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ANHARMONIC ANALYSIS OF A TIME-DEPENDENT PACKED BED THERMOCLINE*

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Abstract. A vectorized separation of variables approach is applied to a coupled pair of parabolic partial differential equations describing the degradation of a thermocline in a packed bed thermal storage tank. The time-dependent quasi-one-dimensional model includes the effects of finite tank length, thermal conduction in the direction parallel to the tank walls, and heat transfer between the fluid and solid components of the bed. For certain classes of boundary conditions, the analysis leads to an eigenvalue problem for the spatial dependence of the fluid and solid temperatures in the bed. The eigenvalues and corresponding eigenfunctions are readily calculated, and completeness of the eigenfunctions follows from a transformation to an integral equation by the construction of a Green's tensor function. The method is illustrated by an example which arises in the analysis of the thermal storage subsystem of a central solar receiver power plant.

1. Introduction. The problem of thermocline degradation in a packed bed thermal storage tank is important for determining precisely how long usable heat energy can be stored in such a system. One particularly important application is in the proposed central solar receiver power plants. By passing hot fluid, heated by the sun in the receiver portion of the system, through a cold bed (charging), heat energy is transferred from the fluid to the solid portions of the tank (Fig. 1). This stored heat energy may then be reclaimed by the reverse process at a later time by passing cold fluid through the hot bed (discharging). Both processes usually result in a moving narrow region, called a thermocline, in which the temperature gradients of the fluid and solid are relatively large and monotonic. For the general case of a partially charged bed, maintaining a thermocline between the hot and cold regions of the bed for reasonably long holding times is essential in order to keep the hot end of the tank at or near its original temperature. In this way, fluid will emerge from the tank during subsequent discharge periods at that temperature and will thus be sufficiently hot to be useful for generating electricity in a turbine. However, heat losses and continuum processes such as thermal conduction have an adverse effect on the tempeature profiles and it is thus desirable to know the inherent limitations on the storage system. Consequently, the purpose of this paper is to describe analytically the time evolution of the fluid and solid temperature during holding periods when no new fluid enters the tank.

The thermal problem of a fluid flowing through a packed bed consisting of crushed material was first treated mathematically by Schumann [1]. There, the assumption of constant fluid and material properties and the neglect of thermal conductivity permitted a closed-form solution of the one-dimensional, semi-infinite problem for nonzero inlet fluid

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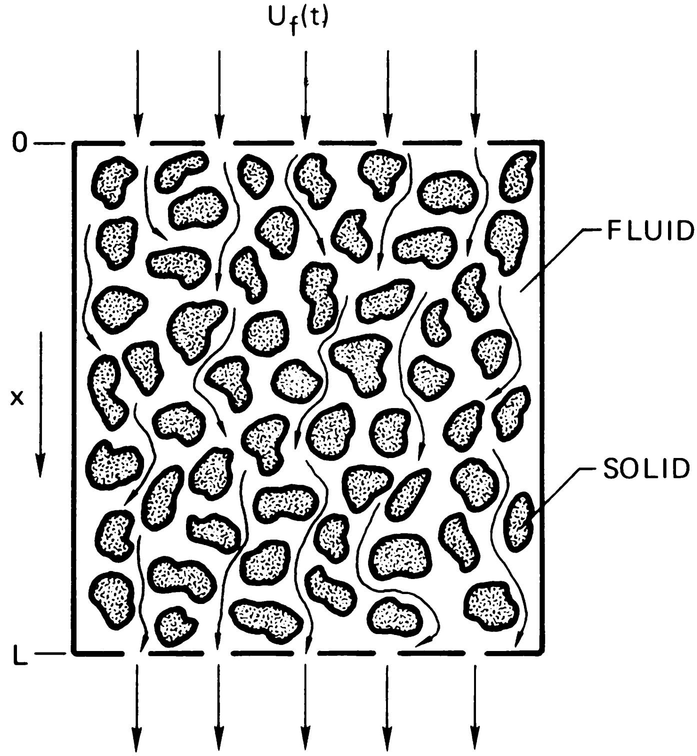


FIG. 1. Sketch of fluid flow through a packed bed.

velocity. More recently, Margolis [2] considered the thermocline degradation problem for zero inlet fluid velocity by including the heat conduction terms. Incorporating the effects of a finite tank length, an analytical solution, applicable to arbitrary boundary conditions, was obtained by use of a Laplace transform method applied to the governing pair of coupled partial differential equations for the fluid and solid temperatures. Although tractable, that type of analysis is rather tedious and thus its chief advantage is its ability to handle completely general boundary conditions for special test problems. The present work describes, for the class of boundary conditions of greatest physical interest, an alternative, more readily usable solution method based on a vectorized separation of variables approach. This method results in an anharmonic Fourier analysis of the spatial dependence of the temperature profiles which can be efficiently implemented on a computer and applied to arbitrary initial conditions.

2. Governing equations. Assuming the fluid to be initially at rest and incompressible and the problem to be globally one-dimensional, the degradation of the thermocline is described by the energy equations for the fluid and solid portions of the bed. Assuming constant thermal properties, these are, in nondimensional variables,

$$\frac{\partial T_f}{\partial t} = h_f(T_s - T_f) + \alpha \frac{\partial^2 T_f}{\partial x^2}, \tag{2.1}$$

$$\frac{\partial T_s}{\partial t} = h_s(T_f - T_s) + \frac{\partial^2 T_s}{\partial x^2}, \tag{2.2}$$

where the terms containing h_f, h_s account for the heat transfer between fluid and solid in the direction(s) perpendicular to the lengthwise heat flow caused by diffusion. In terms of dimensional quantities (denoted by*)

$$T_{f,s} \equiv \frac{T_{f,s}^* \ T_A^*}{T_A^*} \ ; x \equiv \frac{x^*}{L^*} \ ; t^* \equiv \frac{t\lambda_s^*}{\rho_s^* c_s^* L^{*2}} , \tag{2.3a,b,c}$$

$$h_f \equiv \frac{h^* L^{*2} \rho_s^* c_s^*}{\beta \lambda_s^* \rho_f^* c_f^*} \ ; h_s \equiv \frac{h^* L^{*2}}{(1-\beta)\lambda_s^*} \ ; \alpha \equiv \frac{\lambda_f^* \rho_s^* c_s^*}{\lambda_s^* \rho_f^* c_f^*} , \tag{2.4a,b,c}$$

where

T_f^*, T_s^* $\equiv$ temperature of fluid, solid; T_A^* $\equiv$ ambient temperature outside the tank; ρ_f^*, ρ_s^* $\equiv$ densities (constant) of fluid, solid; λ_f^*, λ_s^* $\equiv$ *effective* thermal conductivities (constant) of fluid, solid; h^* $\equiv$ volumetric heat transfer coefficient (constant); β $\equiv$ void fraction of bed; L^* $\equiv$ length of tank; x^* $\equiv$ space coordinate; t^* $\equiv$ time coordinate.

In deriving Eqs. (2.1), (2.2), it has been assumed that the solid particles in the bed are sufficiently small relative to the dimensions of the tank that a continuum formulation is possible. This justifies the use of the Fourier law of heat conduction, provided that the thermal conductivities of the fluid and solid are replaced by their "effective" values to take into account the more intricate paths followed by the heat transport process.

