VIBRATIONS OF LONG NARROW PLATES—II*

By

R. S. CHADWICK

University of California, Los Angeles

Abstract. An asymptotic theory for the natural bending modes of a long narrow plate having variable width and thickness is constructed. A two-variable expansion and a transition layer expansion are matched to find the mode shapes and eigenvalues.

1. Introduction. Previously, Chadwick [1] considered the free vibrations of a long narrow elastic plate with variable width. That problem is motivated by an attempt to understand the mechanics of the cochlea. Basically, a type of boundary layer theory was developed which could accurately determine the first few longitudinal modes. In this paper we use a different expansion procedure which is accurate for higher longitudinal modes and complements the previous calculations. The present work also generalizes the previous study by including the effects of variable thickness.

2. Formulation. The free bending modes are sought for a long narrow plate having the planform shown in Fig. 1a. The ends of the plate at $X = 0, L$ are assumed to be straight and parallel to the Y-axis. The curved edges are given by $Y = \pm BG(X/L)$, where B is some characteristic half-width of the plate. The thickness of the plate, as shown in Fig. 1b, is described by $H = H_0(X/L)$, with H_0 being a characteristic thickness. Our primary interest is the case when the plate stiffens in the X direction. With this in mind, it will be sufficient to require that U/G^2 be a smooth, monotonic increasing function of its argument. Other cases will be discussed briefly in Sec. 7.

In physical coordinates, the deflection of the plate $W(X, Y, T)$ satisfies the linear elastic plate equation for a plate having variable bending rigidity (cf. Vinson [2, p. 17]):

$$\nabla^2 (D \nabla^2 W) - (1 - \sigma) \diamondsuit^4(D, W) + \rho H(\partial^2 W/\partial T^2) = 0, \tag{2.1}$$

$$\nabla^2 = (\partial^2/\partial X^2) + (\partial^2/\partial Y^2), \tag{2.2}$$

$$\diamondsuit^4(D, W) = \frac{\partial^2 D}{\partial X^2}\frac{\partial^2 W}{\partial Y^2} - 2\frac{\partial^2 D}{\partial X\,\partial Y}\frac{\partial^2 W}{\partial X\,\partial Y} + \frac{\partial^2 D}{\partial Y^2}\frac{\partial^2 W}{\partial X^2}, \tag{2.3}$$

$$D = EH^3/12(1 - \sigma^2) \tag{2.4}$$

where E is the modulus of elasticity, σ is the Poisson ratio, and ρ is the density of the material. In what follows, E, σ, and ρ could be taken as smooth functions of the longitudinal coordinate; however, we will treat them as constants and attribute all the

---

* Received November 17, 1977. The author is grateful to Professor J. D. Cole for his advice regarding expansion procedures utilized in this paper. This work is supported by the National Science Foundation under Grant No. ENG 76-81574

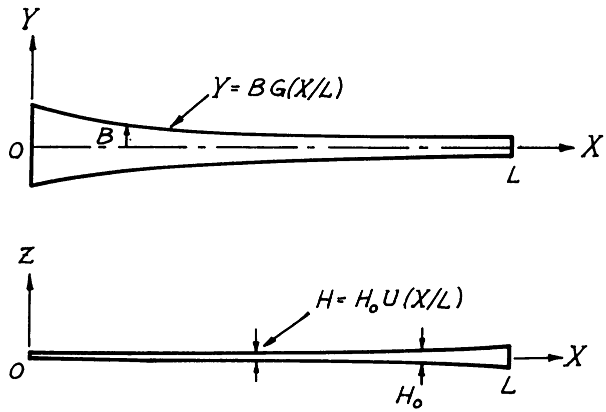

FIG. 1. (a) Planform geometry of plate (top); (b) thickness variation of plate (bottom).

inhomogeneous behavior to the variable thickness. All edges are assumed to be simply supported, which requires the deflection and bending moment to vanish at the edges.

Dimensionless coordinates x, y are introduced with

$$x = X/L, \qquad y = Y/B, \tag{2.5}$$

and since the free vibrations of the plate are to be determined, we look for a solution in the form

$$W(X, Y, T) = w(x, y) \cos \Omega T. \tag{2.6}$$

The plate equation then becomes

$$\left(\frac{\partial^2}{\partial y^2} + \epsilon^2 \frac{\partial^2}{\partial x^2}\right)^2 w + \epsilon^2 U_1(x) \frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial y^2} + \epsilon^2 \frac{\partial^2}{\partial x^2}\right) w + \epsilon^2 U_2(x)\left(\sigma \frac{\partial^2}{\partial y^2} + \epsilon^2 \frac{\partial^2}{\partial x^2}\right) w - U_3(x)\omega^2 w = 0 \tag{2.7}$$

where

$$\begin{aligned} U_1(x) &= 6(U'/U), \\ U_2(x) &= 3[(U''/U) + 2(U'/U)^2], \\ U_3(x) &= 1/U^2 \end{aligned} \tag{2.8}$$

