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FREE BOUNDARY PROBLEMS WITH RADIATION BOUNDARY CONDITIONS*

BY

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Abstract. The paper is concerned with the free boundary problem of a semi-infinite body of arbitrarily prescribed initial temperature, subject to a mixed or radiation boundary condition at its face. The analytically exact solutions of temperature of both phases and the interfacial position are established in series of time and functions of the error integral family. Convergence of these series solutions is studied and proved. A few remarks on the solutions and their simplifications are then offered. A discussion on the analyticity of the solutions is also given. The paper concludes with an illustrative example, the so-called one-phase problem.

1. Introduction. There are many physical problems in which a diffusive process is accompanied by a change of phase of the material. The main feature of these problems is the existence of a moving surface of separation between the two phases. This moving surface of separation is a free surface which is not known a priori and depends upon the states and properties of the two phases of the material. These problems are usually known as the Stefan problems or free boundary problems. A representative example of these problems is the melting or freezing of an ice-water combination.

Free boundary problems have been studied since the nineteenth century by Lamé and Clapeyron in 1831, Neumann in the 1860s and Stefan in 1889. They all considered the problem of a semi-infinite body of a constant temperature which is brought in contact with a different temperature at its face. The exact solution of this free boundary problem was established by Neumann [1]. Since then, seeking solutions of free boundary problems has been the subject of investigations by many researchers using various mathematical methods and techniques. These solutions have been discussed and summarized in several books [2–6] and many survey papers [7–10] in various scientific and engineering areas. However, the only known exact solutions are still those of Neumann and some of their extensions and variations. All these solutions are expressed in terms of the similarity variable $x/t^{1/2}$. This means that the partial differential equations are reducible to ordinary differential equations. This reduction to ordinary differential equations is possible only when the domain is semi-infinite and the initial and boundary conditions are of some special forms; it fails to be possible when the boundary condition of constant temperature is replaced by a constant flux at its face. Establishment of exact solutions of free boundary problems other than those by similarity transforms has been attempted by many researchers.

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Two analytically exact solutions of free boundary problems have been established, very recently, by Tao [11, 12]. One solution is for the Neumann problem with arbitrarily prescribed initial and boundary temperature and the other for the same problem but with arbitrarily prescribed heat flux at its face. Besides these two boundary value problems, there is a third problem with mixed or radiation boundary condition. It is the purpose of this paper to discuss the solutions of the third problem, i.e., to establish an exact solution of the classical free boundary problem of a semi-infinite body of arbitrarily prescribed initial temperature, subject to an arbitrarily prescribed radiation boundary condition at its face.

In Rubinstein's monograph *The Stefan problem* [4] there is a chapter on *still unsolved problems*. One of these problems is concerned with the analyticity of the interfacial boundary. The importance of establishing the analyticity of the interfacial boundary is undeniable. He remarks that this problem is nontrivial. From the exact solution of the present study, the analyticity of the interfacial boundary can be directly confirmed. When the boundary and initial conditions are analytic functions, the interfacial boundary is analytic in $t^{1/2}$.

In the next section we first derive an analytically exact solution to the above-mentioned problem in series of functions in the error integral family and time t. Convergence of these series solutions is then considered and proved in Sec. 3. The solution of this third problem is somewhat more involved than those of the other two problems. In the form of the solution established, it is not possible to include either one of the first two problems as a special case. A few remarks on the solutions and their simplifications are given in Sec. 4. In addition, Sec. 4 includes a discussion on the analyticity of the interfacial boundary and the temperature solutions. For illustrative purposes, we conclude the paper with an application of the solution to the so-called one-phase problem.

2. Mathematical solutions. Consider a material of constant properties in a semi-infinite region, $x > 0$, with an arbitrary initial temperature $V(x)$. At its face $x = 0$, it is subject to a mixed or radiation boundary condition

$$\partial T/\partial x = h[T - U(t)].$$

Due to this change of the temperature at $x = 0$, a new phase of material starts to appear. Let the two phases of the material, new and original, be designated by subscripts I and II, respectively. Then the complete set of equations of this free boundary problem is

$$\begin{aligned}
&\alpha_I \frac{\partial^2 T_I}{\partial x^2} = \frac{\partial T_I}{\partial t}, \quad 0 < x < s(t),\\
&\alpha_{II} \frac{\partial^2 T_{II}}{\partial x^2} = \frac{\partial T_{II}}{\partial t}, \quad s(t) < x < \infty,\\
&\frac{\partial T_I}{\partial x}(0, t) = h[T_I(0, t) - U(t)], \qquad (2.1)\\
&T_{II}(x, 0) = V(x),\ T_I(s, t) = T_{II}(s, t) = T_f,\\
&k_I \left.\frac{\partial T_I}{\partial x}\right|_s - k_{II} \left.\frac{\partial T_{II}}{\partial x}\right|_s = \pm \rho l \frac{ds}{dt},\ s(0) = 0.
\end{aligned}$$

The sign in the heat-balance equation depends upon whether there is a liberation or absorption of heat during the phase change.