The boundary conditions for Eqs. (2.1) and (2.2) are taken to be of the form

$$(\partial T_f/\partial x) + aT_f = 0, (\partial T_s/\partial x) + aT_s = 0 \quad \text{at} \quad x = 0, \tag{2.5}$$

$$(\partial T_f/\partial x) + bT_f = 0, (\partial T_s/\partial x) + bT_s = 0 \quad \text{at} \quad x = 1, \tag{2.6}$$

where a, b are constants such that a ≤ 0, $b \geq 0$ (more general boundary conditions are considered in Sec. 8). These conditions allow for heat losses out the ends at $x = 0, 1$ according to Newton's law of cooling. Although heat losses out the tank side walls are neglected in this quasi-one-dimensional model, practical applications indicate that these losses are small compared to the end losses when the horizontal tank dimension is comparable with the vertical length of the tank. This is due to the presence of piping which allows for the passage of fluid into and out of the bed during charging or discharging cycles and which possesses a relatively large thermal conductivity (cf. [3]). Finally, the (continuous) initial conditions are

$$T_f(x, t = 0) = T_f^{(0)}(x), \tag{2.7}$$

$$T_s(x, t = 0) = T_s^{(0)}(x) \tag{2.8}$$

and are assumed to satisfy the boundary conditions (2.5) and (2.6) and to possess continuous first and piecewise continuous second derivatives.

3. Method of solution. The solution technique is based on a vectorized separation of variables approach. Substituting the assumed representation

$$T_f(x, t) = \sum_k F_k(t)\xi_k(x) \tag{3.1}$$

$$T_s(x, t) = \sum_k S_k(t)\psi_k(x) \tag{3.2}$$

into Eqs. (2.1) and (2.2) and requiring that these equations be satisfied termwise gives, after division of the kth term by $F_k\xi_k$, $S_k\psi_k$, respectively,

$$F_k^{-1}(t)\frac{dF_k}{dt} = h_f\frac{S_k(t)}{F_k(t)}\frac{\psi_k(x)}{\xi_k(x)} - h_f + \alpha\xi_k^{-1}(x)\frac{d^2\xi_k}{dx^2} \tag{3.3}$$

$$S_k^{-1}(t)\frac{dS_k}{dt} = h_s\frac{F_k(t)}{S_k(t)}\frac{\xi_k(x)}{\psi_k(x)} - h_s + \psi_k^{-1}(x)\frac{d^2\psi_k}{dx^2}. \tag{3.4}$$

In order to separate variables, it must be required that either $F_k = S_k$ or $\xi_k = \psi_k$ for all k. For the boundary conditions (2.5) and (2.6), the latter choice turns out to be applicable and one can then write Eqs. (3.3) and (3.4) in the vector form

$$\begin{bmatrix} F_k^{-1} & 0 \\ 0 & S_k^{-1} \end{bmatrix}\begin{bmatrix} \dot{F}_k \\ \dot{S}_k \end{bmatrix} + \begin{bmatrix} F_k^{-1} & 0 \\ 0 & S_k^{-1} \end{bmatrix}\begin{bmatrix} h_f & -h_f \\ -h_s & h_s \end{bmatrix}\begin{bmatrix} F_k \\ S_k \end{bmatrix} = \begin{bmatrix} \alpha\xi_k^{-1} & 0 \\ 0 & \xi_k^{-1} \end{bmatrix}\begin{bmatrix} \xi_k'' \\ \xi_k'' \end{bmatrix}. \tag{3.5}$$

The fact that the left-hand side of Eq. (3.5) is independent of x and the right-hand side is independent of t implies that each side must be equal to a (constant) separation vector

$$\begin{bmatrix} -\gamma^2 \\ -\lambda^2 \end{bmatrix}.$$

Thus, the solution is given by

$$\begin{bmatrix} T_f \\ T_s \end{bmatrix} = \sum_k \begin{bmatrix} F_k(t) \\ S_k(t) \end{bmatrix}\xi_k(x), \tag{3.6}$$

where the basis functions $\xi_k(x)$ satisfy

$$\begin{bmatrix} \xi_k'' \\ \xi_k'' \end{bmatrix} + \begin{bmatrix} \alpha^{-1}\xi_k & 0 \\ 0 & \xi_k \end{bmatrix}\begin{bmatrix} \gamma^2 \\ \lambda^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3.7}$$

and the time-dependent coefficients are determined from

$$\begin{bmatrix} \dot{F}_k \\ \dot{S}_k \end{bmatrix} + \begin{bmatrix} h_f + \gamma^2 & -h_f \\ -h_s & h_s + \lambda^2 \end{bmatrix}\begin{bmatrix} F_k \\ S_k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{3.8}$$

The implicit assumption that the basis functions $\xi_k(x)$ span the solution space of

$$\begin{bmatrix} T_f \\ T_s \end{bmatrix}$$

will be justified later.

4. Spatial dependence. The functions $\xi_k(x)$ are determined by the allowable values of the separation vector. From Eq. (3.7) it is clear that

$$\gamma^2 = \alpha\lambda^2 \tag{4.1}$$

and thus $\xi_k(x)$ satisfies

$$\xi_k'' + \lambda^2\xi_k = 0. \tag{4.2}$$

This has the general solution

$$\xi_k(x) = A\cos\lambda x + B\sin\lambda x, \tag{4.3}$$

which will be normalized by setting $A = 1$ for all k. Requiring that each term in the expansion (3.6) satisfy the boundary conditions (2.5) and (2.6) determines the allowable values (eigenvalues) of λ, $\{\lambda_k\}$, and $B(\lambda_k)$. In particular,

$$\xi_k' + a\xi_k = 0 \quad \text{at} \quad x = 0, \tag{4.4}$$

$$\xi_k' + b\xi_k = 0 \quad \text{at} \quad x = 1, \tag{4.5}$$

where a, b are the same as in Eqs. (2.5) and (2.6). Substituting Eq. (4.3) into these conditions yields

$$B = -a/\lambda, \tag{4.6}$$

$$\left(\lambda + \frac{ab}{\lambda}\right) \sin \lambda = (b - a) \cos \lambda. \tag{4.7}$$

Thus, the roots λ_k of Eq. (4.7) satisfy

$$\tan \lambda_k = (b - a)\lambda_k/(\lambda_k^2 + ab) \tag{4.8}$$

and the basis vectors (eigenvectors) ξ_k are

$$\xi_k(x) = \cos \lambda_k x - \frac{a}{\lambda_k} \sin \lambda_k x. \tag{4.9}$$

The roots λ_k of Eq. (4.8) are easily shown to be real. Setting

$$\lambda_k \equiv x + iy, \tag{4.10}$$

one has that

$$\tan \lambda_k = \frac{\tan x \operatorname{sech}^2 y}{1 + \tan^2 x \tanh^2 y} + i\, \frac{\sec^2 x \tanh y}{1 + \tan^2 x \tanh^2 y}, \tag{4.11}$$

$$\frac{(b - a)\lambda_k}{\lambda_k^2 + ab} = (b - a)\, \frac{x(x + ab) + y^2}{(x + ab)^2 + y^2} + i(b - a)\, \frac{aby}{(x + ab)^2 + y^2}. \tag{4.12}$$

Equating the imaginary parts of Eqs. (4.11) and (4.12) and using the restriction $a \le 0, b \ge 0$ ($\Rightarrow b - a \ge 0$, $ab \le 0$) leads to the condition that $y = 0$.

The transcendental equation (4.8) has an infinite number of real roots $\lambda_{\pm_k}$, $k = 0, 1, 2, \cdots$ (Fig. 2). However, from Eq. (4.7), the root $\lambda_0 = 0$ is an extraneous root unless $a = b = 0$. The positive roots λ_k are characterized in the following manner. Defining n_0, m_0 according to

$$|ab|^{1/2} < n_0\pi < |ab|^{1/2} + \pi \tag{4.13}$$

$$m_0 \frac{\pi}{2} < |ab|^{1/2} < (m_0 + 1)\frac{\pi}{2}, \tag{4.14}$$

then, not counting the possible root $x_0 = 0$, there are $[(m_0 + 1)/2]$ ($\equiv$ greatest integer $\le (m_0 + 1)/2$) distinct roots $\lambda_k < |ab|^{1/2}$ and an infinite number of values $\lambda_k > |ab|^{1/2}$ (see Fig. 2). As $k \to \infty$, $\lambda_k \to (k - 1)\pi$. The exact values of λ_k are easily calculated numerically using a Newton-Raphson method.