and $\omega = [(\rho H_0 B^4)/D_0]^{1/2}\,\Omega$ is a dimensionless frequency, with $D_0 = EH_0^3/[12(1 - \sigma^2)]$. Note that in Eq. (2.8) the primes designate differentiation with respect to x. Also, $\epsilon = B/L$ is the slenderness ratio of the planform. The zero-deflection boundary conditions are

$$w(0, y) = 0, \qquad |y| \le G(0), \tag{2.9}$$

$$w(1, y) = 0, \qquad |y| \le G(1), \tag{2.10}$$

$$w(x, \pm G(x)) = 0, \qquad 0 \le x \le 1. \tag{2.11}$$

The exact forms of the zero-moment boundary conditions are

$$(\partial^2 w/\partial x^2)(0, y) = 0, \qquad |y| \le G(0), \tag{2.12}$$

$$(\partial^2 w/\partial x^2)(1, y) = 0, \qquad |y| \le G(1), \tag{2.13}$$

$$\frac{\partial^2 w}{\partial y^2}(x, \pm G(x)) + \epsilon^2 \sigma \frac{\partial^2 w}{\partial x^2}(x, \pm G(x))$$

$$+ \epsilon^2 G'^2 \left\{\epsilon^2 \frac{\partial^2 w}{\partial x^2}(x, \pm G(x)) + \sigma \frac{\partial^2 w}{\partial y^2}(x, \pm G(x))\right\}$$

$$\mp 2(1-\sigma)\epsilon^2 G' \frac{\partial^2 w}{\partial x\, \partial y}(x, \pm G(x)) = 0, \qquad 0 \le x \le 1. \tag{2.14}$$

(These boundary conditions are identical to those considered in [1], and are reproduced here for convenience.) Eqs. (2.7) through (2.14) define the exact eigenvalue problem.

3. Two-variable expansion procedure. For our immediate application, we want to study the behavior of the solution to the eigenvalue problem posed in Sec. 2 as $\epsilon \to 0$. A fruitful line of approach proves to be a two-variable expansion in the longitudinal coordinate x. The tentative scaling that was made, $x = X/L$, is an appropriate scale for the slowly varying width and thickness functions $G(x)$ and $U(x)$. However, we must anticipate that the deflection w may have short longitudinal wavelengths on the scale of the width, B. We therefore introduce a fast length scale

$$\tilde{x}(x) = \frac{1}{\epsilon} \int^x f(x)\, dx. \tag{3.1}$$

The function $f(x)$ will have to be found, and will involve the slowly modulating effects of $G(x)$ and $U(x)$. The lower limit of integration in Eq. (3.1) will be appropriately chosen later.

The following expansions are introduced:

$$w(x, y) = \Delta(\epsilon)\{w_1(x, \tilde{x}, y) + \epsilon w_2(x, \tilde{x}, y) + \cdots\}, \tag{3.2}$$

$$\omega = \tilde{\alpha} + \epsilon\tilde{\beta} + \cdots \tag{3.3}$$

where $\tilde{\alpha}$, $\tilde{\beta}$, $\cdots$, are unknown constants and $\Delta(\epsilon)$ is an unknown scale factor. The fast variable $\tilde{x}$ enters the equations through the partial derivatives

$$\partial^k/\partial x^k = \left(\frac{\partial}{\partial x} + \frac{d\tilde{x}}{dx}\frac{\partial}{\partial \tilde{x}}\right)^k, \qquad k = 1, 2, 3, 4. \tag{3.4}$$

To order $\Delta(\epsilon)$, the plate equation, Eq. (2.7), becomes

$$Lw_1 \equiv \frac{\partial^4 w_1}{\partial y^4} + 2f^2 \frac{\partial^4 w_1}{\partial y^2 \partial \tilde{x}^2} + f^4 \frac{\partial^4 w_1}{\partial \tilde{x}^4} - \left(\frac{\tilde{\alpha}}{U}\right)^2 w_1 = 0, \tag{3.5}$$

while the boundary conditions, Eqs. (2.11) and (2.14) yield

$$w_1(x, \tilde{x}, \pm G(x)) = 0, \qquad 0 \le x \le 1, \tag{3.6}$$

$$\frac{\partial^2 w_1}{\partial y^2} + \sigma f^2 \frac{\partial^2 w_1}{\partial \tilde{x}^2} = 0 \quad \text{on} \quad y = \pm G(x). \tag{3.7}$$

Considering a solution of the form

$$w_1(x, \tilde{x}, y) = A_1(x, \tilde{x}) \begin{Bmatrix} \cos & (E_s(x)^{1/2} y \\ \sin & (E_a(x))^{1/2} y \end{Bmatrix}, \tag{3.8}$$

both Eqs. (3.6) and (3.7) can be satisfied by choosing

$$\begin{aligned} (E_s(x))^{1/2}G(x) &= (n - \tfrac{1}{2})\pi \\ (E_a(x))^{1/2}G(x) &= n\pi \end{aligned} \qquad n = 1, 2, \cdots . \tag{3.9}$$

The subscripts s and a refer to symmetric and antisymmetric transverse modes, respectively.