The problem is to seek the temperature T_I and T_{II} and the position of the interface $s(t)$.

To be specific, let us consider the solidification problem. Then the original phase is a liquid with an initial temperature $V(x)$, above or equal to the freezing temperature T_f. Solidification will occur only when the environmental temperature $U(t)$ is below T_f. If the environmental temperature $U(t)$ at $t = 0$ is taken as the reference temperature, then $U(0) = 0$. Also, we assume that $U(t)$ and $V(x)$ are analytic functions of their respective argument in their own domain. This means that they are expressible in the form of power series

$$U(t) = \sum_1 u_n t^n/n!, \quad u_n = (d^n U/dt^n)_0,$$

$$V(x) = \sum_0 v_n(x/\alpha_{II}{}^{1/2})^n/n!, \quad v_n = \alpha_{II}{}^{n/2}(d^n V/dx^n)_0 \cdot \tag{2.2}$$

Before we seek the solutions of this free boundary problem, let us first recall some identities of functions in the error integral family [11, 12]:

$$\operatorname{erfc}(-\xi) = 2 - \operatorname{erfc}\xi, \; i\operatorname{erfc}(-\xi) = 2\xi + i\operatorname{erfc}\xi,$$

$$i^n\operatorname{erfc}(-\xi) + (-1)^n\, i^n\operatorname{erfc}\xi = \sum_0^n [1 + (-1)^m]\, i^m \operatorname{erfc}(0)\, \frac{\xi^{n-m}}{(n-m)!}$$

$$= \sum_0^{[n/2]} \frac{\xi^{n-2m}}{2^{2m-1}\, m!\,(n-2m)!}$$

where $[n/2]$ stands for the largest integer of $n/2$. For future convenience, we introduce

$$E_n(\xi) = \tfrac{1}{2}[i^n\operatorname{erfc}(-\xi) + i^n\operatorname{erfc}\xi],$$

$$F_n(\xi) = \tfrac{1}{2}[i^n\operatorname{erfc}(-\xi) - i^n\operatorname{erfc}\xi], \tag{2.3}$$

$$G_n(\xi) = \tfrac{1}{2}[i^n\operatorname{erfc}(-\xi) + (-1)^n\, i^n\operatorname{erfc}\xi].$$

We may notice that $G_n(\xi)$ is a polynomial of order n. These polynomials are closely related to the so-called heat polynomials [13]. It may also be noted that all three functions are undefined at $t = 0$, but their products with $(4\alpha t)^{n/2}$ have a definite limit

$$(4\alpha t)^{n/2}E_n(\xi)|_0 = (4\alpha t)^{n/2}F_n(\xi)|_0 = (4\alpha t)^{n/2}G_n(\xi)|_0$$

$$= x^n/n! \qquad (\xi = x/(4\alpha t)^{1/2}). \tag{2.4}$$

Some other mathematical properties of these functions are given in [11, 12] and will not be repeated here.

Now we are able to express the solutions of T_I and T_{II} in the form of

$$T_I = T_I^{(1)} + T_I^{(2)}, \qquad T_{II} = T_{II}^{(1)} + T_{II}^{(2)} \tag{2.5}$$

where

$$T_I^{(1)} = \sum_1 u_n\,(4t)^n G_{2n}(\xi_I),$$

$$T_I^{(2)} = \sum_0 a_n\,(4t)^{n/2}[E_n(\xi_I) + h(4\alpha_I t)^{1/2}F_{n+1}(\xi_I)],$$

$$T_{II}^{(1)} = \sum_0 v_n(4t)^{n/2}\, G_n(\xi_{II}),$$

$$T_{II}^{(2)} = \sum_0 b_n\,(4t)^{n/2}\, i^n\operatorname{erfc}(\xi_{II}), \tag{2.6}$$

and $\xi_i = x/(4\alpha_i t)^{1/2}$. It is readily seen that every term in the above equations is a solution of its respective diffusion equation. Furthermore, they satisfy their proper boundary and initial condition. At $x = 0$,