The basis functions (4.9) are orthogonal on the interval [0, 1] since they are solutions

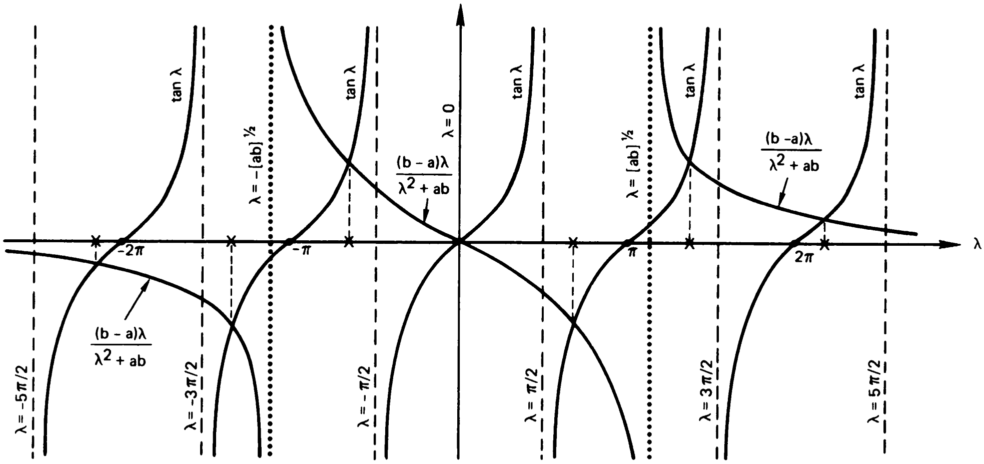


FIG. 2. Graphical solution of $\tan \lambda = (b - a)\lambda/(\lambda^2 + ab)$.

of the Sturm-Liouville problem (4.3) − (4.5). (Note that $\xi_{-k}(x) = \xi_k(x)$, so only positive values of the index need be considered.) That is,

$$\int_0^1 \xi_k(x)\xi_l(x)\,dx = 0, \qquad l \neq k$$
$$= \delta_k, \qquad l = k \tag{4.15}$$

where the normalization constant δ_k is

$$\delta_k = \frac{1}{2} - \frac{a}{2\lambda_k^{\,2}}(1 - \cos 2\lambda_k) + \frac{a^2}{2\lambda_k^{\,2}} + \left(\frac{1}{4\lambda_k} - \frac{a^2}{4\lambda_k^{\,3}}\right)\sin 2\lambda_k, \qquad \lambda_k \neq 0$$
$$= 1, \qquad \lambda_k = 0. \tag{4.16}$$

5. Completeness of eigenfunctions. In order to be able to express the solution (3.6) as a sum of the basis functions $\xi_k(x)$ at any time t, these functions must span the solution space of T_f , T_s . This fact is proven here (for the case where a and b are not both zero) by first converting Eqs. (4.2), (4.4), (4.5) for the $\xi_k(x)$ into an integral equation and then utilizing the properties of the resulting integral operator.

The Green's function $G(x; y)$, $0 \le x \le 1, 0 \le y \le 1$, for the Sturm-Liouville problem

$$(d^2\xi_k/dx^2) + \lambda^2\xi_k = 0, \tag{5.1}$$

$$(d\xi_k/dx) + a\xi_k = 0 \quad \text{at} \quad x = 0, \tag{5.2}$$

$$(d\xi_k/dx) + b\xi_k = 0 \quad \text{at} \quad x = 1, \tag{5.3}$$

satisfies the following conditions:

$$d^2G/dx^2 = 0, \qquad x \neq y, \tag{5.4}$$

$$\lim_{\epsilon \to 0} [G(y + \epsilon; y) - G(y - \epsilon; y)] = 0, \tag{5.5}$$

$$\lim_{\epsilon \to 0} \left[\frac{dG}{dx}\bigg|_{x=y+\epsilon} - \frac{dG}{dx}\bigg|_{x=y-\epsilon}\right] = -1, \tag{5.6}$$

$$(dG/dx) + aG = 0 \quad \text{at} \quad x = 0. \tag{5.7}$$

$$(dG/dx) + bG = 0 \quad \text{at} \quad x = 1. \tag{5.8}$$

The general solution of Eq. (5.4) is

$$\begin{aligned} G(x; y) &= A + Bx, \qquad 0 \le x < y \\ &= C + Dx, \qquad y < x \le 1 \end{aligned} \tag{5.9}$$

and applying the four conditions (5.5)–(5.8) enables one to determine the four unknowns A, B, C, D. The result is

$$\begin{aligned} G(x; y) &= \frac{-b - 1 + by}{b - a(1 + b)}(-1 + ax), \qquad x < y, \\ &= \frac{-1 + ay}{b - a(1 + b)}(-b - 1 + bx), \qquad x > y. \end{aligned} \tag{5.10}$$

It is readily verified that solving the problem (5.1)–(5.3) is equivalent to solving the integral equation

$$\xi(x) = \lambda^2 \int_0^1 G(x; y)\xi(y)\, dy, \tag{5.11}$$

since Eqs. (5.4) and (5.6) imply that

$$d^2G/dx^2 = -\delta(x - y), \tag{5.12}$$

where $\delta(x - y)$ is the Dirac delta function:

$$\delta(x - y) = 0, \qquad x \neq y, \tag{5.13}$$

$$\int_0^1 \delta(x - y)\, dx = 1, \qquad y \in (0, 1) \tag{5.14}$$

The integral operator $\mathbf{H}$ defined by

$$\mathbf{H}\phi \equiv \int_0^1 G(x; y)\phi(y)\, dy, \tag{5.15}$$

where ϕ is any element in the domain of $\mathbf{H}$, is a Hilbert-Schmidt operator. It is self-adjoint, since the kernel $G(x; y)$ is symmetric:

$$G(x; y) = G^*(y; x), \tag{5.16}$$

where * denotes the complex conjugate. It is also compact (cf. Helmberg [4]), and therefore bounded:

$$\|\mathbf{H}\|^2 \leqq \int_0^1 \int_0^1 G(x, y)G^*(x, y)\, dx\, dy < \infty. \tag{5.17}$$

The spectrum of a compact self-adjoint operator is well-characterized, and most of the results of the previous section follow immediately from the theory of such operators. In particular, the eigenvalues $\{1/\lambda_k^2\}$ are bounded above by $\|\mathbf{H}\|$ and have the origin as the only accumulation point (if the number of eigenvalues is infinite). The eigenfunctions $\{\xi_k(x)\}$ corresponding to distinct eigenvalues are orthogonal and span the range of $\mathbf{H}$ (cf. Helmberg [4]). This last result leads to the desired completeness property of the eigenfunctions. Any continuous function $\phi(x)$ satisfying the boundary conditions (5.2) and (5.3) and having a continuous first and a piecewise continuous second derivative lies in the range of **H**, for one can write

$$\phi(x) = \int_0^1 G(x; y)[-\phi''(y)]\, dy \tag{5.18}$$

(To see this, note that because of (5.6) an identity is obtained if Eq. (5.18) is differentiated twice. The fact that both sides of Eq. (5.18) satisfy the same boundary conditions (5.2) and (5.3) then gives the result.) Hence, $\phi(x)$ may be represented as an infinite series (uniformly convergent) of eigenfunctions:

$$\phi(x) = \sum_k \alpha_k \xi_k(x) = \sum_k \alpha_k \left(\cos \lambda_k x - \frac{a}{\lambda_k} \sin \lambda_k x\right), \tag{5.19}$$

where the orthogonality of the $\xi_k(x)$ and the normalization integral gives

$$\alpha_k = \frac{\int_0^1 \phi(x)\xi_k(x)\, dx}{\frac{1}{2} - \frac{a}{2\lambda_k^2}(1 - \cos 2\lambda_k) + \frac{a^2}{2\lambda_k^2} + \left(\frac{1}{4\lambda_k} - \frac{a^2}{4\lambda_k^3}\right)\sin 2\lambda_k} \tag{5.20}$$

(Note also that $\lambda_k \neq 0$ for any k since $1/\lambda_k^2 \leqq \|\mathbf{H}\| < \infty \Rightarrow \lambda_k^2 \geqq 1/\|\mathbf{H}\| > 0$.)