Eq. (3.8) is a solution of Eq. (3.5) provided that $A_1(x, \tilde{x})$ satisfies

$$f^4 \frac{\partial^4 A_1}{\partial \tilde{x}^4} - 2f^2 E \frac{\partial^2 A_1}{\partial \tilde{x}^2} + \left[E^2 - \left(\frac{\tilde{\alpha}}{U} \right)^2 \right] A_1 = 0 \tag{3.10}$$

where the subscripts on E have been dropped for convenience. Looking for a solution of the form $A_1 \sim \exp \lambda \tilde{x}$, we find

$$\lambda^2 = \frac{1}{f^2} \left(E \pm \frac{\tilde{\alpha}}{U} \right). \tag{3.11}$$

To prevent secular terms in $\tilde{x}$ from appearing in the second-order problem, λ must be independent of x and therefore a constant. For oscillatory solutions, λ^2 must be negative. Without loss of generality, both these requirements can be satisfied by choosing

$$f^2 = (\tilde{\alpha}/U) - E \tag{3.12}$$

which makes $\lambda^2 = -1$. The first-order solution can now be written as

$$w_1(x, \tilde{x}, y) = \{a_1(x) \sin \tilde{x} + b_1(x) \cos \tilde{x}\} \begin{Bmatrix} \cos & (E_s(x))^{1/2} y \\ \sin & (E_a(x))^{1/2} y \end{Bmatrix} . \tag{3.13}$$

The functions $a_1(x)$, $b_1(x)$, and the eigenvalue approximation $\tilde{\alpha}$ are still undetermined.

The order $\epsilon\Delta(\epsilon)$ problem is

$$\begin{aligned} Lw_2 = {} & 2\tilde{\alpha}\tilde{\beta}\, U_3\, w_1 - 4f \frac{\partial^4 w_1}{\partial \tilde{x}\, \partial x\, \partial y^2} - (2f' + fU_1) \frac{\partial^3 w_1}{\partial \tilde{x}\, \partial y^2} \\ & - (6f^2 f' + f^3 U_1) \frac{\partial^3 w_1}{\partial \tilde{x}^3} - 4f^3 \frac{\partial^4 w_1}{\partial \tilde{x}^3\, \partial x}, \end{aligned} \tag{3.14}$$

$$w_2(x, \tilde{x}, \pm G(x)) = 0 \qquad 0 \le x \le 1, \tag{3.15}$$

$$\frac{\partial^2 w_2}{\partial y^2} + \sigma f^2 \frac{\partial^2 w_2}{\partial \tilde{x}^2} = 2(1 - \sigma)\, G' f \frac{\partial^2 w_1}{\partial \tilde{x}\, \partial y} - \sigma f' \frac{\partial w_1}{\partial \tilde{x}} - 2\sigma f \frac{\partial^2 w_1}{\partial \tilde{x}\, \partial x} \quad \text{on} \quad y = \pm G(x). \tag{3.16}$$

Particular solutions of Eq. (3.14) are of the form

$$w_2 \sim y^2 \begin{Bmatrix} \sin \tilde{x} \\ \cos \tilde{x} \end{Bmatrix} \begin{Bmatrix} \cos & E_s\, y \\ \sin & E_a\, y \end{Bmatrix} \tag{3.17}$$

where the four combinations have coefficients which depend on x. Eq. (3.17) satisfies Eq. (3.15), and applying Eq. (3.16) yields two coupled differential equations for $a_1(x)$ and $b_1(x)$:

$$(da_1/dx) + p(x) a_1 = -\frac{1}{2} \frac{\tilde{\beta}}{fU} b_1 \tag{3.18a}$$

$$(db_1/dx) + p(x) b_1 = \frac{1}{2} \frac{\tilde{\beta}}{fU} a_1 \tag{3.18b}$$

where

$$p(x) = \frac{9}{4}\frac{U'}{U} - \frac{1}{4}\frac{E'}{E} - \frac{1}{4}\frac{(EU)'}{\tilde{\alpha} - EU} \quad . \tag{3.19}$$

The solution of Eqs. (3.18a, b) is

$$a_1(x) = (C_1 \sin \tilde{\beta}\xi + C_2 \cos \tilde{\beta}\xi) \exp\left(-\int p(x)dx\right) \tag{3.20a}$$

$$b_1(x) = (-C_1 \cos \tilde{\beta}\xi + C_2 \sin \tilde{\beta}\xi) \exp\left(-\int p(x)dx\right) \tag{3.20b}$$

with C_1 and C_2 arbitrary constants, and

$$\xi = \frac{1}{2}\int^x \frac{dx}{f(x)} \quad . \tag{3.21}$$

With $a_1(x)$ and $b_1(x)$ now determined, the first-order solution takes the simple form

$$w_1(x, \tilde{x}, y) = \frac{C_0}{(fGU^5)^{1/2}} \cos(\tilde{x} + \tilde{\beta}\xi + \theta_0) \begin{Bmatrix} \cos & E_s\, y \\ \sin & E_a\, y \end{Bmatrix} \tag{3.22}$$

with C_0, θ_0 arbitrary constants, still to be determined. Notice that $\tilde{\alpha}$, which is imbedded in f (cf. Eq. (3.12)), is still unknown, as well as $\tilde{\beta}$. Eq. (3.22) cannot be expected to be valid in the whole interval $0 \le x \le 1$, since it is possible that f can vanish at some point. In fact, this is the interesting case that will be examined.