$$\left[T_I^{(1)} - \frac{1}{h}\frac{\partial T_I^{(1)}}{\partial x}\right]_{x=0} = \sum_1 u_n (4t)^n G_{2n}(0) - \frac{1}{h\alpha_I^{1/2}} \sum_1 u_n (4t)^{n-1/2} G_{2n-1}(0)$$

$$= \sum_1 u_n t^n/n! = U(t)$$

and

$$\left[T_I^{(2)} - \frac{1}{h}\frac{\partial T_I^{(2)}}{\partial x}\right]_{x=0} = a_0 h F_1(0) + \sum_1 a_n \Big\{(4t)^{n/2}[E_n(0) + h(4\alpha_I t)^{1/2} F_{n+1}(0)]$$

$$- \frac{1}{h\alpha_I^{1/2}} (4t)^{(n-1)/2} [F_{n-1}(0) + h(4\alpha_I t)^{1/2} E_n(0)]\Big\} = 0.$$

Here we have used $G_{2n-1}(0) = F_n(0) = 0$. As $t \to 0$,

$$T_{II}^{(1)}(x, 0) = \sum_0 v_n (4t)^{n/2} G_n(\xi_{II})|_0 = \sum_0 v_n (x/\alpha_{II}^{1/2})^n/n! = V(x),$$

$$T_{II}^{(2)}(x, 0) = \sum_0 b_n (4t)^{n/2} i^n\mathrm{erfc}(\infty) = 0.$$

To complete the solutions, we need to determine the coefficients a_n and b_n and the function $s(t)$ satisfying the interface conditions:

$$T_I(s, t) = T_{II}(s, t) = T_f\,,$$

$$k_I \left.\frac{\partial T_I}{\partial x}\right|_s - k_{II}\left.\frac{\partial T_{II}}{\partial x}\right|_s = \rho l \frac{ds}{dt}\,. \tag{2.7}$$

We write the solution of $s(t)$ in the form of

$$s(t) = 2\alpha_I^{1/2} t^\nu \chi(t)$$

where $\nu > 0$ and $\chi(t)$ is a function of t with $\chi(0) \neq 0$. Substitution of this equation in (2.7) and evaluation at $t = 0$ reveal that $\nu = \frac{1}{2}$. The usual approach to find the coefficients would have been to express $\chi(t)$ in a power series and to equate equal powers of t. A series of $\chi(t)$ in integer powers of t, however, will not be able to satisfy all three interface equations. It is necessary to express the function $\chi(t)$ in a power series of $t^{1/2}$.

Let us introduce

$$\tau = t^{1/2}, \qquad \omega = (\alpha_I/\alpha_{II})^{1/2}$$

and write

$$s(t) = (4\alpha_I)^{1/2}\tau\,\chi(t) = (4\alpha_I)^{1/2} \sum_0 c_n \tau^{n+1},$$

$$\eta_I(t) = s/(4\alpha_I t)^{1/2} = \sum_0 c_n \tau^n, \tag{2.8}$$

$$\eta_{II}(t) = s/(4\alpha_{II} t)^{1/2} = \omega \sum_0 c_n \tau^n.$$

The coefficients a_n, b_n and c_n can now be determined by matching the powers of τ, after all functions have been expanded in power series of τ. However, we prefer to differentiate the interface equations successively with respect to τ and evaluate them at $\tau = 0$. Also, to circumvent terms of negative powers of τ in the heat-balance equation, we differentiate this equation after it is multiplied by τ. This is permissible since these equations are valid for all time. Differentiations yield

$$T_I(s, t) = T_{II}(s, t) = T_f ,$$
$$D_\tau^N T(s, t)|_0 = D_\tau^N T_{II}(s, t)|_0 = 0, \qquad N = 1, 2, 3, \cdots , \tag{2.9}$$

$$k_I D_\tau^N \left(\tau \left.\frac{\partial T_I}{\partial x}\right|_s\right)_0 - k_{II} D_\tau^N \left(\tau \left.\frac{\partial T_{II}}{\partial x}\right|_s\right)_0 = \rho l \alpha_I^{1/2}(N+1)!\, c_n , \qquad N = 0, 1, 2, \cdots \cdot$$

Taking $N = 0$, we obtain

$$a_0 = T_f , \qquad b_0 = (T_f - v_0)/\text{erfc}(\omega c_0)$$

and c_0 satisfies

$$b_0 \omega k_{II} \Phi_1(\omega c_0) = 2\rho l \alpha_I c_0 \qquad (\Phi_1(\xi) = d\,\text{erf}\xi/d\xi). \tag{2.10}$$

