VIBRATIONS OF LONG NARROW PLATES—I*

By

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Abstract. An asymptotic theory for the determination of the natural flexural modes and eigenvalues of a long narrow plate having a quite general planform shape is presented. A local transition layer exists in the vicinity of the widest portion of the plate, which reveals the essential structure of the flexural modes. Mode shapes are computed for trapezoidal and semi-elliptical planforms. The theory is relevant to an understanding of frequency discrimination in the cochlea (inner ear).

1. Introduction. An exact analytical description of the natural flexural modes of a vibrating plate is possible in those few cases when the planform geometry admits a separation of variables solution. The simplest examples are the rectangle and circular sector. It will be shown, however, that under certain conditions when the plate is long and narrow, but otherwise has a quite general planform, an asymptotic theory can be constructed which has a simple separable form. The theory to be presented is basically a slender body theory for plates, so naturally the slenderness or aspect ratio of the planform shape plays an essential role.

Our interest in this problem stems from cochlear mechanics [1]. The main elastic element responsible for frequency discrimination in the inner ear is the basilar membrane, which seems to be best described as a plate since it lacks static tension. The basilar membrane is long and narrow with a slowly changing width, and is helicoidal in shape. It is also bounded on both sides by an incompressible liquid. As a first step towards understanding the vibrations of the basilar membrane, we omit the fluid coupling and helicoidal geometry effects.

2. Formulation. Consider a long narrow plate having the planform shown in Fig. 1. The ends of the plate at $X = 0, L$ are assumed to be straight and parallel to the Y-axis. The curved edges are $Y = \pm BG(X/L)$, where $G(X/L)$ is arbitrarily smooth and B is some characteristic half-width of plate. A precise specification of B is not required.

In physical coordinates, the deflection of the plate $W(X, Y, T)$ satisfies the linear elastic plate equation

$$D\left\{\frac{\partial^4 W}{\partial X^4} + 2\,\frac{\partial^4 W}{\partial X^2\,\partial Y^2} + \frac{\partial^4 W}{\partial Y^4}\right\} + \rho H\,\frac{\partial^2 W}{\partial T^2} = 0 \tag{2.1}$$

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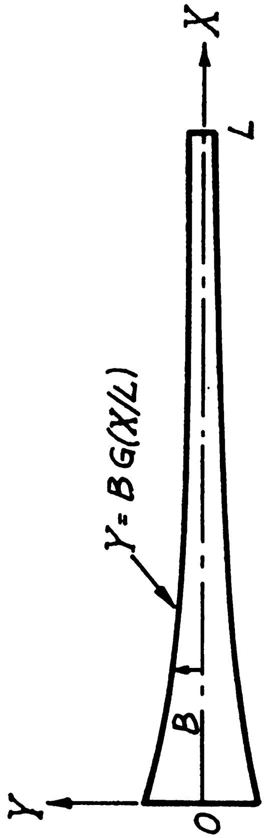

FIG. 1. Planform geometry of plate.

where

$$D = \frac{1}{12} EH^3/(1 - \sigma^2). \tag{2.2}$$

E is the modulus of elasticity, H is the plate thickness, σ is Poisson's ratio, and ρ is the density of the material. All edges are assumed to be simply supported, which requires that the deflection and bending moment vanish on the edges.

Dimensionless coordinates (x, y) are introduced by scaling the longitudinal coordinate with the plate length, and the transverse coordinate with the characteristic half-width:

$$x = X/L, \qquad y = Y/B. \tag{2.3}$$

Since the free vibrations of the plate are to be determined, we look for a solution of the form

$$W(X, Y, T) = w(x, y) \cos \Omega T. \tag{2.4}$$

Then the plate equation (2.1) becomes

$$\frac{\partial^4 w}{\partial y^4} - \omega^2 w = -2\epsilon^2 \frac{\partial^4 w}{\partial x^2\, \partial y^2} - \epsilon^4 \frac{\partial^4 w}{\partial x^4} \tag{2.5}$$

where $\epsilon = B/L$ is the slenderness ratio and $\omega = (\rho H B^4/D)^{1/2}\Omega$ is a dimensionless frequency.

The zero-deflection boundary conditions are written as

$$w(0, y) = 0, \qquad |y| \le G(0); \tag{2.6}$$

$$w(1, y) = 0, \qquad |y| \le G(1); \tag{2.7}$$

$$w(x, \pm G(x)) = 0, \qquad 0 \le x \le 1. \tag{2.8}$$

The exact form of the zero-moment boundary conditions are

$$(\partial^2 w/\partial x^2)(0, y) = 0, \qquad |y| \le G(0); \tag{2.9}$$

$$(\partial^2 w/\partial x^2)(1, y) = 0, \qquad |y| \le G(1); \tag{2.10}$$

$$\begin{aligned} &\frac{\partial^2 w}{\partial y^2}(x, \pm G(x)) + \epsilon^2 \sigma \frac{\partial^2 w}{\partial x^2}(x, \pm G(x)) \\ &\quad + \epsilon^2 G'^2(x) \left\{ \epsilon^2 \frac{\partial^2 w}{\partial x^2}(x, \pm G(x)) + \sigma \frac{\partial^2 w}{\partial y^2}(x, \pm G(x)) \right\} \\ &\quad \mp 2(1-\sigma)\epsilon^2 G'(x) \frac{\partial^2 w}{\partial x\, \partial y}(x, \pm G(x)) = 0, \qquad 0 \le x \le 1. \end{aligned} \tag{2.11}$$