Let x_t be a zero of f referred to as the transition point. The monotonicity of U/G^2 insures that for fixed $\tilde{\alpha}$, at most one transition point exists. Also, from the construction of f in Eq. (3.12), Eq. (3.22) will be valid in the flexible end of the plate $0 \le x < x_t$. Assuming that x_t exists, another expansion valid in the neighborhood of x_t must be constructed which will asymptotically match with the two-variable expansion. This matching process will, in fact, determine $\tilde{\beta}$, θ_0, C_0, and $\Delta(\epsilon)$; and then satisfying the boundary conditions on $x = 0$ will determine $\tilde{\alpha}$.

4. Transition layer expansion. Assuming the existence of x_t, which is a zero of $f(x)$, we introduce the boundary layer coordinate

$$x^* = (x - x_t)/\delta(\epsilon) \tag{4.1}$$

and we consider the limit process $\epsilon \to 0$ with (x^*, y) fixed. $\delta(\epsilon)$ is a measure of the thickness of the transition layer and its dependence on ϵ must be determined. The following expansions are introduced into Eq. (2.7):

$$w(x, y) = q_1(x^*, y) + \gamma(\epsilon)\, q_2(x^*, y) + \cdots, \tag{4.2}$$

$$\omega = \alpha + \tau(\epsilon)\beta + \cdots \tag{4.3}$$

where $\gamma(\epsilon), \tau(\epsilon), \cdots$, are unknown scale factors and $\alpha, \beta, \cdots$, are unknown constants. To first order the plate equation becomes

$$L_b\, q_1 = (\partial^4 q_1/\partial y^4) - (\alpha/U_t)^2 q_1 = 0 \tag{4.4}$$

where $U_t = U(x_t)$. Notice that L_b is just the beam operator in the transverse direction. In obtaining Eq. (4.4) we have assumed $\epsilon/\delta(\epsilon) \to 0$ as $\epsilon \to 0$; otherwise Eq. (2.7) would retain its full structure and we would not obtain a distinguished limit equation. To the same order, Eqs. (2.11) and (2.14) are

$$q_1(x^*, \pm G_t) = 0, \qquad |x^*| < \infty, \tag{4.5}$$

$$(\partial^2 q_1/\partial y^2)(x^*, \pm G_t) = 0, \qquad |x^*| < \infty, \tag{4.6}$$

where we have written $G_t = G(x_t)$. In this approximation the planform is an infinite strip of width $2G_t$.

The solution of Eq. (4.4), subject to Eqs. (4.5) and (4.6), is simply

$$q_1(x^*, y) = Q_1(x^*) \begin{Bmatrix} \cos (\alpha_s/U_t)^{1/2} y \\ \sin (\alpha_a/U_t)^{1/2} y \end{Bmatrix} \tag{4.7}$$

with

$$\begin{matrix} (\alpha_s/U_t)^{1/2}\, G_t = (n - 1/2)\pi \\ (\alpha_a/U_t)^{1/2} G_t = n\pi \end{matrix} \qquad n = 1, 2, \cdots . \tag{4.8}$$

At this point $Q_1(x^*)$ is still arbitrary and must be determined from the second-order problem. If the location of x_t were known, Eq. (4.8) would determine the first approximation to the eigenvalues which correspond to the natural frequencies of a beam having the length $2G_t$.

The most complete equation of order $\gamma(\epsilon)$ is obtained if

$$\gamma(\epsilon) = \tau(\epsilon) = \epsilon^2/\delta^2(\epsilon) = \delta(\epsilon) = \epsilon^{2/3},$$

giving

$$L_b q_2 = \left(\frac{2\alpha\beta}{U_t} - \frac{2\alpha^2 U_t'}{U_t^3} x^*\right) q_1 - 2\,\frac{\partial^4 q_1}{\partial x^{*2}\,\partial y^2} = R q_1 . \tag{4.9}$$

To the same order, the boundary conditions, Eqs. (2.11) and (2.14), become[1]

$$q_2(x^*, \pm G_t) = \mp\, x^*\, G_t'\, \frac{\partial q_1}{\partial y} (x^*, \pm G_t), \qquad |x^*| < \infty, \tag{4.10}$$

$$\frac{\partial^2}{\partial y^2} q_2(x^*, \pm G_t) = \mp\, x^*\, G_t'\, \frac{\partial^3 q_1}{\partial y^3} (x^*, \pm G_t), \qquad |x^*| < \infty, \tag{4.11}$$

where use has been made of the fact that

$$(\partial^2 q_1/\partial x^{*2})(x^*, \pm G_t) = 0. \tag{4.12}$$

Application of the solvability condition

$$\int_{-G_t}^{G_t} \begin{Bmatrix} \cos\ (\alpha_s/U_t)^{1/2} y \\ \sin\ (\alpha_a/U_t)^{1/2} y \end{Bmatrix} L_b q_2\, dy = \int_{-G_t}^{G_t} \begin{Bmatrix} \cos\ (\alpha_s/U_t)^{1/2} y \\ \sin\ (\alpha_a/U_t)^{1/2} y \end{Bmatrix} R q_1\, dy \tag{4.13}$$

yields directly the equation for $Q_1(x^*)$:

$$\frac{d^2 Q_1(x^*)}{dx^{*2}} + \frac{1}{U_t} \left\{\beta - \alpha \left(\frac{U_t'}{U_t} - 2\,\frac{G_t'}{G_t}\right) x^*\right\} Q_1(x^*) = 0. \tag{4.14}$$