The case $a = b = 0$ must be treated separately due to the fact that a Green's function satisfying (5.4)-(5.8) cannot be constructed. This behavior is caused by the fact that $\lambda = 0$ is an eigenvalue of Eq. (5.1) for this case (the corresponding eigenfunction, from Eq. (4.3), is $\xi_k(x) = \text{const.}$, which of course satisfies (5.1)-(5.3) when $a = b = 0$). This special case can be remedied by constructing a generalized Green's function (cf. Courant and Hilbert [5]), but the completeness result follows directly by noting from Eq. (4.7) that the eigenvalues are

$$\lambda_k = k\pi, \qquad k = 0, 1, 2, \cdots \tag{5.21}$$

and the resulting eigenfunctions are

$$\xi_k(x) = \cos k\pi, \qquad k = 0, 1, 2, \cdots \tag{5.22}$$

One can extend $\phi(x)$ to the interval $-1 \leqq x \leqq 1$ by requiring that $\phi(x)$ be an even function; i.e.,

$$\phi(x) = \phi(-x). \tag{5.23}$$

The completeness of the ordinary trigonometric basis

$$\cos kx, \sin kx, \qquad k = 0, 1, 2, \cdots \tag{5.24}$$

on $[-1, 1]$ guarantees that

$$\phi(x) = \sum_{k=0}^{\infty} (\alpha_k \cos kx + \beta_k \sin kx). \tag{5.25}$$

But the evenness of $\phi(x)$ and oddness of $\sin kx$ give, by use of the orthogonality relations among the basis functions (5.24),

$$\beta_k = \int_{-1}^{1} \phi(x) \sin kx\, dx = 0 \quad \text{for all} \quad k \tag{5.26}$$

and thus the eigenfunctions (5.22) span the space of those functions satisfying (5.1)-(5.3).

As a final note, it is desirable to know for computational purposes how rapidly the "anharmonic" Fourier coefficients α_k in (5.19) tend to zero as k gets large. Assuming that $\phi(x)$, $\phi'(x)$ are continuous and that $\phi''(x)$ is at least piecewise continuous, one can integrate the integral in Eq. (5.20) by parts twice and obtain

$$\int_0^1 \phi(x)(\cos \lambda_k x - \frac{a}{\lambda_k} \sin \lambda_k x)\, dx = \frac{1}{\lambda_k}\phi(x)(\sin \lambda_k x + \frac{a}{\lambda_k} \cos \lambda_k x)\Big|_{x=0}^{1}$$

$$+ \frac{1}{\lambda_k^2}\phi'(x)(\cos \lambda_k x - \frac{a}{\lambda_k} \sin \lambda_k x)\Big|_{x=0}^{1} - \frac{1}{\lambda_k^2}\int_0^1 \phi''(x)(\cos \lambda_k x - \frac{a}{\lambda_k} \sin \lambda_k x)\, dx. \quad (5.27)$$

For large k, $\lambda_k \approx (k-1)\pi$ and thus from (4.8) the term $(1/\lambda_k)\phi(x) \sin \lambda_k x \,|^1_{x=0}$ tends to zero like $O(1/\lambda_k^2)$. Consequently, the asymptotic behavior of α_k is

$$\alpha_k \sim O(1/\lambda_k^2) \quad \text{as} \quad k \to \infty. \quad (5.28)$$

6. Time dependence. The time dependence of the solution (3.6) is determined by the evolution equation (3.8) with $\lambda^2 = \lambda_k^2$, $\gamma^2 = \alpha\lambda_k^2$:

$$\frac{d}{dt}\begin{bmatrix} F_k \\ S_k \end{bmatrix} = \begin{bmatrix} -h_f - \alpha\lambda_k^2 & h_f \\ h_s & -h_s - \lambda_k^2 \end{bmatrix}\begin{bmatrix} F_k \\ S_k \end{bmatrix}. \quad (6.1)$$

This has the formal solution

$$\begin{bmatrix} F_k(t) \\ S_k(t) \end{bmatrix} = \exp(\mathbf{M}t)\begin{bmatrix} F_k^{(0)} \\ S_k^{(0)} \end{bmatrix} \quad (6.2)$$

where

$$\mathbf{M} \equiv \begin{bmatrix} -h_f - \alpha\lambda_k^2 & h_f \\ h_s & -h_s - \lambda_k^2 \end{bmatrix} \quad (6.3)$$

and

$$\begin{bmatrix} F_k^{(0)} \\ S_k^{(0)} \end{bmatrix} = \frac{1}{\delta_k}\int_{-1}^1 \begin{bmatrix} T_f^{(0)}(x) \\ T_s^{(0)}(x) \end{bmatrix}\xi_k(x)\, dx, \quad (6.4)$$

where δ_k is given by Eq. (4.26) and $T_f^{(0)}$, $T_s^{(0)}$ are the initial conditions (2.7) and (2.8).

The convenient evaluation of the solution (6.2) requires a knowledge of the eigenvalues and eigenvectors of $\mathbf{M}$. Defining

$$\tilde{a} \equiv -h_f - \alpha\lambda_k^2, \qquad \tilde{b} \equiv h_f, \quad (6.5a, b)$$

$$\tilde{c} \equiv h_s, \qquad \tilde{d} \equiv -h_s - \lambda_k^2, \quad (6.5c, d)$$

the eigenvalues $\mu_\pm$ of $\mathbf{M} = \begin{bmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{bmatrix}$ are

$$\begin{aligned} \mu_\pm &= \frac{1}{2}(\tilde{a} + \tilde{d}) \pm \frac{1}{2}[(\tilde{a} - \tilde{d})^2 + 4\tilde{b}\tilde{c}]^{1/2} \\ &= -\frac{1}{2}[(h_s + h_f) + \lambda_k^2(1 + \alpha)] \\ &\quad \pm \frac{1}{2}\{[(h_s - h_f) + \lambda_k^2(1 - \alpha)]^2 + 4h_f h_s\}^{1/2}. \end{aligned} \quad (6.6)$$

It is readily shown that $\mu_\pm$ are both real and negative and that for large k,

$$\begin{aligned}\mu_+ &\sim -\alpha\lambda_k^2 \sim -\alpha\pi^2(k-1)^2, & \alpha \leqq 1,\\ &\sim -\lambda_k^2 \sim -\pi^2(k-1)^2, & \alpha \geqq 1,\end{aligned} \tag{6.7}$$

$$\begin{aligned}\mu_- &\sim -\lambda_k^2 \sim -\pi^2(k-1)^2, & \alpha \leqq 1,\\ &\sim -\alpha\lambda_k^2 \sim -\alpha\pi^2(k-1)^2, & \alpha \geqq 1.\end{aligned} \tag{6.8}$$

The eigenvectors $e_\pm$ corresponding to $\mu_\pm$ are

$$e_\pm = \begin{bmatrix} 1 \\ \dfrac{\mu_\pm - \tilde{a}}{\tilde{b}} \end{bmatrix} = \begin{bmatrix} 1 \\ \dfrac{\tilde{c}}{\mu_\pm - \tilde{d}} \end{bmatrix}.$$

The columns of the similarity transformation $\mathbf{R}$ which diagonalizes $\mathbf{M}$ are the eigenvectors $e_\pm$. Thus, defining

$$\mathbf{R} \equiv \begin{bmatrix} 1 & 1 \\ \dfrac{\mu_+ - \tilde{a}}{\tilde{b}} & \dfrac{\mu_- - \tilde{a}}{\tilde{b}} \end{bmatrix}, \tag{6.10}$$

one calculates

$$\mathbf{R}^{-1} = \begin{bmatrix} \dfrac{\mu_- - \tilde{a}}{\mu_- - \mu_+} & \dfrac{-\tilde{b}}{\mu_- - \mu_+} \\ \dfrac{-(\mu_+ - \tilde{a})}{\mu_- - \mu_+} & \dfrac{\tilde{b}}{\mu_- - \mu_+} \end{bmatrix} \tag{6.11}$$