To determine the other coefficients, we use a general formula for the material time derivative of an arbitrary order [14]. Let $f = f(x, t)$ and $x = x(t)$; then

$$D^n f = \sum_{n=0}^{N} \sum_{m=0}^{N-n} \left(\frac{N!}{n!}\right) Z_m^{N-n}(x)\, \partial_t^n \partial_x^m f$$

where $D \equiv d/dt$, $\partial_t \equiv \partial/\partial t$, $\partial_x \equiv \partial/\partial x$ and

$$Z_m^n(x) = \sum\nolimits_{\beta_k} \frac{1}{\beta_1!\,\beta_2! \cdots \beta_n!} (Dx)^{\beta_1} (D^2x/2!)^{\beta_2} \cdots (D^n x/n!)^{\beta_n}$$

The summation $\sum_{\beta_k}$ is the sum extended to all non-negative integers $\beta_k = 0, 1, 2, \cdots$ such that

$$\sum_1^n \beta_k = m, \qquad \sum_1^n k\beta_k = n.$$

This formula is a generalization of Faa de Bruno's formula for the derivative of a function of a function [15]. Using this formula, we find

$$D_\tau^N [(2\tau)^n G_n(\eta_{II})]_{\tau=0} = 2^n N! \sum_0^{N-n} Z_m^{N-n}(\eta_{II})\, G_{n-m}(\eta_{II})|_{\tau=0}$$
$$= 2^n N! \sum_0^{p} Z_m^{N-n}(\eta_{II})\, G_{n-m}(\eta_{II})|_{\tau=0}$$

where $p = \min(N - n, n)$. In the last step, we have used that $G_{n-m}(\eta) = 0$ where $m > n$. In summary, we have

$$D_\tau^N [(2\tau)^n G_n(\eta_{II})]_{\tau=0} = N!\, A_n^N(\omega) \tag{2.11}$$

with

$$A_n^N(\omega) = 2^n \sum_{m=0}^{p} \omega^m G_{n-m}(\omega c_0) \sum\nolimits_{\beta_r} \left(\prod_{r=1}^{p} \frac{C_r^{\beta_r}}{\beta_r!}\right)$$

subject to

$$\sum_{1}^{N-n} \beta_r = m, \ \sum_{1}^{N-n} r\beta_r = N - n.$$

Similarly,

$$\begin{aligned} D_\tau^N\, [(2\tau)^n\, i^n \operatorname{erfc}(\eta_{II})]_{\tau=0} &= N!\, B_n^N(\omega), \\ D_\tau^N\, [(2\tau)^n\, E_n(\eta_{II})]_{\tau=0} &= N!\, [A_n^N(\omega) + \tfrac{1}{2}(1 - (-1)^n)\, B_n^N(\omega)], \\ D_\tau^N\, [(2\tau)^n\, F_n(\eta_{II})]_{\tau=0} &= N!\, [A_n^N(\omega) - \tfrac{1}{2}(1 + (-1)^n)\, B_n^N(\omega)], \end{aligned} \tag{2.12}$$

where

$$B_n^N(\omega) = 2^n \sum_{m=0}^{N-n} (-\omega)^m\, i^{n-m} \operatorname{erfc}(\omega c_0) \textstyle\sum_{\beta_r} \left(\displaystyle\prod_{r=1}^{N-n} c_r^{\beta r}/\beta r! \right)$$

subject to

$$\sum_{1}^{N-n} \beta_r = m, \ \sum_{1}^{N-n} r\beta_r = N - n.$$

Here we have, for convenience, used the notation

$$i^{-m} \operatorname{erfc} \xi = (-1)^{m+1}\, \Phi_m(\xi),\ m \geq 1,$$

where $\Phi_m(\xi) = d^m \mathrm{erf}\xi/d\xi^m$. Also we find

$$\begin{aligned} D_\tau^N[2^n \tau^{n+1} G_n(\eta_{II})]_{\tau=0} &= N\, D_\tau^{N-1}[(2\tau)^n Gn(\eta_{II})] = N!\, A_n^{N-1}(\omega), \\ D_\tau^N\, [2^n \tau^{n+1} i^n \mathrm{erfc}(\eta_{II})]_{\tau=0} &= N!\, B_n^{N-1}(\omega), \\ D_\tau^N\, [2^n \tau^{n+1}\, E_n(\eta_{II})]_{\tau=0} &= N!\, [A_n^{N-1}(\omega) + \tfrac{1}{2}(1 - (-1)^n) B_n^{N-1}(\omega)], \\ D_\tau^N\, [2^n \tau^{n+1}\, F_n(\eta_{II})]_{\tau=0} &= N!\, [A_n^{N-1}(\omega) - \tfrac{1}{2}(1 + (-1)^n)\, B_n^{N-1}(\omega)]. \end{aligned} \tag{2.13}$$