Eq. (4.14) can be transformed into the standard form of Airy's equation:

[1] The case $G_t' = 0$ requires special treatment and is not considered here.

$$(d^2/dz^2)Q_1(z) - zQ_1(z) = 0 \tag{4.15}$$

where

$$x^* = x_0 + \lambda_0 z, \tag{4.16}$$

$$\lambda_0 = \left\{ \frac{U_t}{\alpha\left(\dfrac{U_t'}{U_t} - 2\dfrac{G_t'}{G_t}\right)} \right\}^{1/3} \tag{4.17a}$$

$$x_0 = \frac{\beta}{\alpha\left(\dfrac{U_t'}{U_t} - 2\dfrac{G_t'}{G_t}\right)} \tag{4.17b}$$

The appropriate solution of Eq. (4.15), which decays exponentially as $x^* \to \infty$ and oscillates as $x^* \to -\infty$, is the Airy function $A_i(z)$:

$$Q_1(z) = A_i(z) = A_i\left[\frac{1}{\lambda_0}(x^* - x_0)\right]. \tag{4.18}$$

To get this behavior, the coefficient of x^* is Eq. (4.14) must be positive. By Eq. (4.8), α is positive, and the expression in parentheses is also positive since $(U/G^2)'$ is positive by assumption.

5. Matching and the determination of eigenvalues. The transition layer expansion and the two-variable expansion must match asymptotically in some overlap domain. To facilitate the matching process we introduce the intermediate variable

$$x_\eta = (x_t - x)/\eta(\epsilon) \tag{5.1}$$

where $\epsilon^{2/3} \ll \eta(\epsilon) \ll 1$ as $\epsilon \to 0$. Both expansions are to be expressed in terms of x_η and we consider the limit $\epsilon \to 0$ with x_η fixed. The two expansions are said to be matched to $O(\mu)$ if

$$\lim_{\substack{\epsilon \to 0 \\ x_\eta \text{ fixed}}} \frac{\Delta(\epsilon)\{w_1 + \epsilon w_2 + \cdots\} - \{q_1 + \epsilon^{2/3} q_2 + \cdots\}}{\mu(\epsilon)} = 0. \tag{5.2}$$

Considering first the transition layer expansion, we note that

$$x^* = -\frac{\eta(\epsilon)\, x_\eta}{\epsilon^{2/3}} \to -\infty, \qquad x_\eta > 0. \tag{5.3}$$

Using the asymptotic expansion of the Airy function for large negative arguments, we have

$$q_1(x_\eta, y) = \frac{1}{\sqrt{\pi}}\left(\frac{\gamma\lambda_0}{x_\eta}\right)^{1/4}\left[\sin\left\{\frac{2}{3}\left(\frac{x_\eta}{\gamma\lambda_0}\right)^{3/2} + \left(\frac{x_\eta}{\gamma\lambda_0}\right)^{1/2}\frac{x_0}{\lambda_0} + \frac{\pi}{4}\right\} \times \begin{Bmatrix} \cos\ (\alpha_s/U_t)^{1/2}y \\ \sin\ (\alpha_a/U_t)^{1/2}y \end{Bmatrix} + O(\gamma)^{1/2}x_0 + O(\gamma)^{3/2}\right] \tag{5.4}$$

where $\gamma = \epsilon^{2/3}/\eta \to 0$. We will also require the leading term of $q_2(x_\eta, y)$ for higher-order matching. This term comes from the homogeneous solution which has the form

$$Q_2(x^*)\begin{Bmatrix} \cos\ (\alpha_s/U_t)^{1/2}y \\ \sin\ (\alpha_a/U_t)^{1/2}y \end{Bmatrix}. \tag{5.5}$$

$Q_2(x^*)$ must be determined from the *third-order* transition layer problem, in the same way $Q_1(x^*)$ in Eq. (4.7) had to be found from the second-order problem. Details of the third-order problem are worked out in Chadwick [1]. For our purposes here, we note that the dominant part of $Q_2(x^*)$ comes from the term

$$a_{11}\, x^{*2} \frac{d}{dx^*} Q_1(x^*)$$

where a_{11} is a constant involving first and second derivatives of U and G evaluated at x_t.