Hence,

$$\exp(\mathbf{M}t) = \mathbf{R}\exp(\mathbf{R}^{-1}\mathbf{M}\mathbf{R})\mathbf{R}^{-1} = \mathbf{R}\begin{bmatrix} \exp(\mu_+ t) & 0 \\ 0 & \exp(\mu_- t) \end{bmatrix}\mathbf{R}^{-1}$$

$$= \frac{1}{\mu_- - \mu_+}\begin{bmatrix} (\mu_- - \tilde{a})\exp(\mu_+ t) - (\mu_+ - \tilde{a})\exp(\mu_- t) & -\tilde{b}\exp(\mu_+ t) + \tilde{b}\exp(\mu_- t) \\ \dfrac{1}{\tilde{b}}(\mu_+ - \tilde{a})(\mu_- - \tilde{a})[\exp(\mu_+ t) - \exp(\mu_- t)] & -(\mu_+ - \tilde{a})\exp(\mu_+ t) + (\mu_- - \tilde{a})\exp(\mu_- t) \end{bmatrix}. \tag{6.12}$$

Substituting Eq. (6.12) into Eq. (6.2) finally gives

$$\begin{aligned} F_k(t) = {} & \frac{1}{\mu_- - \mu_+}[(\mu_- - \tilde{a})F_k^{(0)} - \tilde{b}S_k^{(0)}]\exp(\mu_+ t) \\ & + \frac{1}{\mu_- - \mu_+}[-(\mu_+ - \tilde{a})F_k^{(0)} - \tilde{b}S_k^{(0)}]\exp(\mu_- t), \end{aligned} \tag{6.13}$$

$$\begin{aligned} S_k(t) = {} & \frac{\mu_+ - \tilde{a}}{(\mu_- - \mu_+)}\left[\frac{1}{\tilde{b}}(\mu_- - \tilde{a})F_k^{(0)} - S_k^{(0)}\right]\exp(\mu_+ t) \\ & + \frac{\mu_- - \tilde{a}}{(\mu_- - \mu_+)}\left[-\frac{1}{\tilde{b}}(\mu_+ - \tilde{a})F_k^{(0)} + S_k^{(0)}\right]\exp(\mu_- t). \end{aligned} \tag{6.14}$$

The complete analytic solution for $T_f(x, t)$, $T_s(x, t)$ is now given by Eqs. (3.6), (4.9), (6.13), and (6.14):

$$\begin{bmatrix} T_f(x, t) \\ T_s(x, t) \end{bmatrix} = \sum_k \begin{bmatrix} F_k(t) \\ S_k(t) \end{bmatrix} \left(\cos \lambda_k x - \frac{a}{\lambda_k} \sin \lambda_k x\right). \tag{6.15}$$

Assuming the initial data is continuous, has a continuous first and piecewise continuous second derivative, and satisfies the boundary conditions (2.5), (2.6), the series (6.15) and all partial derivatives with respect to x and t are uniformly convergent for all x and t lying in the semi-infinite strip $0 \leqq x \leqq 1$, $t \geqq 0$. This last result follows from the asymptotic behavior (6.7), (6.8) of $\mu_\pm$ as $k \to \infty$ (cf. Petrovskii [6]).

7. Example. In order to use the above analysis, one must be given the initial profiles (2.7), (2.8). The theoretical specification of these profiles arising from a consideration of the general problem in which fluid flows through the tank is still unsolved for a finite tank length. However, Schumann's analysis (cf. [1]) for the semi-infinite tank coupled with various laboratory measurements (cf. [3]) and numerical experiments indicate that the profiles of Fig. 3a are typical. The actual specification of these profiles has been artifically constructed according to Eqs. (7.1)-(7.12) below so as to mimic the general features of the initial profiles in a partially charged bed immediately after hot fluid has been passed through the tank. (The finite rate of heat transfer between fluid and solid accounts for the differences in the two profiles.) However, the solution technique is applicable to arbitrary initial conditions and is easily implemented on a computer. The $O(\lambda_k^{-2})$ decay of the anharmonic Fourier coefficients permits an early truncation of the infinite series solution and thus results in an efficient algorithm.

The particular initial profiles of Fig. 3a were obtained from the following equations:

$$T_{f,s}^{(0)}(x) = D_1 + C_1(x_0 - x)^2 + E_1(x_0 - x)^4, \qquad 0 \leqq x \leqq x_0, \tag{7.1}$$

$$T_f^{(0)}(x) = \alpha_1 + (\alpha_2 - \alpha_1)\left[1 - \cos^6\left(\frac{\pi}{2} \cdot \frac{x - x_0}{y_0 - x_0}\right)\right], \qquad x_0 \leqq x \leqq y_0, \tag{7.2a}$$

$$T_s^{(0)}(x) = \alpha_1 + (\alpha_2 - \alpha_1)\left[1 - \cos^4\left(\frac{\pi}{2} \cdot \frac{x - x_0}{y_0 - x_0}\right)\right], \qquad x_0 \leqq x \leqq y_0, \tag{7.2b}$$

$$T_{f,s}^{(0)}(x) = D_2 + C_2(x - y_0)^2 + E_2(x - y_0)^4, \qquad y_0 \leqq x \leqq 1. \tag{7.3}$$

Eqs. (7.1), (7.3) allow the boundary conditions to be satisfied and (7.2a, b) give the thermocline-like behavior to the profiles. In order to satisfy boundary and continuity restrictions, it is required that

$$T_{f,s}^{(0)}(x = 0) = \alpha_1 - \epsilon_1, \tag{7.4}$$

$$\partial T_{f,s}^{(0)}/\partial x\big|_{x=0} + aT_{f,s}^{(0)}\big|_{x=0} = 0, \tag{7.5}$$

