value problem:

\[ \frac{d}{dx} \left( h^3 \frac{dp}{dx} \right) = \frac{dh}{dx}, \]

\[ p(0) = p(1) = 0. \]

In Eq. (1) \( p \) is the dimensionless film pressure and \( h(x) \geq 1 \) is the film profile. We observe that for each \( h \geq 1 \) a film pressure \( p_h \) can be obtained and the load capacity functional \( W = W[h] \) can be written as

\[ W[h] = \int_0^1 p_h dx. \]

Lord Rayleigh [2] investigated the effect of different forms of \( h \) on \( W \) and discovered the optimum profile, i.e., the profile which maximized \( W[h] \) over all profiles satisfying \( h(x) \geq 1 \). That profile is called the Rayleigh step and is shown in Fig. 1.

Due to the generation of high pressures, it is recognized that the bearing components in a practical situation may deflect and hence the film thickness becomes functionally related to the film pressure; i.e., we write

\[ h(x) = h_0(x) + \mathcal{L} p \]

where \( \mathcal{L} \) is a linear operator relating the bearing component deformations to the applied pressure distribution. This situation is called elastohydrodynamic lubrication (EHD) [1]. Note that by substituting Eq. (3) into Eq. (1) a nonlinear integrodifferential equation typically results. In this note we consider the optimization problem considered by Rayleigh for the EHD case. We again require that \( h \) as given by (3) satisfies \( h \geq 1 \). For notational convenience we will denote \( p_h \), where \( h \) is given by Eq. (3), by \( p_{h_0} \) and \( W[h] \) by \( W[h_0] \).

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Denote by \( h_{\text{opt}} \) the optimum film thickness when \( \xi = 0 \) (Rayleigh's result) and by \( p_{\text{opt}} \) the corresponding film pressure:

\[
W_{\text{opt}} = W[h_{\text{opt}}] = \int_0^1 p_{\text{opt}} \, dt. \tag{4}
\]

Let \( h_1(x) \) be such that

\[
h_{\text{opt}} = h_1 + \xi p_{\text{opt}}. \tag{5}
\]
Then we have $\tilde{p}_{h_1} = p_{opt}$ and hence the optimum solution $h^*$ for the EHD case satisfies

$$W[h^*] \geq W[h_1] = W_{opt}.$$  \hspace{1cm} (6)

On the other hand, consider the problem of finding an $a^* \geq 1$ such that $W[a]$ is maximized, where we do not require that $a(x) = h_g(x) + \xi \rho_a$. We thus have

$$W[a^*] \geq W[h^*].$$  \hspace{1cm} (7)

But clearly $a^* = h_{opt}$ and $W[a^*] = W_{opt}$. Hence, combining Eqs. (6) and (7), we have

$$W_{opt} = W[h^*],$$  \hspace{1cm} (8)

and we may set $h^* = h_1$. In view of Eq. (5) we have the simple result that

$$h^* = h_{opt} - \xi \rho_{opt}.$$  \hspace{1cm} (9)

Eq. (9) allows us to trivially calculate $h^*$ for different operators $L$. Figs. 2 and 3 show some typical results. Note that nowhere in our derivation have we made use of the fact that Eq. (1) is one-dimensional. Hence the result (9) applies equally well to two-dimensional problems and we may use the recent results [3,4] to calculate two-dimensional EHD optimum profiles. The extension to the case in which the viscosity is pressure-dependent is more complicated and will be the topic of a future note.

**References**