Expanding this term in the intermediate variable, we obtain

$$q_2(x_\eta, y) = -\frac{1}{\sqrt{\pi}} a_{11} \lambda_0 \left(\frac{x_\eta}{\gamma\lambda_0}\right)^{9/4} \cos\left\{\frac{2}{3}\left(\frac{x_\eta}{\gamma\lambda_0}\right)^{3/2} + \frac{\pi}{4}\right\} \times \begin{Bmatrix} \cos & (\alpha_s/U_t)^{1/2} y \\ \sin & (\alpha_a/U_t)^{1/2} y \end{Bmatrix} + O\left(\frac{1}{\gamma}\right)^{5/4}. \tag{5.6}$$

Proceeding to the two-variable expansion, it is first noted that

$$x = x_t - \eta(\epsilon) x_\eta \to x_t \quad \text{as} \quad \epsilon \to 0 \quad \text{with} \quad x_\eta \quad \text{fixed.} \tag{5.7}$$

It is clear that $\tilde{\alpha} = \alpha$ for the transverse modes shapes to agree in both expansions. Then the expansion for the phase function f near the transition point x_t takes the form

$$f \simeq (\eta\, x_\eta/\lambda_0^3)^{1/2} + \frac{1}{2} C''(\lambda_0\, \eta\, x_\eta)^{3/2} + \cdots \tag{5.8}$$

where $C'' = [f^2(x_t)]''$. It is convenient to choose x_t as the lower limit of integration in the integrals defining $\tilde{x}$ and ξ, which then gives

$$\tilde{x} = \frac{1}{\epsilon} \int_{x_t}^{x_t - \eta x_\eta} f(\tau) d\tau \simeq -\frac{2}{3}\left(\frac{x_\eta}{\gamma\lambda_0}\right)^{3/2} - \frac{1}{5}\epsilon^{2/3} C'' \lambda_0^4 \left(\frac{x_\eta}{\gamma\lambda_0}\right)^{5/2} + \cdots, \tag{5.9}$$

and

$$\xi = \frac{1}{2} \int_{x_t}^{x_t - \eta x_\eta} f^{-1}(\tau) d\tau \simeq -(\eta\, x_\eta\, \lambda_0^3)^{3/2} + 0(\eta^{3/2}). \tag{5.10}$$

The intermediate limit of the two-variable expansion can then be written:

$$w_1(x_\eta, y) \simeq \frac{\epsilon^{-1/6} C_0}{\sqrt{\pi}} \left(\frac{\lambda_0 \pi}{G_t\, U_t^5}\right)^{1/2} \left(\frac{\gamma\lambda_0}{x_\eta}\right)^{1/4} \left[\cos\left\{-\frac{2}{3}\left(\frac{x_\eta}{\gamma\lambda_0}\right)^{3/2} + \theta_0\right\} + \left\{\frac{1}{5}\epsilon^{2/3} C'' \lambda_0^4 \left(\frac{x_\eta}{\gamma\lambda_0}\right)^{5/2} + \tilde{\beta}(\eta x_\eta \lambda_0^3)^{1/2} + \cdots\right\} \times \sin\left\{-\frac{2}{3}\left(\frac{x_\eta}{\gamma\lambda_0}\right)^{3/2} + \theta_0\right\}\right] \begin{Bmatrix} \cos & (\alpha_s/U_t)^{1/2} y \\ \sin & (\alpha_a/U_t)^{1/2} y \end{Bmatrix} + \cdots. \tag{5.11}$$

The matching condition, Eq. (5.2), is now applied with $\mu(\epsilon) = \gamma^{1/4}$ and with $\eta(\epsilon)$ chosen such that $\epsilon^{2/3} \ll \eta \ll \epsilon^{2/5}$. For this limit to vanish, we find

$$\Delta(\epsilon) = \epsilon^{1/6}; \qquad C_0 = (\pi\lambda_0/G_t\, U_t^5)^{1/2}$$
$$\theta_0 = \pi/4; \qquad x_0 = 0 \qquad (\beta = 0). \tag{5.12}$$

Nothing can be said about $\tilde{\beta}$ to this order of matching. To proceed to the higher-order matching we choose $\mu(\epsilon) = \gamma^{1/4}\eta^{1/2}$ and consider the more restricted overlap domain, $\sqrt{\epsilon} \ll \eta \ll \epsilon^{2/5}$. In this domain the $O(\gamma)^{3/2}$ term in Eq. (5.4) is filtered out and the term involving a_{11} in Eq. (5.6) stays. This a_{11} term matches identically with the C'' term in Eq. (5.11), leaving the term involving $\tilde{\beta}$ unmatchable. We must conclude, therefore, that $\tilde{\beta} = 0$.

The final form for the first term of the two-variable expansion can now be written

$$w(x, y) = \frac{\epsilon^{1/6}\, C_0}{(fGU^5)^{1/2}} \cos\left(\frac{1}{\epsilon}\int_{x_t}^{x} f(\tau)d\tau + \frac{\pi}{4}\right)\left\{\begin{matrix}\cos & \sqrt{E_s}\, y\\ \sin & \sqrt{E_n}\, y\end{matrix}\right\} + \cdots. \tag{5.13}$$

The boundary conditions on $x = 0$, given by Eq. (2.9) and (2.12), require

$$\frac{1}{\epsilon}\int_0^{x_t} f(\tau)d\tau = \left(m - \frac{1}{4}\right)\pi, \qquad m = 1, 2, \cdots. \tag{5.14}$$