$$T_{f,s}^{(0)}(x = x_0) = \alpha_1, \tag{7.6}$$

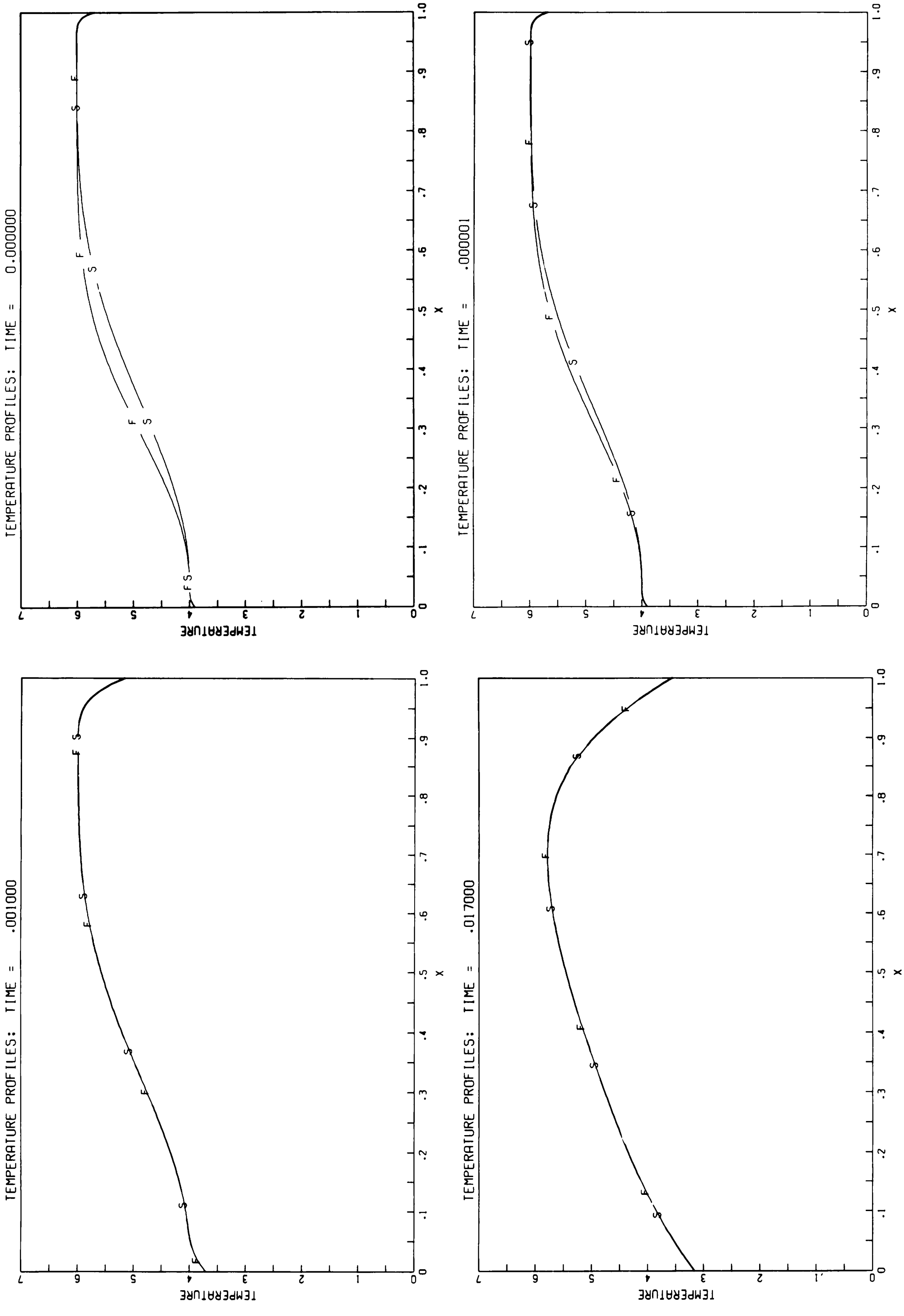
$$T_{f,s}^{(0)}(x = y_0) = \alpha_2, \tag{7.7}$$

$$\partial T_{f,s}^{(0)}/\partial x\big|_{x=1} + bT_{f,s}^{(0)}\big|_{x=1} = 0, \tag{7.8}$$

$$T_{f,s}^{(0)}(x = 1) = \alpha_2 - \epsilon_2. \tag{7.9}$$

This gives

$$D_1 = \alpha_1; \qquad D_2 = \alpha_2, \tag{7.10}$$


TEMPERATURE PROFILES: TIME = 0.000000
TEMPERATURE PROFILES: TIME = .000001
TEMPERATURE PROFILES: TIME = .001000
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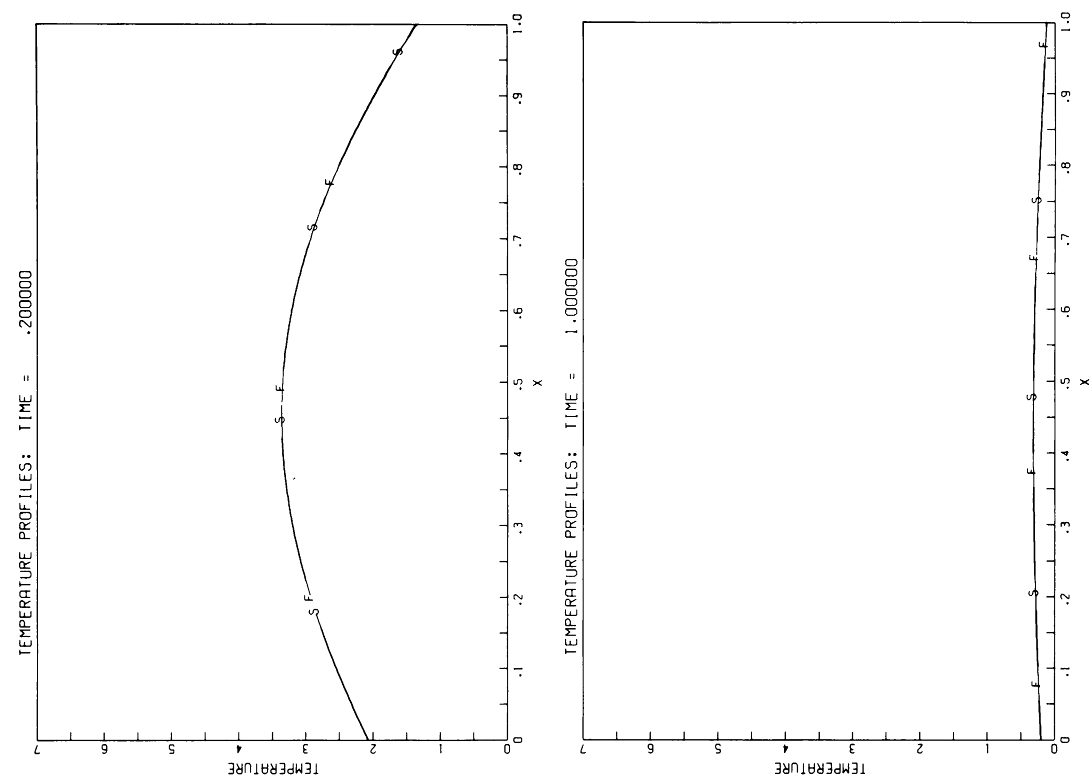

FIG. 3a-f. Time evolution of temperature profiles for the case $a = -2.5$, $b = 5.0$.

$$C_1 = \frac{-4\epsilon_1 - ax_0(\alpha_1 - \epsilon_1)}{2x_0^2}; \qquad C_2 = \frac{-4\epsilon_2 + b(1 - y_0)(\alpha_2 - \epsilon_2)}{2(1 - y_0)^2}, \tag{7.11}$$

$$E_1 = \frac{2\epsilon_1 + ax_0(\alpha_1 - \epsilon_1)}{2x_0^4}; \qquad E_2 = \frac{2\epsilon_2 - b(1 - y_0)(\alpha_2 - \epsilon_2)}{2(1 - y_0)^4}, \tag{7.12}$$

where α_1, α_2, x_0, y_0, ϵ_1, ϵ_2 are yet to be specified and allow some further freedom in the choice of initial profiles.

For an ambient outside temperature T_A^* of 80°F (see Eq. (2.3a)), the choice $\alpha_1 = 4$, $\alpha_2 = 6$ gives a typical pair of initial thermocline profiles between temperatures of 400°F and 560°F over the interval $x_0 \leqq x \leqq y_0$. For a tank length L^* of 50 feet and a typical mean bed conductivity λ^* of 0.674 BTU/hr.-ft.°F (as reported in [3] for an oil/granite bed with a 25% void fraction), the choices of dimensionless heat transfer coefficients (Nusselt numbers) $a = -2.5$, $b = 5.0$ correspond to dimensional values of $a\lambda^*/L^* = -0.034$ BTU/hr,-ft.°F, $b\lambda^*/L^* = 0.068$ BTU/hr.-ft.^{2}°F, respectively. These values are roughly an order of magnitude less than those measured in [3], but the results presented here serve to indicate a best possible performance for the thermal storage system which might be attainable with much improved insulation. Choosing $\epsilon_1 = 0.1$, $\epsilon_2 = 0.2$, $x_0 = 0.04$, $y_0 = 0.96$ gives the profiles shown in Fig. 3a, which of course satisfy the boundary conditions (7.5), (7.8). Figs. 3b-3f show the time evolution of the thermocline when the typical values (for a 50-foot tank) of h_f, h_s, α are taken to be 5×10^5, 2.5×10^5, 0.1, respectively. Note that the large values of h_f , h_s in this particular example prevent the maintenance or formation of any significant temperature difference between fluid and solid, as one would expect.