The prime designates differentiation with respect to x. Eqs. (2.9) through (2.11) are obtained from the dimensional relation for the edge moment M_E given by

$$\begin{aligned} M_E = -D &\left\{ \cos^2\theta \left(\frac{\partial^2 W}{\partial X^2} + \sigma \frac{\partial^2 W}{\partial Y^2} \right) + \sin^2\theta \left(\frac{\partial^2 W}{\partial Y^2} + \sigma \frac{\partial^2 W}{\partial X^2} \right) \right. \\ &\left. + (1-\sigma) \sin 2\theta \frac{\partial^2 W}{\partial X\, \partial Y} \right\} = 0 \end{aligned} \tag{2.12}$$

where θ is the angle between the outward normal to an edge and the X axis (cf. Love [2, p. 465]). Eqs. (2.5) through (2.11) define the exact eigenvalue problem.

3. Expansion procedure. The full eigenvalue problem thus defined does not lend itself to an exact analytical solution for a general $G(x)$. To proceed, we must systematically utilize the slenderness of the plate, i.e., $\epsilon \ll 1$, for our immediate application. To begin with, we also consider the special case where $G(x)$ is a smooth monotonic decreasing function with $G'(0) < 0$.

We postulate the existence of a boundary layer transition near $x = 0$, where the maximal vibratory activity occurs. This is reasonable on physical grounds since we expect the largest amplitudes to occur near the wide end of the plate where the compliance is greatest. We are also motivated by the known behavior of the exact solution for a narrow circular sector plate (cf. Sec. 8).

The longitudinal coordinate in this region is stretched by introducing a new variable

$$x^* = x/\delta(\epsilon) \tag{3.1}$$

and we consider the limit process $\epsilon \to 0$ with (x^*, y) fixed. $\delta(\epsilon)$ is a measure of the thickness of the transition layer and its dependence on ϵ must be determined during the course of the solution.

We assume expansions for $w(x, y)$ and ω in the form

$$w(x, y) = w_0(x^*, y) + \gamma_1(\epsilon) w_1(x^*, y) + \gamma_2(\epsilon) w_2(x^*, y) + \cdots \tag{3.2}$$

$$\omega = \alpha + \tau_1(\epsilon)\beta + \tau_2(\epsilon)\mu + \cdots \tag{3.3}$$

where α, β, μ, $\cdots$ are unknown constants, and $\gamma_1(\epsilon)$, $\gamma_2(\epsilon)$, $\cdots$; $\tau_1(\epsilon)$, $\tau_2(\epsilon)$, $\cdots$, are unknown scale factors.

4. First-order problem. The plate equation is written in terms of the stretched longitudinal variable x^* by replacing $\partial/\partial x$ with $\delta^{-1}(\epsilon)\ \partial/\partial x^*$. Substitution of the expansions, Eqs. (3.2) and (3.3), into Eq. (2.5) then yields to first order

$$L_b w_0 = \frac{\partial^4 w_0}{\partial y^4} - \alpha^2 w_0 = 0. \tag{4.1}$$

In obtaining Eq. (4.1), we have assumed $\epsilon/\delta(\epsilon) \to 0$; otherwise Eq. (2.5) would retain its full structure and we would not obtain a distinguished limit equation. Note that L_b is just the beam operator. To the same order, the zero-deflection boundary conditions are

$$w_0(0, y) = 0 \qquad |y| \le a \tag{4.2}$$

$$w_0(x^*, y) \to 0 \quad \text{as} \quad x^* \to \infty \tag{4.3}$$

$$w_0(x^*, \pm a) = 0 \qquad 0 \le x^* < \infty \tag{4.4}$$

where we have written $G(0) = a$. Similarly, the zero-moment conditions to first order are

$$(\partial^2 w_0/\partial x^{*2})(0, y) = 0 \qquad |y| \le a, \tag{4.5}$$

$$\frac{\partial^2 w_0}{\partial x^{*2}} \to 0 \qquad \text{as} \quad x^* \to \infty, \tag{4.6}$$

$$\frac{\partial^2 w_0}{\partial y^2}(x^*, \pm a) = 0 \qquad 0 \le x^* < \infty. \tag{4.7}$$

In this approximation the planform is a semi-infinite strip of width 2a.

The solution to Eq. (4.1) is simply

$$w_0(x^*, y) = A_0(x^*) \begin{Bmatrix} \cos \sqrt{\alpha_n}\, y \\ \sin \sqrt{\alpha_n}\, y \end{Bmatrix} \tag{4.8}$$

where the cosine and sine correspond to symmetric and antisymmetric transverse modes, respectively. To satisfy boundary conditions on $y = \pm a$, it is necessary that

$$\begin{aligned} \sqrt{\alpha_n} &= (n - 1/2)\frac{\pi}{a}; \quad n = 1, 2, \cdots \text{ symmetric modes} \\ &= n\left(\frac{\pi}{a}\right); \qquad n = 1, 2, \cdots \text{ antisymmetric modes.} \end{aligned} \tag{4.9}$$

At this point, $A_0(x^*)$ is still arbitrary and must be determined from the second-order problem. We do know, however, from Eqs. (4.2) through (4.6) that $A_0(x^*)$ must satisfy

$$A_0(0) = (d^2/dx^{*2})A_0(0) = 0, \tag{4.10}$$

$$A_0(x^*),\ (d^2/dx^{*2})A_0(x^*) \to 0 \quad \text{as} \quad x^* \to \infty. \tag{4.11}$$

Eq. (4.9) provides the first approximation to the eigenvalues, which correspond to the natural frequencies of a beam having length 2a. The α_n provide a lower bound to the exact eigenvalues, since the corrections will be shown to be positive. This is to be expected, since $A_0(x^*)$ represents, in effect, a coupling between "adjacent transverse beams" which tends to increase the potential energy stored in bending.