Eq. (5.14) is the equation which determines the locations of the transition points x_t. The eigenvalues are then determined from

$$\alpha_{nm} = E_n(x_{t_{nm}})\, U1(x_{t_{nm}}), \qquad n, m = 1, 2, \cdots. \tag{5.15}$$

6. An example. For a numerical example illustrating the use of the formulas developed in the preceding analysis, we consider a trapezoidal planform with length 30 mm, maximum half-width 0.26 mm, and minimum half-width 0.04 mm, corresponding to the nominal dimensions of the basilar membrane. The thickness of the plate will be taken as uniform, so a comparison can be made with previous numerical results. Corresponding to these dimensions, we have $G(x) = 1 - \kappa^2 x$, with $\kappa^2 = 0.84692$ and $\epsilon = 0.008667$.

The first term of the two-variable expansion, $\epsilon^{1/6} w_1$, takes the shape

$$w(x, y) \sim \frac{\epsilon^{1/6}C_0}{(k_n \tan\beta)^{1/2}} \cos\left\{\frac{k_n}{\epsilon\kappa^2}(\tan\beta - \beta) - \frac{\pi}{4}\right\}\left\{\begin{matrix}\cos & \sqrt{E_s}\, y\\ \sin & \sqrt{E_a}\, y\end{matrix}\right\} \tag{6.1}$$

where

$$\cos\beta = \frac{G_t}{G} = \frac{1 - \kappa^2 x_t}{1 - \kappa^2 x} \tag{6.2}$$

and

$$k_n = \left\{\begin{matrix}(n - 1/2)\pi\\ n\pi\end{matrix}\right\}; n = 1, 2, \cdots, \left\{\begin{matrix}\text{symmetric transverse modes}\\ \text{antisymmetric transverse modes}\end{matrix}\right. \tag{6.3}$$

with C_0 given by Eq. (5.12).

The transition points x_t are determined from the transcendental equations,

$$\tan\beta_0 - \beta = \epsilon\pi\frac{\kappa^2}{k_n}(m - 1/4); \qquad n, m = 1, 2, \cdots, \tag{6.4}$$

and

$$\cos\beta_0 = G_t = 1 - \kappa^2 x_t. \tag{6.5}$$

The corresponding eigenvalues are then determined by Eq. (5.15) with U taken to be unity. The second column of Table I shows the first five eigenvalues computed by this method for the case $k_n = \pi/2$, which corresponds to the first symmetric transverse mode. The first column lists the eigenvalues computed from three terms of the transition layer expansion, and the last column is computed from the exact frequency equation of a circular sector plate (cf. Chadwick [1]). The eigenvalues obtained from the transition layer expansion gradually deviate from the "exact" eigenvalues. It was originally thought that the two-variable expansion would be accurate only for large longitudinal mode numbers, but it is seen that it is remarkably accurate even for the first eigenvalue.

The centerline deflection of the symmetric mode, $n = 1, m = 5$, is shown in Fig. 2. The curve is a composite of the first term of the two-variable expansion, Eq. (6.1), and the first term of the transition layer expansion, Eq. (4.18). The matching of the expansions in an intermediate overlap domain can be seen.

7. Discussion. We have discussed in detail the case when $(U/G^2)' > 0$, and found that the longitudinal mode shape is oscillatory for $x < x_t$ and decays exponentially for $x > x_t$. The case $(U/G^2)' < 0$, is essentially the same but with oscillatory soluation for $x > x_t$ and exponential decay for $x < x_t$. The case when $(U/G^2)' = 0$ has a special character and should be briefly discussed. Physically, this situation corresponds to the case when each transverse beam element has the same natural frequency. Therefore, the plate is not divided into "soft" and "stiff" regions, so we expect that no transition points exist. The appropriate expansions for this case take the form

$$w(x, y) = w_1(x, y) + \epsilon^2 w_2(x, y) + \cdots, \tag{7.1a}$$

$$\omega = \alpha + \epsilon^2 \beta + \cdots \tag{7.1b}$$

and we find

$$w_1(x, y) = A_1(x) \begin{Bmatrix} \cos & (\alpha_s/U)^{1/2} y \\ \sin & (\alpha_a/U)^{1/2} y \end{Bmatrix} \tag{7.2}$$

where

$$\begin{aligned} \alpha_s &= \left(n - \frac{1}{2}\right)^2 \pi^2 \frac{U}{G^2} \\ \alpha_a &= n^2 \pi^2 \frac{U}{G^2}. \end{aligned} \qquad n = 1, 2, \cdots. \tag{7.3}$$

TABLE I. Eigenvalues for trapezoidal planform (first two columns) and circular sector (last column).