Using these formulas, we may write the interface conditions as

$$\begin{aligned} &\sum_{1}^{[N/2]} u_n\, A_{2n}^N(1) + \sum_{0}^{N} a_n\, [A_n^N(1) + h\alpha_I^{1/2}\, A_{n+1}^N(1)] \\ &\quad + \sum_{0}^{[N-1]} a_{2n+1}\, B_{2n+1}^N(1) - h\alpha_I^{1/2} \sum_{1}^{[N/2]} a_{2n-1}\, B_{2n}^N(1) = 0, \\ &\sum_{1}^{N} v_n\, A_n^N(\omega) + \sum_{0}^{N} b_n\, B_n^N(\omega) = 0, \\ &\sum_{1}^{[N/2]} u_n\, A_{2n-1}^{N-1}(1) + \sum_{1}^{N} a_n\, [A_{n-1}^{N-1}(1) + h\alpha_I^{1/2}\, A_n^{N-1}(1) \\ &\quad - \sum_{0}^{[(N-1)/2]} a_{2n+1}\, B_{2n}^{N-1}(1) + h\alpha_I^{1/2} \sum_{1}^{[N/2]} a_{2n-1}\, B_{2n-1}^{N-1}(1) \\ &\quad - \omega(k_{II}/k_I) \left[\sum_{1}^{N} v_n\, A_{n-1}^{N-1}(\omega) - \sum_{0}^{N} b_n\, B_{n-1}^{N-1}(\omega) \right] = (\rho l \alpha_I/k_I)(N+1)\, c_N\,. \end{aligned} \tag{2.14}$$

From this set of algebraic equations, we may determine, starting from $N = 1$ step by step, the coefficients a_n, b_n and c_n.

3. Convergence. In the preceding section we have formally established the solutions of the free boundary problem with a radiation boundary condition. It is the purpose of this section to consider the convergence of the solutions. Though there are many existence and uniqueness proofs [4, 16–18], a direct proof of the convergence of the series solutions is necessary to complete the present study. To this end, let us adopt an order of magnitude approach analogous to that used by Widder [13] for problems of the heat equation without moving boundaries. This method is also used in [11, 12], but the present one is an improved version.

Let s_0 be the interfacial position at time t_0. The series

$$\sum a_n(4t_0)^{n/2}\,\mathcal{E}_n(\xi_0, t_0) = \sum a_n(4t_0)^{n/2}[E_n(\xi_0) + h(4\alpha t_0)F_{n+1}(\xi_0)$$

converges, since a_n are determined from (2.7a) of the interface equations. Thus we deduce that a_n are at most of the order

$$a_n = O\{[(4t_0)^{n/1}\,\mathcal{E}_n(\xi_0\,, t_0)]^{-1}\}. \tag{3.1}$$

Also we notice that when $0 < \xi < \xi_0$, $E_n(\xi)$, $F_n(\xi)$ and $G_n(\xi)$ are positive and monotonically increasing. The positiveness follows from their definitions and the fact that

$$i^n \operatorname{erfc}(-\xi) > i^n \operatorname{erfc} \xi > 0.$$

The monotonicity follows from the fact that the derivatives of $E_n(\xi)$, $F_n(\xi)$ and $G_n(\xi)$ are respectively $F_{n-1}(\xi)$, $E_{n-1}(\xi)$ and $G_{n-1}(\xi)$, which are all positive. Therefore

$$0 < \mathcal{E}_n(\xi, t) < \mathcal{E}_n(\xi_0\,, t_0). \tag{3.2}$$

Using (3.1) and (3.2), we have

$$\left|\sum a_n(4t)^{n/2}\,\mathcal{E}_n(\xi, t)\right| < M_1{}_1\left(\frac{t}{t_0}\right)^{n/2} \frac{\mathcal{E}_n(\xi, t)}{\mathcal{E}_n(\xi_0\,, t_0)} < M_1 \sum \left(\frac{t}{t_0}\right)^{n/2}, \tag{3.3}$$