5. Second-order problem. The most complete equation of order $\gamma_1(\epsilon)$ is obtained if $\tau_1(\epsilon) = \gamma_1(\epsilon) = \epsilon^2/\delta^2(\epsilon)$, giving

$$L_0 w_1 = 2\alpha_n \beta w_0 - 2 \frac{\partial^4 w_0}{\partial x^{*2}\, \partial y^2} \equiv R_0 w_0 \,. \tag{5.1}$$

Assuming $G(x)$ has a Taylor series expansion near $x = 0$, we can expand Eq. (2.8) as follows:

$$w_0(x^*, \pm a) \pm \delta(\epsilon) x^* G'(0) \frac{\partial w_0}{\partial y}(x^*, \pm a) + \gamma_1(\epsilon) w_1(x^*, \pm a) + O(\partial^2) = 0. \tag{5.2}$$

A similar expansion exists for the zero moment condition on the curved edge (Eq. (2.11)):

$$\frac{\partial^2 w_0}{\partial y^2}(x^*, \pm a) \pm \delta(\epsilon) x^* G'(0) \frac{\partial^3 w_0}{\partial y^3}(x^*, \pm a) + \gamma_1(\epsilon) \frac{\partial^2 w_1}{\partial y^2}(x^*, \pm a) + O(\delta^2) = 0 \tag{5.3}$$

where we have used

$$(\partial^2 w_0/\partial x^{*2})(x^*, \pm a) = 0 \tag{5.4}$$

from Eq. (4.8). Notice the dominant term in each of these expansions has already been set equal to zero in the first-order problem.

To achieve the most complicated balance of terms in Eqs. (5.2) and (5.3), we set $\delta(\epsilon) = \gamma_1(\epsilon)$, from which we can now determine

$$\delta(\epsilon) = \gamma_1(\epsilon) = \tau_1(\epsilon) = \epsilon^2/\delta^2(\epsilon) = \epsilon^{2/3}. \tag{5.5}$$

The boundary conditions for the second-order problem can now be written as

$$w_1(0, y) = 0, \qquad |y| \le a, \tag{5.6}$$

$$w_1(x^*, y) \to 0 \qquad \text{as} \quad x^* \to \infty, \tag{5.7}$$

$$w_1(x^*, \pm a) = \mp x^* G'(0)(\partial w_0/\partial y)(x^*, \pm a), \qquad 0 \le x^* < \infty, \tag{5.8}$$

$$\frac{\partial^2 w_1}{\partial x^{*2}}(0, y) = 0, \qquad |y| \le a, \tag{5.9}$$

$$\frac{\partial^2 w_1}{\partial x^{*2}}(x^*, y) \to 0 \qquad \text{as} \quad x^* \to \infty, \tag{5.10}$$

$$\frac{\partial^2 w_1}{\partial y^2}(x^*, \pm a) = \mp x^* G'(0) \frac{\partial^3 w_0}{\partial y^3}(x^*, \pm a), \qquad 0 \le x^* < \infty. \tag{5.11}$$

The general solution of Eq. (5.1), substituting Eq. (4.8) in the right-hand side, is

$$\begin{aligned} w_1(x^*, y) &= A_1(x^*) \begin{Bmatrix} \cos \sqrt{\alpha_n}\, y \\ \sin \sqrt{\alpha_n}\, y \end{Bmatrix} + B_1(x^*) \begin{Bmatrix} \cosh \sqrt{\alpha_n}\, y \\ \sinh \sqrt{\alpha_n}\, y \end{Bmatrix} \\ &\quad \mp \frac{1}{2} \frac{1}{\sqrt{\alpha_n}} \left(\frac{d^2 A_0}{dx^{*2}} + \beta A_0 \right) y \begin{Bmatrix} \sin \sqrt{\alpha_n}\, y \\ \cos \sqrt{\alpha_n}\, y \end{Bmatrix} . \end{aligned} \tag{5.12}$$

Choosing $B_1(x^*) = 0$, we can satisfy Eq. (5.8) provided that

$$d^2 A_0/dx^{*2} + \left(\beta - 2\alpha_n \frac{\kappa^2 x^*}{a} \right) A_0 = 0 \tag{5.13}$$

where we have written $\kappa^2 = -G'(0)$. Eq. (5.13) holds for both the symmetric and antisymmetric transverse modes. The zero-moment condition, Eq. (5.11), leads (fortunately) to exactly the same equation for $A_0(x^*)$.

Eq. (5.13) can be obtained more directly by considering the following solvability condition for Eq. (5.1):

$$\int_{-a}^{a} \left\{ \begin{matrix} \cos \sqrt{\alpha_n}\, y \\ \sin \sqrt{\alpha_n}\, y \end{matrix} \right\} L_b w_1 \, dy = \int_{-a}^{a} \left\{ \begin{matrix} \cos \sqrt{\alpha_n}\, y \\ \sin \sqrt{\alpha_n}\, y \end{matrix} \right\} R_0 w_0 \, dy \tag{5.14}$$

where the cosine is used for the symmetric case and the sine is used for the antisymmetric case.