One measure of a time τ to thermocline breakdown is the time it takes for the maximum temperature in the bed to shrink to α_1 plus a certain percentage, say 90%, of $\alpha_2 - \alpha_1$. For the data used here, $\tau \approx 0.017$ (850 hr., assuming a solid diffusivity of 0.05 ft^2/hr.), as shown in Fig. 3d. Thus, the important physical result is the quantitative prediction of $T_f(x, t)$, $T_s(x, t)$ which allows one to determine the point at which no useful heat may be extracted from the bed due to the decay of the initial profiles.

8. More general boundary conditions. If the boundary conditions for T_f and T_s are different, one cannot choose $\xi_k(x) = \psi_k(x)$ in (3.3) and (3.4). That is, the basis functions for T_f and T_s are different, since each ξ_k must satisfy the boundary conditions on T_f and each ψ_k must satisfy those on T_s . It turns out, as will be shown in this section, that if one chooses $F_k(t) = S_k(t)$ for all k (instead of $\xi_k = \psi_k$) in (3.3) and (3.4), then one can handle, for the special case $\alpha = 1$, $h_f = h_s$, general self-adjoint boundary conditions of the form

$$\frac{\partial}{\partial x}\begin{bmatrix} T_f \\ T_s \end{bmatrix} + \begin{bmatrix} a_0 & c_0 \\ c_0 & b_0 \end{bmatrix}\begin{bmatrix} T_f \\ T_s \end{bmatrix} = 0 \quad \text{at} \quad x = 0, \tag{8.1}$$

$$\frac{\partial}{\partial x}\begin{bmatrix} T_f \\ T_s \end{bmatrix} + \begin{bmatrix} a_1 & c_1 \\ c_1 & b_1 \end{bmatrix}\begin{bmatrix} T_f \\ T_s \end{bmatrix} = 0 \quad \text{at} \quad x = 1 \tag{8.2}$$

(if $c_0 = c_1 = 0$, the restriction $\alpha = 1$ is unnecessary).

Choosing $F_k(t) = S_k(t)$ in Eqs. (3.3) and (3.4) in order to separate variables, one obtains

$$\begin{aligned} &\begin{bmatrix} F_k^{-1} & 0 \\ 0 & F_k^{-1} \end{bmatrix}\begin{bmatrix} \dot{F}_k \\ \dot{F}_k \end{bmatrix} + \begin{bmatrix} F_k^{-1} & 0 \\ 0 & F_k^{-1} \end{bmatrix}\begin{bmatrix} h_f & 0 \\ 0 & h_s \end{bmatrix}\begin{bmatrix} F_k \\ F_k \end{bmatrix} \\ &= \begin{bmatrix} \alpha\xi_k^{-1} & 0 \\ 0 & \psi_k^{-1} \end{bmatrix}\begin{bmatrix} \xi_k'' \\ \psi_k'' \end{bmatrix} + \begin{bmatrix} \xi_k^{-1} & 0 \\ 0 & \psi_k^{-1} \end{bmatrix}\begin{bmatrix} 0 & h_f \\ h_s & 0 \end{bmatrix}\begin{bmatrix} \xi_k \\ \psi_k \end{bmatrix}. \end{aligned} \tag{8.3}$$

Again, the left-hand side of Eq. (8.3) is independent of x and the right-hand side is independent of t and hence each side must be equal to a constant separation vector

$$\begin{bmatrix} -\gamma^2 \\ -\lambda^2 \end{bmatrix}.$$

Thus,

$$\begin{bmatrix} T_f \\ T_s \end{bmatrix} = \sum_k \begin{bmatrix} \xi_k(x) \\ \psi_k(x) \end{bmatrix} F_k(t), \tag{8.4}$$

where

$$\begin{bmatrix} \dot{F}_k \\ \dot{F}_k \end{bmatrix} + \begin{bmatrix} h_f + \gamma^2 & 0 \\ 0 & h_s + \lambda^2 \end{bmatrix}\begin{bmatrix} F_k \\ F_k \end{bmatrix} = 0, \tag{8.5}$$

$$\begin{bmatrix} \xi_k'' \\ \psi_k'' \end{bmatrix} + \begin{bmatrix} \alpha^{-1}\gamma^2 & \alpha^{-1}h_f \\ h_s & \lambda^2 \end{bmatrix}\begin{bmatrix} \xi_k \\ \psi_k \end{bmatrix} = 0. \tag{8.6}$$

Clearly, Eq. (8.5) requires that

$$h_f + \gamma^2 = h_s + \lambda^2, \tag{8.7}$$

and thus the equations for F_k, ξ_k, and ψ_k are

$$\frac{d}{dt} F_k + (h_f + \gamma^2)F_k = 0, \tag{8.8}$$

$$\frac{d^2}{dx^2}\begin{bmatrix}\xi_k \\ \psi_k\end{bmatrix} + \begin{bmatrix}\alpha^{-1}\gamma^2 & \alpha^{-1}h_f \\ h_s & h_f - h_s + \gamma^2\end{bmatrix}\begin{bmatrix}\xi_k \\ \psi_k\end{bmatrix} = 0. \tag{8.9}$$

The formal solution of Eq. (8.9) is

$$\begin{bmatrix}\xi_k \\ \psi_k\end{bmatrix} = \cos(\mathbf{M}x)\begin{bmatrix}A_1 \\ B_1\end{bmatrix} + \sin(\mathbf{M}x)\begin{bmatrix}A_2 \\ B_2\end{bmatrix} \tag{8.10}$$

where

$$\mathbf{M} \equiv \begin{bmatrix}\alpha^{-1}\gamma^2 & \alpha^{-1}h_f \\ h_s & h_f - h_s + \gamma^2\end{bmatrix}. \tag{8.11}$$

Here,

$$\cos(\mathbf{M}x) = \mathbf{R}\begin{bmatrix}\cos\Lambda_1 x & 0 \\ 0 & \cos\Lambda_2 x\end{bmatrix}\mathbf{R}^{-1}, \tag{8.12}$$

$$\sin(\mathbf{M}x) = \mathbf{R}\begin{bmatrix}\sin\Lambda_1 x & 0 \\ 0 & \sin\Lambda_2 x\end{bmatrix}\mathbf{R}^{-1}, \tag{8.13}$$

where Λ_1, Λ_2 are the eigenvalues of $\mathbf{M}$ and $\mathbf{R}$ is the similarity transformation matrix whose columns are the eigenvectors of $\mathbf{M}$.

The solution (8.10) may be normalized by setting $A_1 \equiv 1$ and the two boundary conditions (8.1), (8.2) on

$$\begin{bmatrix}\xi_k \\ \psi_k\end{bmatrix}$$

then determine the allowable values of γ, denoted by γ_k, and $A_2(\gamma_k)$, $B_1(\gamma_k)$, $B_2(\gamma_k)$.