where M_1 is a constant independent of n. This is a geometric series. It implies that the given series converges absolutely, and hence T_I is bounded for all $\xi < \xi_0$, and $t < t_0$. Since the series is majorized by a convergent series with constant terms, its convergence is also uniform. From (2.7) we have

$$\sum b_n\,(4t_0)^{n/2}\, i^n \operatorname{erfc} \xi_0$$

converges for $t_0 > 0$. With

$$b_n = 0\,[(2^n t_0{}^{n/2}\, i^n \operatorname{erfc} \xi_0)^{-1}] \tag{3.4}$$

and

$$0 < i^n \operatorname{erfc} \xi < i^n \operatorname{erfc} \xi_0\,, \qquad x > s_0 \text{ and } t < t_0\,,$$

we conclude that

$$\left|\sum b_n(4t)^{n/2}\, i^n \operatorname{erfc} \xi\right| < M_2 \sum \left(\frac{t}{t_0}\right)^{n/2} \frac{i^n \operatorname{erfc} \xi}{i^n \operatorname{erfc} \xi_0} < M_2 \sum (t/t_0)^{n/2}. \tag{3.5}$$

This establishes the boundedness of T_{II}.

The series of $s(t)$ can be similarly discussed. From the heat-balance equation (2.7b), we observe that both $\partial T_I/\partial x$ and $\partial T_{II}/\partial x$ exist at the interface. This implies that

$$\sum a_n(4t_0)^{(n-1)/2}\mathfrak{F}_{n-1}(\xi_0, t_0) = \sum a_n(4t_0)^{(n-1)/2}[F_{n-1}(\xi_0) + h(4\alpha t_0)^{1/2}E_n(\xi_0)]$$

and

$$\sum b_n (4t_0)^{(n-1)/2}\, i^{n-1} \operatorname{erfc} \xi_0$$

converge for all finite $t_0 > 0$. Therefore as $n \to \infty$,

$$a_{n+1} = O\{[(4t_0)^{n/2}\mathfrak{F}_n(\xi_0, t_0)]^{-1}\},$$
$$b_{n+1} = O\{[(4t_0)^{n/2}\, i^n \operatorname{erfc} \xi_0]^{-1}\}.$$

Also, we observe that

$$\begin{aligned} &\mathfrak{F}_n(\xi, t) < \mathfrak{F}_n(\xi_0, t_0), \qquad \xi < \xi_0,\ t < t_0, \\ &i^n \operatorname{erfc} \xi < i^n \operatorname{erfc} \xi_0, \qquad x > s_0,\ t < t_0. \end{aligned} \tag{3.6}$$

Thus

$$\left|\sum a_{n+1}(4t)^{n/2}\mathfrak{F}_n(\xi, t)\right| < M_3 \sum (t/t_0)^{n/2},$$
$$\left|\sum b_{n+1}(4t)^{n/2}\, i^n \operatorname{erfc} \xi\right| < M_4 \sum (t/t_0)^{n/2},$$

where M_3 and M_4 are independent of n. Hence $\partial T_I/\partial x$ and $\partial T_{II}/\partial x$ are bounded in their respective domain. This, in turn, shows from (2.7b) that $s(t)$ is absolutely and uniformly convergent for all $t < t_0$.

4. Remarks. In the preceding sections we have established an analytically exact solution to the free boundary problem of a semi-infinite body with an arbitrarily prescribed radiation boundary condition at its face and an arbitrary initial condition. The solution is applicable only when the heat transfer coefficient is finite and positive. When $h = 0$, the given problem is reduced to that with an insulated face at $x = 0$, which will have no phase change. On the other hand, as $h \to \infty$, $T_I^{(2)}$ of (2.6) is unbounded; the solution is meaningless. If it is necessary to accommodate the case of $h \to \infty$, the solution of $T_I^{(2)}$ must be modified. The determination of the coefficients will be more involved and complicated. In view of the fact that the solutions of free boundary problems with either prescribed temperature or prescribed flux have been found [11, 12], we will not pursue these limiting cases further.