Eq. (5.13) can be transformed into the standard form of Airy's equation with the transformation

$$x^* = x_0 + \lambda z. \tag{5.15}$$

Choosing

$$x_0 = \beta a / 2\alpha_n \kappa^2; \qquad \lambda = (a/2\alpha_n \kappa^2)^{1/3}, \tag{5.16}$$

Eq. (5.13) then takes the form

$$\frac{d^2}{dz^2} A_0(z) - z A_0(z) = 0. \tag{5.17}$$

The appropriate solution which satisfies Eq. (4.11) is

$$A_0(z) = A_i(z) = A_i\left[\frac{1}{\lambda}(x^* - x_0)\right] \tag{5.18}$$

where $A_i(z)$ denotes the Airy function. To satisfy Eq. (4.10) we must choose

$$-x_0/\lambda = \rho_m, \qquad m = 1, 2, \cdots, \tag{5.19}$$

where ρ_m is the mth root of the Airy function. From Eqs. (5.16) we can determine the first correction to the eigenvalues

$$\beta_{nm} = -\rho_m (2\alpha_n \kappa^2 / a)^{2/3}, \qquad n, m = 1, 2, \cdots. \tag{5.20}$$

Since all $\rho_m < 0$, the β_{nm} are necessarily positive.

With $A_0(x^*)$ now determined, we have fully determined $w_0(x^*, y)$ and will not be concerned with exponentially small deflection and bending moment at $x = 1$. $A_0(x^*)$ is shown by the solid line in Fig. 2 for the case $n = 1$, $m = 5$, and a planform geometry discussed in Sec. 8. For this particular case the transition point $\delta x_0 = 0.208$. For $x > \delta x_0$, the deflection decays exponentially, and for $x < \delta x_0$ the function has an oscillatory behavior. The longitudinal extent of the transition layer is $O(\epsilon^{2/3})$. For this case, $\epsilon^{2/3} \simeq 0.042$, and we see that $m\epsilon^{2/3} \simeq 0.211$ gives quite accurately the thickness of the transition region, as well as locating the transition point.

We now restate the essential results which emerge from the solution to the order which we have calculated. The first-order solution directly yields both the first approximation α_n to the eigenvalues which depend only on the maximum width of the plate, and the transverse shape of w_0. The longitudinal shape of w_0, and the second approximation β_{nm} to the eigenvalues, are determined from the second-order problem. The slope of the curved edge at the maximum width enters at this stage, but further details of the shape have only a higher-order effect. The difference between successive eigenvalues is $\epsilon^{2/3}\beta_{nm}$, indicating that the spectrum is relatively dense. The point of maximum amplitude (located slightly to the left of δx_0) moves monotonically to the right with increasing m. This contrasts with the behavior of a long narrow rectangular plate which has a slightly denser spectrum (order ϵ between successive eigenvalues), but the points of maximum deflection jump back and forth from one end to the other.

6. Third-order problem. For numerical results, it can be important to carry out the calculations to third order (cf. Sec. 8). Also, the third-order problem illustrates the need for introducing a thinner, order-ϵ edge layer near $x = 0$, which is needed to satisfy the boundary conditions. If the plate were clamped at $x = 0$, $A_0(x^*)$ could not satisfy the zero slope condition at $x = 0$, and an edge layer would be required even in the second-order problem.

The most complete third-order problem is obtained by choosing

$$\gamma_2(\epsilon) = \tau_2(\epsilon) = \epsilon^{4/3} \tag{6.1}$$

which yield, after some reduction, the following problem:

$$L_b w_2 = (\beta_{nm}{}^2 + 2\alpha_n\mu)w_0 - \frac{\partial^4 w_0}{\partial x^{*4}} + 2\alpha_n \beta_{nm} w_1 - 2\frac{\partial^4 w_1}{\partial x^{*2}\,\partial y^2} \equiv R_1\{w_0, w_1\} \tag{6.2}$$

where

$$w_2(0, y) = 0 \qquad |y| \le a, \tag{6.3}$$

$$w_2(x^*, y) \to 0 \qquad \text{as} \quad x^* \to \infty, \tag{6.4}$$

$$\begin{aligned} w_2(x^*, \pm a) = {} & \pm x^*\kappa^2 \frac{\partial w_1}{\partial y}(x^*, \pm a) \\ & \mp \tfrac{1}{2} x^{*2} G''(0) \frac{\partial^2 w_0}{\partial y^2}(x^*, \pm a) \qquad 0 \le x^* < \infty, \end{aligned} \tag{6.5}$$

$$(\partial^2 w_2/\partial x^{*2})(0, y) = 0 \qquad |y| \le a, \tag{6.6}$$

$$(\partial^2 w_2/\partial x^{*2})(x^*, y) \to 0 \qquad \text{as} \quad x^* \to \infty, \tag{6.7}$$

$$\begin{aligned} \frac{\partial^2 w_2}{\partial y^2}(x^*, \pm a) = {} & \pm x^*\kappa^2 \frac{\partial^3 w_1}{\partial y^3}(x^*, \pm a) \mp 2\kappa^2 \frac{\partial^2 w_0}{\partial x^*\,\partial y}(x^*, \pm a) \\ & \mp \tfrac{1}{2} x^{*2} G''(0) \frac{\partial^3 w_0}{\partial y^3}(x^*, \pm a), \qquad 0 \le x^* < \infty. \end{aligned} \tag{6.8}$$