	3 terms of transition layer expansion	1 term of two-variable expansion	"Exact" circular sector
ω_{11}	2.734	2.732	2.734
ω_{12}	2.947	2.948	2.948
ω_{13}	3.131	3.133	3.133
ω_{14}	3.300	3.305	3.305
ω_{15}	3.460	3.468	3.468

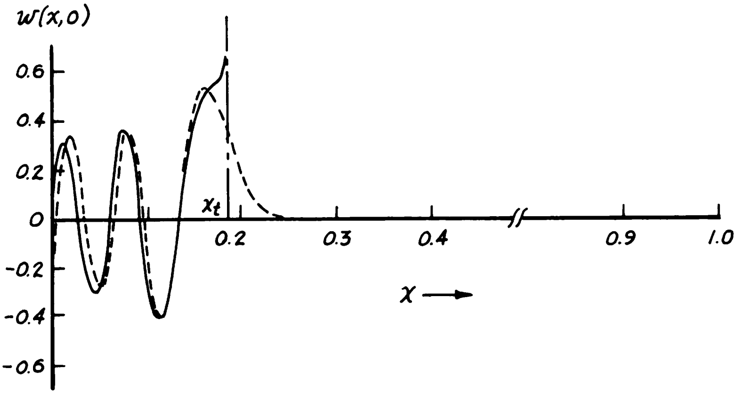

FIG. 2. First-order centerline mode shape of trapezoidal planform, $n = 1$, $m = 5$.

The function $A_1(x)$ is determined from the $O(\epsilon^2)$ problem, and is found to satisfy the differential equation

$$UA_1'' + (4 - \sigma)U'A_1' \left\{\frac{3}{2}\sigma U'' + 3\sigma \frac{U'^2}{U} + \beta\right\} A_1 = 0 \tag{7.4}$$

subject to the boundary conditions

$$A_1(0) = A_1(1) = A_1''(0) = A_1''(1) = 0. \tag{7.5}$$

The eigenvalue correlations, β, must be determined from the solution of Eq. (7.4). Edge layers must exist, in general, at the ends $x = 0, 1$, since a second-order equation can satisfy four boundary conditions only in the case of good fortune.

Finally, we can qualitatively discuss the case when f has more than one zero, or equivalently, when more than one transition point exists. This situation would occur when U/G^2 is not monotonic. A somewhat exaggerated case of this type is sketched in Fig. 3.

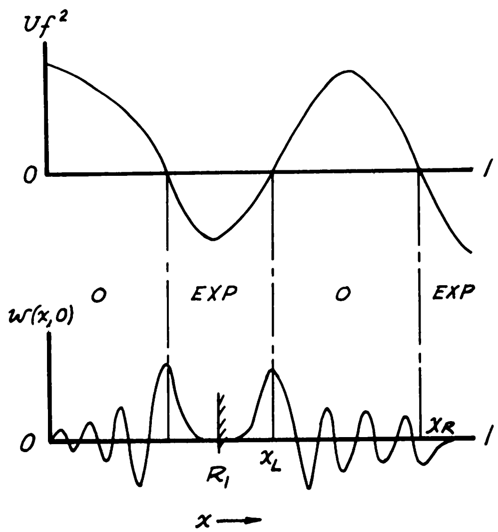

FIG. 3. Non-monotonic stiffness effects.

The upper part of the figure shows Uf^2 plotted against x, where

$$Uf^2 = \tilde{\alpha} - EU \tag{7.6}$$

and EU is proportional to U/G^2 (cf. Eq. (3.9). So for fixed $\tilde{\alpha}$, $Uf^2 > 0$ corresponds to regions where the two-variable oscillatory solutions are allowable, which are denoted by the symbol "O." Conversely, $Uf^2 < 0$ corresponds to regions of exponential decay which are marked by the symbol "EXP." The corresponding longitudinal mode shapes are sketched in the lower part of the figure. The interior of EXP regions are nodes, if transcendentally small terms are neglected. The plate is then effectively divided into separate portions which can be excited independently of each other. Thus, the regions $0 \leq x < R_1$, and $R_1 < x \leq 1$, are uncoupled from each other, in general.

The process of determining an eigenvalue $\tilde{\alpha}$ is equivalent to finding a positioning of the zeros of Uf^2 such that in at least one region the phase of the oscillatory solutions match the neighboring regions. The details of finding an eigenvalue for a region such as $0 \leq x < R_1$ has been discussed in Sec. 5. Supposing such an eigenvalue has been determined, it is unlikely that this would also be an eigenvalue for either of the other two regions. In such a case, only the region $0 \leq x < R_1$, would be excited and the others would remain at rest. Transition points for a region such as $R_1 < x \leq 1$ are determined by the equation

$$\int_{x_L(x_R)}^{x_R} f(\tau)\, d\tau = \epsilon\left(m - \frac{1}{2}\right)\pi, \qquad m = 1, 2, \cdots . \tag{7.7}$$

The function $x_L(x_R)$ can be determined once the function U/G^2 is specified. It is interesting that Eq. (7.7) is analogous to the Bohr-Sommerfeld quantization condition commonly used in one-dimensional wave mechanics problems (cf. Fermi [3]). Once the transition points are found, the eigenvalues can be calculated from Eq. (5.15).

References

[1] R. S. Chadwick, *Vibration of long narrow plates*—I, Quart. Appl. Math., this issue

[2] J. R. Vinson, *Structural mechanics: the behavior of plates and shells*, Wiley & Sons, New York, 1974

[3] E. Fermi, *Notes on quantum mechanics*, University of Chicago Press, Chicago, 1961