If the initial temperature $V(x)$ has the value $V(0) = v_0 = T_f$, we may readily see that $b_0 = c_0 = 0$ from (2.10). Then the solution of $s(t)$ would be of the order of t, instead of $t^{1/2}$, as $t \to 0$. If, in addition to $V(0) = T_f$, the initial temperature $V(x)$ contains only even powers of x, then the solutions are expressible in power series of t, instead of $t^{1/2}$, i.e.

$$a_{2n+1} = b_{2n+1} = c_{2n} = 0.$$

Though this conclusion can be established by reducing the previous result, we prefer to derive it directly by writing the solutions in the form

$$\begin{aligned} T_I &= \sum_1 u_n (4t)^n G_{2n}(\xi_I) + \sum_0 a_{2n}(4t)^n [G_{2n}(\xi_I) + h(4\alpha_I t)^{1/2}G_{2n+1}(\xi_I)], \\ T_{II} &= T_f + \sum_1 v_{2n}(4t)^n G_{2n}(\xi_{II}) + \sum_1 b_{2n}(4t)^n\, i^n \operatorname{erfc}(\xi_{II}), \\ s(t) &= (4\alpha_I)^{1/2} \sum_0 c_{2n+1} t^{n+1}. \end{aligned} \tag{4.1}$$

Here we have used $E_{2n}(\xi) = G_{2n}(\xi)$ and $F_{2n+1}(\xi) = G_{2n+1}(\xi)$. Parallel to the previous derivations, with the exception that the heat balance equation is not multiplied by $t^{1/2}$, we obtain

$$
\begin{aligned}
&a_0 = T_f\ , c_1 = hk_I\, T_f/2\rho l\alpha^{1/2},\\
&\cdot \sum_1^N u_n\, A_{2n}^{2N}(1) + \sum_0^N a_{2n}\,[A_{2n}^{2N}(1) + h\alpha_I{}^{1/2}\, A_{2n+1}^{2N}(1)] = 0,\\
&\cdot \sum_1^N v_{2n}\, A_{2n}^{2N}(\omega) + \sum_1^N b_{2n}\, B_{2n}^{2N}(\omega) = 0,\\
&\cdot \sum_1^N u_n\, A_{2n-1}^{2N}(1) + \sum_1^N a_{2n}\,[A_{2n-1}^{2N}(1) + h\alpha_I{}^{1/2}\, A_{2n}^{2N}(1)]\\
&- \omega(k_{II}/k_I)\left[\sum_1^N v_{2n}\, A_{2n-1}^{2N}(\omega) - \sum_1^N b_{2n}\, B_{2n-1}^{2N}(\omega)\right]\\
&= 2\,(\rho l\alpha_I/k_I)\,(N+1)\,c_{2N+1}\,.
\end{aligned}
\tag{4.2}
$$

This is the required result.

From the solutions established in this paper the question of the analyticity of the solutions can be answered. It is readily seen from (2.8) that when the boundary and initial conditions of the problem are analytic functions, the interfacial boundary is analytic in $t^{1/2}$. Also since the terms of $t^{n/2}G_n(\xi)$ and others of (2.7) are defined for all finite $x > 0$ and $0 < t < t_0 < \infty$, the temperature solutions are analytic functions of x and $t^{1/2}$. Moreover, when $V(0) = T_f$ and $V(x)$ contains only even powers of x, the interfacial boundary is analytic in t, i.e., a power series of t. The corresponding solutions of the temperature are analytic functions of x and t.

5. An illustrative example. We intend now to apply the result to the case of a liquid which is initially at T_f and subject to a radiation condition

$$\partial T_I/\partial x - hT_I = 0$$

at $x = 0$. This is the so-called one-phase problem; the temperature T_{II} is identically equal to T_f. Since the material properties of phase II are not involved, the subscript I for all material constants will, hereafter, be omitted.

In this problem, only the coefficients a_{2n} and c_{2n+1} need be determined. We first calculate $A_{2n}^{2N}(1)$ and $A_{2n+1}^{2N}(1)$ from (2.11). By direct evaluations [12], we find

$$
\begin{aligned}
&A_{2n}^{2N} = A_{2n+1}^{2N} = 0 \qquad \text{when } n > N,\\
&A_{2N}^{2N} = 1/N!, \qquad A_0^{2N} = 0,\\
&A_1^2 = 2c_1\,, \qquad A_1^4 = 2c_3\,,\\
&A_2^4 = 2c_1{}^2, \qquad A_3^4 = 2c_1\,,\\
&A_1^6 = 2c_5\,, \qquad A_2^6 = 4c_1c_3\,,\\
&A_3^6 = 2c_3 + 4c_1{}^3/3, \qquad A_4^6 = 2c_1{}^2, \qquad A_5^6 = c_1\,.
\end{aligned}
$$