The quantities which we would like to determine are $A_1(x^*)$, which would completely specify w_1, and μ, which is the third approximation to the eigenvalues. This can be achieved most simply by considering the solvability condition for w_2:

$$\int_{-a}^{a} \begin{Bmatrix} \cos \sqrt{\alpha_n}\, y \\ \sin \sqrt{\alpha_n}\, y \end{Bmatrix} L_b w_2\, dy = \int_{-a}^{a} \begin{Bmatrix} \cos \sqrt{\alpha_n}\, y \\ \sin \sqrt{\alpha_n}\, y \end{Bmatrix} R_1\{w_0, w_1\}\, dy \tag{6.9}$$

where, again, the cosine is used for the symmetric transverse modes, and the sine for the antisymmetric modes. From Eq. (6.9) we find for both cases

$$\frac{d^2 A_1}{dx^{*2}} + \left(\beta_{nm} - \frac{2\kappa^2\alpha_n}{a} x^*\right) A_1 = \frac{\kappa^2}{a}\frac{d}{dx^*} A_0 - \mu A_0 + \left(\frac{3\kappa^2\alpha_n}{a^2} - \frac{G''(0)\alpha_n}{a}\right) x^{*2} A_0 . \tag{6.10}$$

The particular solution of Eq. (6.10) has the form

$$A_1(x^*) = a_{11}x^{*2}\frac{d}{dx^*}A_0 + a_{12}x^*\frac{d}{dx^*}A_0 + a_{13}x^*A_0 + a_{14}\frac{d}{dx^*}A_0\,, \tag{6.11}$$

where the coefficients are

$$\begin{aligned}
a_{11} &= \frac{3}{10}\frac{\kappa^2}{a} - \frac{1}{10}\frac{G''(0)}{\kappa^2}\,,\\
a_{12} &= \frac{1}{5}\frac{\beta_{nm}}{\alpha_n} - \frac{1}{15}\frac{\beta_{nm}G''(0)a}{\alpha_n\kappa^4}\,,\\
a_{13} &= \frac{1}{5}\frac{\kappa^2}{a} + \frac{1}{10}\frac{G''(0)}{\kappa^2}\,,\\
a_{14} &= \frac{a}{2\kappa^2\alpha_n}\left(\frac{2}{5}\frac{\beta_{nm}{}^2}{\alpha_n} - \frac{2}{15}\frac{\beta_{nm}{}^2G''(0)a}{\alpha_n\kappa^4} - \mu\right).
\end{aligned} \tag{6.12}$$

The homogeneous solution is proportional to $A_0(x^*)$ and can be added to the first-order solution which has an arbitrary amplitude.

$A_1(x^*)$ must satisfy the boundary conditions

$$A_1(0) = \frac{d^2}{dx^{*2}}A_1(0) = 0. \tag{6.13}$$

From Eq. (6.11) it is seen that to satisfy $A_1(0) = 0$, we must choose $a_{14} = 0$, since $(d/dx^*)A_0(0) \neq 0$. However, a problem quickly arises when we notice that even with the choice $a_{14} = 0$, $(d^2/dx^{*2})A_1(0) = 0$ cannot be satisfied. Thus, we must seek another solution, valid in an edge layer near $x = 0$, which satisfies zero-deflection and zero-moment boundary conditions on $x = 0$ and asymptotically matches the transition layer solution.

Before proceeding to the edge layer problem, we expand the transition layer solution in an intermediate variable and obtain an expression which must match, term by term, with the edge layer solution expressed in the same variables. We therefore introduce

$$x_\eta = x/\eta(\epsilon) \tag{6.14}$$

where $\epsilon \ll \eta(\epsilon) \ll \epsilon^{2/3}$ as $\epsilon \to 0$. Notice that the transition layer variable, when expressed in terms of x_η, is

$$x^* = (\eta(\epsilon)/\epsilon^{2/3})x_\eta\,. \tag{6.15}$$

So $x^* \to 0$ as $\epsilon \to 0$, with x_η fixed. Calculating the limit, we find

$$\begin{aligned}
\lim_{\substack{\epsilon\to 0\\ x_\eta \text{ fixed}}} w(x_\eta, y) = \begin{Bmatrix}\cos\sqrt{\alpha_n}\,y\\ \sin\sqrt{\alpha_n}\,y\end{Bmatrix}\Bigg[&\frac{\eta}{\epsilon^{2/3}}\frac{x_\eta}{\lambda}A_i'(\rho_m) + \frac{1}{3!}\frac{\eta^3}{\epsilon^2}\frac{x_\eta{}^3}{\lambda^3}A_i'''(\rho_m)\\
&+ \epsilon^{2/3}\frac{a_{14}}{\lambda}A_i'(\rho_m) + \eta x_\eta\frac{a_{12}}{\lambda}A_i'(\rho_m)\Bigg] + O\left(\frac{\eta^2}{\epsilon^{2/3}}\right)
\end{aligned} \tag{6.16}$$

where the primes designate differentiation with respect to the argument of the Airy function.