The determination of the eigenvalues γ_k and corresponding eigenfunctions

$$\begin{bmatrix}\xi_k(x) \\ \psi_k(x)\end{bmatrix}$$

can be related to the solution of an integral equation in a manner similar to that employed in Sec. 5, if one adopts a fully vectorized approach to the earlier technique. In particular a Green's tensor function

$$\mathbf{G}(x; y) = \begin{bmatrix}G_\xi^{(1)}(x; y) & G_\xi^{(2)}(x; y) \\ G_\psi^{(1)}(x; y) & G_\psi^{(2)}(x; y)\end{bmatrix} \tag{8.14}$$

(cf. Courant and Hilbert [5]) is constructed for the vector Sturm-Liouville problem

$$\frac{d}{dx}\begin{bmatrix}\alpha & 0 \\ 0 & 1\end{bmatrix}\frac{d}{dx}\begin{bmatrix}\xi_k \\ \psi_k\end{bmatrix} + \begin{bmatrix}0 & h_f \\ h_s & h_f - h_s\end{bmatrix}\begin{bmatrix}\xi_k \\ \psi_k\end{bmatrix} + \gamma^2\begin{bmatrix}\xi_k \\ \psi_k\end{bmatrix} = 0, \tag{8.15}$$

obtained from multiplying (8.9) by the (constant) matrix

$$\begin{bmatrix}\alpha & 0 \\ 0 & 1\end{bmatrix},$$

subject to the boundary conditions (8.1), (8.2). Each of the vectors

$$\begin{bmatrix} G_\xi^{(i)} \\ G_\psi^{(i)} \end{bmatrix}, \qquad i = 1, 2,$$

is required to satisfy (8.1), (8.2) and

$$\frac{d}{dx}\begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix}\frac{d}{dx}\begin{bmatrix} G_\xi^{(i)} \\ G_\psi^{(i)} \end{bmatrix} + \begin{bmatrix} 0 & h_f \\ h_s & h_f - h_s \end{bmatrix}\begin{bmatrix} G_\xi^{(i)} \\ G_\psi^{(i)} \end{bmatrix} = 0, \qquad x \neq y, \tag{8.16}$$

$$\lim_{\epsilon\to 0}\left\{\begin{bmatrix} G_\xi^{(i)} \\ G_\psi^{(i)} \end{bmatrix}_{x=y+\epsilon} - \begin{bmatrix} G_\xi^{(i)} \\ G_\psi^{(i)} \end{bmatrix}_{x=y-\epsilon}\right\} = 0, \tag{8.17}$$

$$\lim_{\epsilon\to 0}\left\{\frac{d}{dx}\begin{bmatrix} G_\xi^{(i)} \\ G_\psi^{(i)} \end{bmatrix}_{x=y+\epsilon} - \frac{d}{dx}\begin{bmatrix} G_\xi^{(i)} \\ G_\psi^{(i)} \end{bmatrix}_{x=y-\epsilon}\right\}$$

$$= -\begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix}^{-1}\begin{bmatrix} \kappa_{1i} \\ \kappa_{2i} \end{bmatrix} = -\begin{bmatrix} \kappa_{1i}\alpha^{-1} \\ \kappa_{2i} \end{bmatrix}, \tag{8.18}$$

where

$$\begin{aligned} \kappa_{ij} &= 1, \qquad i = j, \\ &= 0, \qquad i \neq j. \end{aligned} \tag{8.19}$$

The result is that

$$\begin{bmatrix} \xi(x) \\ \psi(x) \end{bmatrix} = \gamma^2 \int_0^1 \mathbf{G}(x; y)\begin{bmatrix} \xi(y) \\ \psi(y) \end{bmatrix} dy. \tag{8.20}$$

Using the implication from (8.16) − (8.18) that

$$\frac{d}{dx}\begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix}\frac{d}{dx}\mathbf{G}(x; y) + \begin{bmatrix} 0 & h_f \\ h_s & h_f - h_s \end{bmatrix}\mathbf{G}(x; y) = \begin{bmatrix} -\delta(x-y) & 0 \\ 0 & -\delta(x-y) \end{bmatrix}, \tag{8.21}$$

where $\delta(x)$ is the delta function, it is readily verified that the right-hand side of Eq. (8.20) is a solution of Eq. (8.15) and satisfies the boundary conditions in Eqs. (8.1) and (8.2). Those values of γ for which Eq. (8.20) is satisfied are the eigenvalues γ_k, and the corresponding solutions

$$\begin{bmatrix} \xi_k \\ \psi_k \end{bmatrix}$$

are the eigenvectors.

In order to prove orthogonality and completeness of the eigenvectors, one imposes the restriction

$$h_f = h_s \equiv h \tag{8.22}$$

so that the operator

$$\mathfrak{L} \equiv \frac{d}{dx}\begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix}\frac{d}{dx} + \begin{bmatrix} 0 & h_f \\ h_s & h_f - h_s \end{bmatrix} = \frac{d}{dx}\begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix}\frac{d}{dx} + \begin{bmatrix} 0 & h \\ h & 0 \end{bmatrix} \tag{8.23}$$

is self-adjoint. Consequently, any two vectors

$$\begin{bmatrix} u_1(x) \\ u_2(x) \end{bmatrix}, \begin{bmatrix} v_1(x) \\ v_2(x) \end{bmatrix}$$

in the domain of $\mathcal{L}$ satisfy

$$\int_0^1 \left\{[u_1^*, u_2^*]\mathcal{L} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} - [v_1, v_2]\,\mathcal{L} \begin{bmatrix} u_1^* \\ u_2^* \end{bmatrix}\right\} dx$$
$$= \left\{[u_1^*, u_2^*]\,\frac{d}{dx}\begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} - [v_1, v_2]\,\frac{d}{dx}\begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} u_1^* \\ u_2^* \end{bmatrix}\right\}_{x=0}^{1} \tag{8.24}$$

The right-hand side of Eq. (8.22) vanishes if the matrices in Eqs. (8.1) and (8.2) pre-multiplied by

$$\begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix}$$

are self-adjoint. This condition is fulfilled when

$$\text{either } \alpha = 1 \text{ or } c_0 = c_1 = 0. \tag{8.25}$$

Under this assumption, replacing

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

with

$$\begin{bmatrix} G_\xi^{(i)}(x; \eta) \\ G_\psi^{(i)}(x; \eta) \end{bmatrix}, \begin{bmatrix} G_\xi^{(j)}(x; \xi) \\ G_\psi^{(j)}(x; \xi) \end{bmatrix},$$

$i, j = 1, 2$, in Eq. (8.24) and using Eq. (8.21) leads to the result that

$$\mathbf{G}(\xi; \eta) = \mathbf{G}^T(\eta; \xi) \tag{8.26}$$

where $\mathbf{G}^T(\eta; \xi)$ is the adjoint of $\mathbf{G}(\xi; \eta)$ (the superscript "T" denotes the conjugate transpose of the tensor).

The fact that the (Hilbert-Schmidt) integral operator in Eq. (8.20) is compact and self-adjoint implies, as before, that the eigenfunctions are orthogonal; that is,

$$\int_0^1 [\xi_k^*(x), \psi_k^*(x)] \begin{bmatrix} \xi_l(x) \\ \psi_l(x) \end{bmatrix} dx = 0, \qquad k \neq l. \tag{8.27}$$

Also, the fact that the eigenfunctions of a compact self-adjoint operator span the range of that operator implies that the eigenfunctions

$$\begin{bmatrix} \xi_k(x) \\ \psi_k(x) \end{bmatrix}$$

span the space of all continuous vector functions

$$\begin{bmatrix} u(x) \\ v(x) \end{bmatrix}$$

which have continuous first and piecewise continuous second derivatives and satisfy the boundary conditions (8.1) and (8.2). In particular, one has that

$$\begin{bmatrix} u(x) \\ v(x) \end{bmatrix} = \int_0^1 G(x; y) \begin{bmatrix} u''(y) \\ v''(y) \end{bmatrix} dy. \tag{8.28}$$

Finally, the time behavior $F_k(t)$ is, from Eq. (8.8),

$$F_k(t) = F_k^{(0)} \exp(-h - \gamma^2)t,$$

where the anharmonic coefficients $F_k^{(0)}$ are given by

$$F_k^{(0)} = \frac{1}{\delta_k} \int_0^1 [\xi_k^*(x), \psi_k^*(x)] \begin{bmatrix} T_f(x, t = 0) \\ T_s(x, t = 0) \end{bmatrix} dx, \tag{8.29}$$

$$\delta_k = \int_0^1 [\xi_k^*(x), \psi_k^*(x)] \begin{bmatrix} \xi_k(x) \\ \psi_k(x) \end{bmatrix} dx = \int_0^1 [|\xi_k(x)|^2 + |\psi_k(x)|^2]\, dx. \tag{8.30}$$

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