From (4.2) we then have

$$a_0 = T_f\,, \qquad c_1 = \frac{hkT_f}{2\rho l\alpha^{1/2}} = \lambda\Omega/2,$$

$$a_2 = -\lambda^2\Omega T_f\,, \qquad c_3 = -\lambda^3\Omega^2(\Omega + 1)/4,$$

$$a_4 = \lambda^4\Omega^2(2\Omega + 3)T_f\,,\ c_5 = \lambda^5\Omega^3(5\Omega^2 + 8\Omega + 3)/12,$$

$$a_6 = -\lambda^6\Omega^3(14\Omega^2 + 28\Omega + 15)T_f\,,\ c_7 = -\lambda^7\Omega^4(51\Omega^3 + 109\Omega^2 + 73\Omega + 15)/48,$$

where $\lambda = h\alpha^{1/2}$ and $\Omega = kT_f/\rho l\alpha$. Therefore the solution of this one-phase problem is

$$\begin{aligned} T_1/T_f = {} & (1 + hx) - \lambda^2\Omega[G_2(\xi) + 2\lambda t^{1/2}G_3(\xi)](4t) \\ & + \lambda^4\Omega^2(2\Omega + 3)[G_4(\xi) + 2\lambda t^{1/2}G_5(\xi)](4t)^2 \\ & + \lambda^6\Omega^3(14\Omega^2 + 28\Omega + 15)[G_6(\xi) + 2\lambda t^{1/2}G_7(\xi)](4t)^3 + \cdots, \end{aligned}$$

$$\begin{aligned} s(t) = {} & (\Omega/h)[\lambda^2 t - \tfrac{1}{2}\Omega(\Omega + 1)(\lambda^2 t)^2 + \tfrac{1}{6}\Omega^2(5\Omega^2 + 8\Omega + 3)(\lambda^2 t)^3 \\ & - \tfrac{1}{48}\Omega^3(51\Omega^3 + 109\Omega^2 + 73\Omega + 15)(\lambda^2 t)^4 + \cdots]. \end{aligned}$$

A result of only the first two terms has previously been found by a double series expansion in [2].

References

[1] G. F. B. Riemann and H. Weber, *Die partiellen Differentialgleichungen der mathematischen Physik*, Bd.2, 5-fe Aufl. Braunschweig, 1912

[2] H. S. Carslaw and J. C. Jaeger, *Conduction of heat in solids*, 2nd ed., Clarendon Press, Oxford, 1959

[3] J. Crank, *The mathematics of diffusion*, 2nd ed., Clarendon Press, Oxford, 1975

[4] L. I. Rubinstein, *The Stefan problem* (English translation), Am. Math. Soc., 1971

[5] J. R. Ockendon and W. R. Hodgkin (eds.), *Moving boundary problems in heat flow and diffusion*, Clarendon Press, Oxford, 1975

[6] R. W. Ruddle, *The solidification of castings*, Inst. Metals, London, 1957

[7] S. G. Bankoff, *Heat conduction or diffusion with change of phase*, Adv. in Chem. Eng. **5**, 75–150 (1964)

[8] J. C. Muehbauer and J. E. Sunderland, *Heat conduction with freezing or melting*, Appl. Mech. Rev. **18**, 951–959 (1965)

[9] W. A. Tiller, *Principles of solidification*, in *Arts and sciences of growing crystals*, ed. J. J. Gilman, Wiley, New York, 1963

[10] B. A. Boley, *Survey of recent developments in the fields of heat conduction in solids and thermo-elasticity*, Nuclear Eng. Des. **18**, 377–399 (1972)

[11] L. N. Tao, *The Stefan problem with arbitrary initial and boundary conditions*, Quart. Appl. Math. **36**, 223–233 (1978)

[12] L. N. Tao, *On free boundary problems with arbitrary initial and flux conditions*, to be published

[13] D. V. Widder, *The heat equation*, Academic Press, New York, 1975

[14] L. N. Tao, *On the material time derivative of arbitrary order*, Quart. Appl. Math. **36**, 323–324 (1978)

[15] C. Jordan, *Calculus of finite differences*, Chelsea, New York, 1965

[16] A. Friedman, *Free boundary problems for parabolic equations, I, Melting of solids*, J. Math. Mech. **8**, 499–517 (1959)

[17] A. Friedman, *Partial differential equations of parabolic type*, Prentice Hall, Englewood Cliffs, N. J., 1964

[18] J. R. Cannon and C. D. Hill, *Existence, uniqueness, stability, and monotone dependence in a Stefan problem for the heat equation*, J. Math. Mech. **17**, 1–20 (1967)