For the edge layer we introduce the following stretched longitudinal variable:

$$\tilde{x} = x/\epsilon \tag{6.17}$$

and the expansions

$$w(x, y) = \Delta_0(\epsilon)h_0(\tilde{x}, y) + \Delta_1(\epsilon)h_1(\tilde{x}, y) + \cdots, \tag{6.18a}$$

$$\omega = \alpha + \epsilon^{2/3}\beta + \epsilon^{4/3}\mu + \cdots. \tag{6.18b}$$

Substitution of Eqs. (6.17) and (6.18) into the full problem defined by Eqs. (2.5) through (2.11) then yields, to first order,

$$L_p h_0 \equiv \frac{\partial^2 h_0}{\partial y^4} + 2\frac{\partial^4 h_0}{\partial y^2\,\partial \tilde{x}^2} + \frac{\partial^4 h_0}{\partial \tilde{x}^4} - \alpha^2 w_0 = 0, \tag{6.19}$$

where

$$h_0(0, y) = 0, \qquad |y| \le a, \tag{6.20}$$

$$h_0(\tilde{x}, \pm a) = 0, \qquad 0 \le \tilde{x} < \infty, \tag{6.21}$$

$$(\partial^2 h_0/\partial \tilde{x}^2)(0, y) = 0, \qquad |y| \le a, \tag{6.22}$$

$$(\partial^2 h_0/\partial y^2)(\tilde{x}, \pm a) = 0, \qquad 0 \le \tilde{x} < \infty. \tag{6.23}$$

In $(\tilde{x}, y)$ coordinates the full plate equation must be solved, but in a simplified domain which is a semi-infinite strip having constant width $2a$.

The appropriate solution to the first-order problem with no exponential growth is

$$h_0(\tilde{x}, y) = C_0\tilde{x}\begin{Bmatrix}\cos\sqrt{\alpha_n}\,y\\ \sin\sqrt{\alpha_n}\,y\end{Bmatrix}, \tag{6.24}$$

where C_0 is an arbitrary constant. Rewriting Eq. (6.24) in terms of x_η, we have

$$\lim_{\substack{\epsilon\to 0\\ x_\eta\ \text{fixed}}} w(x_\eta, y) = C_0\Delta_0(\epsilon)\frac{\eta}{\epsilon}x_\eta\begin{Bmatrix}\cos\sqrt{\alpha_n}\,y\\ \sin\sqrt{\alpha_n}\,y\end{Bmatrix} + \cdots. \tag{6.25}$$

Comparing Eq. (6.25) with Eq. (6.16), we must choose

$$\Delta_0(\epsilon) = \epsilon^{1/3}, \tag{6.26a}$$

$$C_0 = \frac{1}{\lambda}A_i'(\rho_m) \tag{6.26b}$$

so the leading term will be matched.

Proceeding to the second-order edge layer problem, we have:

$$L_p h_1 = 2\alpha\beta\, h_0 \tag{6.27}$$

and the same boundary conditions as Eqs. (6.20) through (6.23) with h_0 replaced by h_1. In obtaining Eq. (6.27) we have set

$$\Delta_1(\epsilon) = \epsilon^{2/3}\Delta_0(\epsilon) = \epsilon. \tag{6.28}$$

The appropriate solution of Eq. (6.27) with no exponential growth is

$$h_1(\tilde{x}, y) = \begin{Bmatrix}\cos\sqrt{\alpha_n}\,y\\ \sin\sqrt{\alpha_n}\,y\end{Bmatrix}\left\{-\frac{\beta C_0}{6}\tilde{x}^3 + C_1\tilde{x}\right\} \tag{6.29}$$

where C_1 is an arbitrary constant that will be determined by matching. The edge layer solution to second order, when written in terms of x_η , is

$$\lim_{\substack{\epsilon\to 0\\ x_\eta \text{ fixed}}} = \begin{Bmatrix} \cos\sqrt{\alpha_n}\,y \\ \sin\sqrt{\alpha_n}\,y \end{Bmatrix} \left[C_0 \frac{\eta}{\epsilon^{2/3}} x_\eta - \frac{\beta C_0}{6} \frac{\eta^3}{\epsilon^2} x_\eta{}^3 + C_1 \eta x_\eta \right] + \cdots . \tag{6.30}$$

Comparing Eqs. (6.16) and (6.30), we see that the first terms match by virtue of Eq. (6.26). The cubic terms match, which can be checked by using the identity $A_i'''(\rho_m) = \rho_m A_i'(\rho_m)$ and Eqs. (5.16) and (5.20). The remaining linear term matches if we set

$$C_1 = \frac{a_{12}}{\lambda} A_i'(\rho_m). \tag{6.31}$$

Since a constant term does not appear in Eq. (6.30), we must set

$$a_{14} = 0, \tag{6.32}$$

which finally yields a condition for the determination of μ, the third approximation to the eigenvalues. Thus, from Eq. (6.12),

$$\mu_{nm} = \frac{2}{5}\frac{\beta_{nm}{}^2}{\alpha_n} - \frac{2}{15}\frac{\beta_{nm}{}^2 G''(0) a}{\alpha_n \kappa^4} \tag{6.33}$$

for both the symmetric and antisymmetric transverse modes. Notice that $\mu_{nm} \le 0$ if $G''(0) \ge 3\kappa^4/a$.

7. Case with two transition points. When a planform has a local maximum width someplace away from the ends (such as a long narrow, elliptical planform) the same methods can be applied. Suppose the width attains a maximum at $x = \xi$, $0 < \xi < 1$, with the local geometry given by $G(\xi) = a$, $G'(\xi) = 0$ and $G''(\xi) = -r^2$. Introducing, as before, a stretched longitudinal variable

$$x^* = \frac{x - \xi}{\delta(\epsilon)} \tag{7.1}$$

and assuming the expansions given by Eq. (3.2) and (3.3), we find

$$\delta(\epsilon) = \epsilon^{1/2}, \tag{7.2a}$$

$$\gamma_1(\epsilon) = \tau_1(\epsilon) = \epsilon. \tag{7.2b}$$

The transition layer is now slightly thicker than the previous case, whereas the magnitude of the first eigenvalue and eigenfunction corrections are smaller. The first-order solution is again of the form

$$w_0(x^*, y) = A_0(x^*) \begin{Bmatrix} \cos\sqrt{\alpha_n}\,y \\ \sin\sqrt{\alpha_n}\,y \end{Bmatrix} \tag{7.3}$$

with the α_ns given by Eq. (4.9). However, the amplitude equation for $A_0(x^*)$ is now, for both the symmetric and antisymmetric transverse modes,

$$\frac{d^2}{dx^{*2}} A_0(x^*) + \left(\beta - \frac{\alpha_n r^2}{a} x^{*2}\right) A_0(x^*) = 0 \tag{7.4}$$

which has two turning points located at

$$x^* = \pm(\beta a/\alpha_n r^2)^{1/2}. \tag{7.5}$$

The boundary conditions for $A_0(x^*)$ are

$$A_0(x^*), \frac{d^2}{dx^{*2}} A_0(x^*) \to 0 \quad \text{as} \quad |x^*| \to \infty. \tag{7.6}$$

At first sight, it appears that since the domain is the entire real axis, β could take on continuous values and still satisfy the boundary conditions of Eq. (7.6). This would contradict the fact that the spectrum is known to be discrete for the physical (unstretched) domain which is finite. This apparent difficulty is resolved by noticing that Eq. (7.4) has the same form as Schrödinger's equation for the wave function of a simple harmonic oscillator, where exactly the same problem arises. Following the discussion of Mott and Sneddon [3], we see that Eq. (7.6) can be satisfied only if β takes on the particular discrete values

$$\beta_{nm} = (2m - 1)\left(\frac{\alpha_n r^2}{a}\right)^{1/2}, \qquad n, m = 1, 2, \cdots \tag{7.7}$$

which is the analogue of the quantized energy levels of an oscillator. The solution of Eq. (7.4) is then

$$A_0(x^*) = \exp(-\tfrac{1}{2}z^2) H_m(z) \tag{7.8}$$

where $H_m(z)$ are the Hermite polynomials

$$H_m(z) = (-1)^m \exp(z^2) \frac{d^m}{dz^m} \exp(-z^2) \tag{7.9}$$

where

$$z = \left(\frac{\alpha_n r^2}{a}\right)^{1/4} x^*. \tag{7.10}$$

A plot of $A_0(x)$ is shown for the case $n = 1$, $m = 9$, for a half-elliptical planform shown in Fig. 4. When m is odd, the zero-deflection and zero-moment boundary conditions are satisfied with $\xi = 0$. For this case ($\epsilon = 0.00867$), the positive transition point is located at $x \simeq 0.217$.

8. Numerical results. As an example, we consider a trapezoidal platform with length 30 mm, maximum half-width 0.26 mm, and a minimum half-width 0.04 mm, corresponding to the dimensions of the basilar membrane. The results for the non-dimensional eigenvalues computed from the series expansion, Eq. (6.18), are given in the first three columns of Table I. Shown in the last column are the results for a circular sector planform

TABLE 1. Eigenvalues for trapezoidal planform (first three columns) and circular sector (last column).

	1 term	2 terms	3 terms	Circular sector
ω_{11}	2.467	2.723	2.734	2.734
ω_{12}	2.467	2.915	2.947	2.948
ω_{13}	2.467	3.072	3.131	3.133
ω_{14}	2.467	3.210	3.300	3.305
ω_{15}	2.467	3.337	3.460	3.468

having the same opening angle and maximum half-width as the basilar membrane. The eigenvalues are computed from the exact frequency equation for a simply supported sectorial plate

$$2u\,J_\nu(u)\,I_\nu(u) - (1-\sigma)\,I_{\nu+1}(u)\,J_\nu(u) - (1-\sigma)\,J_\nu(u)\,I_{\nu+1}(u) = 0 \tag{8.1}$$

where $\omega = \theta_0^2 u^2$, with $2\theta_0$ being the opening angle, and $\nu = \pi/2\theta_0$. For the case considered $\theta_0 = 0.00734$ radians and $\nu = 214$. The J_ν are Bessel functions of the first kind of order ν, and are computed using the Hewlett Packard HP-65 MATH PAC 2-21A Bessel function program. The I_ν are modified Bessel functions of the first kind and are evaluated using their asymptotic expansions for larger orders. The roots of Eq. (8.1) are found by trial and error using linear interpolation. By comparing the third and fourth columns in Table I it is seen that the three-term expansion for the eigenvalues compares favorably with the exact solution for the circular sector. As the longitudinal mode number increases, the error increases. For $m = 5$ the error is eight parts in a thousand. It was found that for $\sigma < 1$, the roots of Eq. (8.1) are independent of σ. This agrees with the asymptotic theory where, to third order, the eigenvalues are independent of σ. The numerical results also indicate that the difference between the circular sector planform and the trapezoidal planform has no effect on the eigenvalues, as is also suggested by the asymptotic theory. The reason for this is that the deflections are exponentially small in the region near the vertex of the circular sector.

The solid line in Fig. 2 shows the first-order centerline deflection of the trapezoidal planform for $n = 1$ and $m = 5$, with the amplitude in arbitrary units. The second-order centerline deflection is shown in Fig. 3. The dotted line in Fig. 2 is the sum of the first- and second-order centerline deflections. Fig. 4 shows the first-order centerline deflection of a half-elliptical planform having the same length and maximum width as the trapezoidal planform, for the case $n = 1$ and $m = 9$. The qualitative similarity between this curve and the solid curve of Fig. 2 should be noted.

When calculating the roots of Eq. (8.1) for $\nu \gg 1$, it was found numerically that the second and third terms have a negligible effect. That is, to the accuracy of the calculations, the roots of $J_\nu(u) = 0$ have been found. This suggests that the series expansion

$$\omega_{nm} = \alpha_n + \epsilon^{2/3}\beta_{nm} + \epsilon^{4/3}\mu_{nm} + \cdots \tag{8.2}$$

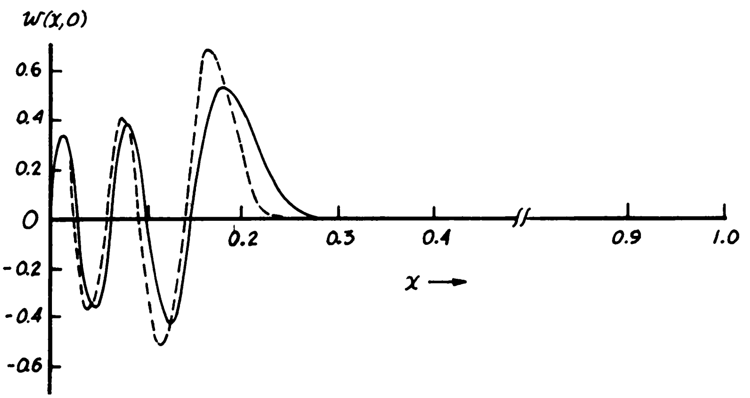

FIG. 2. Centerline mode shape of trapezoidal planform; $n = 1$, $m = 5$, $\epsilon = 0.00867$.——— one term of asymptotic expansion; ------ two terms of asymptotic expansion.

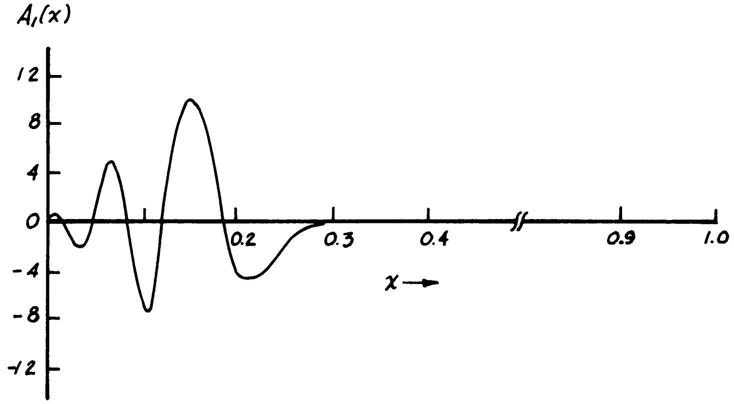

FIG. 3. Second-order centerline mode shape of trapezoidal planform; same parameters as in Fig. 2.

can be used to generate an asymptotic expansion for the zeros of J_ν when $\nu \gg 1$. Denoting the mth root of J_ν by $j_{\nu,m}$, we have

$$\omega_{1m} = j_{\nu,m}{}^2\, \epsilon^2 + O(\theta^4{}_0). \tag{8.3}$$

Substituting Eq. (8.3) into Eq. (8.2) with $n = 1$, and using Eq. (4.9) for symmetric transverse modes, Eqs. (5.20) and (6.33), and solving for $j_{\nu,m}$, we find

$$j_{\nu,m} = \nu - \rho_m(\nu/2)^{1/3} + \frac{3}{20}\rho_m{}^2(\nu/2)^{-1/3} + O(\nu^{-1}). \tag{8.4}$$

Eq. (8.4) provides an explicit relation for the mth zero of J_ν when $\nu \gg 1$, in terms of ρ_m, the mth root of the Airy function A_i which is tabulated; this equation agrees with a formula given by Olver [4] which he obtained by an independent method.

9. Some further comments. An asymptotic theory is developed for the determination of the eigenfunctions and eigenvalues of a long narrow, simply supported plate having a general planform shape. The theory is accurate provided the longitudinal mode number is not too large. For large longitudinal mode numbers, the present theory does not apply and

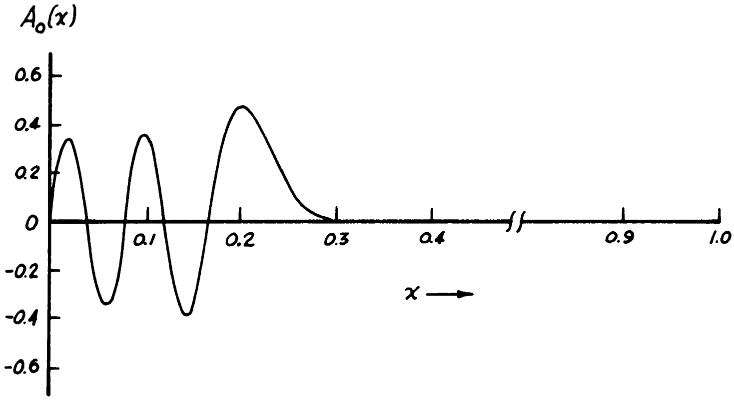

FIG. 4. First-order centerline mode shape of semi-elliptical planform; $n = 1$, $m = 9$, $\epsilon = 0.00867$.

a different expansion procedure must be devised. This problem is considered by Chadwick [5] in a companion paper. It should be emphasized that the idea of the transition layer near the wide end of the plate is still valid for other boundary conditions, although each case must be examined separately.

The methods presented here also can be used to determine the modes of long narrow membranes, as well as other problems set in "thin domains" such as the acoustic modes of a gas in a long thin tapered duct.

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