RESONANT FREQUENCIES IN AN ELECTROMAGNETIC ECCENTRIC SPHERICAL CAVITY*

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Abstract. The interior boundary-value electromagnetic (vector) problem in the region between two perfectly conducting spheres of radii R_1, R_2 and distance d between their centers is considered. Surface singular integral equations are used to formulate the problem. Use of spherical vector wave functions and related addition theorems reduces the solution of the integral equations to the problem of solving an infinite set of linear equations. Their determinant is evaluated in powers of $kd = 2\pi d/\lambda$ to a few terms. It is then specialized to the axially symmetric case and set equal to zero. This yields closed-form expressions for the coefficients g_{ns} in the resulting relations $\omega_{ns}(kd) = \omega_{ns}(0)\,[1 + g_{ns}(kd)^2 + \cdots]$ for the natural frequencies of the cavity. Numerical results, comparisons and possible generalizations are also included.

Introduction. The interior boundary-value acoustic problem in the region between two spheres of radii R_1, R_2 and distance d between their centers (see Fig. 1) for both Dirichlet and Neumann boundary conditions, has been solved elsewhere [1]. The present paper deals with the corresponding electromagnetic problem between perfectly conducting eccentric spheres and should be read in conjuction with [1]. The motivation for considering analytical and exact solutions to such problems, their advantages over numerical ones and the possible generalizations to other shapes are discussed in [1] and will not be repeated here. An updated and extensive reference list may also be found in [1]. However, the significant differences between acoustic and electromagnetic cavities that will be brought to light in this paper require a separate treatment.

As in [1], let small letters (p_1, q_2) denote points on S_1 and S_2 and capitals (P, Q) points not on S_1, S_2. A fixed point on S_1 or S_2 is designated by p_1 (θ_1, ϕ_1) or $p_2(\theta_2\nabla$, $\phi_2)$, a variable one by $q_1(\theta_1'$, $\phi_1')$ or $q_2(\theta_2'$, $\phi_2')$, with the primes indicating, in particular, variables of integration. Under this notation R_1, θ_1, ϕ_1 (or R_2, θ_2', ϕ_2') indicate spherical coordinates of p_1 (or q_2) with respect to centers O_1 (or O_2) of the surface S_1 (or S_2) to which the point belongs. However, $p_1(R_{12}$, θ_{12}, $\phi_{12})$ or $q_2(R_{21}'$, θ_{21}', $\phi_{21}')$ are spherical coordinates of the same points with respect to the other center, O_2 or O_1, indicated by the second subscript. Normal unit vectors $\hat{n}_{p1}$, $\hat{n}_{q2}$ on S_1, S_2 are directed out of V, as shown in Fig. 1.

The electromagnetic cavity problem in V, with perfectly conducting boundaries S_1 and

* Received December 19, 1977. Partially supported by the National Research Foundation of Greece. The authors wish to thank Dr. E. Lecatsas for his assistance in the numerical evaluation of the g_{vs}.

S_2, can be formulated in terms of homogeneous surface integral equations on S_1, S_2 [2, 3, 4] for the unknown surface current densities $\bar{j}_1(p_1)$, $\bar{j}_2(p_2)$:

$$-2\pi\bar{j}_1(p_1) = \hat{n}_{p1} \times \int_{S_1} [\bar{j}_1(q_1) \times \nabla' G(p_1, q_1)]\, dS_{q1} + \hat{n}_{p1} \times \int_{S_2} [\bar{j}_2(q_2) \times \nabla' G(p_1, q_2)]\, dS_{q2}, \tag{1}$$

$$-2\pi\bar{j}_2(p_2) = \hat{n}_{p2} \times \int_{S_1} [\bar{j}_1(q_1) \times \nabla' G(p_2, q_1)]\, dS_{q1} + \hat{n}_{p2} \times \int_{S_2} [\bar{j}_2(q_2) \times \nabla' G(p_2, q_2)]\, dS_{q2}, \tag{2}$$

where

$$G(P, Q) = G(R) = \exp(ikR)/R \qquad (R = PQ) \tag{3}$$

is the free space Green's function and ∇' operates on the prime coordinates of the point q_1 (or q_2). The integrals involving $\nabla' G(p_1, q_1)$ or $\nabla' G(p_2, q_2)$ are convergent singular surface integrals [2, 4, 5].

In the following, analytical solutions of the preceding integral equations are obtained, after properly evaluating the singular integrals by a convenient limiting process. Use also is made of translational addition theorems for spherical vector wave functions [6]. In the limit of small kd and in the particular but important case of axially symmetric fields, an exact evaluation of the elements of an infinite determinant and of the determinant itself is achieved that yields the values of the resonant frequencies $\omega_{\nu s}(kd)$ to second order in kd, in exact analogy with the methods of [1].

Solution of the integral equations. The unknown surface current densities $\bar{j}_1(q_1)$, $\bar{j}_2(q_2)$, being tangential to the surfaces S_1 and S_2 of the spheres, may be conveniently expanded in terms of a complete set of complex spherical surface vector functions $\bar{B}$, $\bar{C}$ of the angles θ, ϕ [7]. The latter are directly related to the complex spherical eigenvectors $\bar{m}$, $\bar{n}$, $\bar{l}$ by the definitions:

$$\bar{m}_{mn}(r, \theta, \phi) = [n(n+1)]^{1/2} z_n(kr) \bar{C}_{nm}(\theta, \phi) = z_n(kr) \exp(im\phi) \left[\frac{im}{\sin\theta} P_n^m(\cos\theta)\, \hat{\theta} - \frac{\partial P_n^m}{\partial\theta}\, \hat{\phi} \right], \tag{4}$$

$$\bar{n}_{mn}(r, \theta, \phi) = n(n+1) \frac{z_n(kr)}{kr} \bar{P}_{mn}(\theta, \phi) + [n(n+1)]^{1/2} \frac{z_n^d(kr)}{kr} \bar{B}_{mn}(\theta, \phi)$$
$$= \frac{\exp(im\phi)}{kr} \left\{ n(n+1) z_n(kr) P_n^m(\cos\theta) \hat{r} + z_n^d(kr) \cdot \left[\frac{\partial P_n^m}{\partial\theta}\, \hat{\theta} + \frac{im}{\sin\theta} P_n^m(\cos\theta) \hat{\phi} \right] \right\}, \tag{5}$$

where $m = -n, -n+1, \cdots 0, 1, \cdots n$, $z_n(x)$ is the general spherical Bessel function and $z_n^d(x) = [x z_n(x)]'$. In this notation, similar to Cruzan's [6], $P_n^{-m}(x)$ $(m \geq 0)$ is defined by

$$P_n^{-m}(x) = (-1)^m \frac{(n-m)!}{(n+m)!} P_n^m(x) \tag{6}$$

differing by a $(-1)^m$ factor from another common definition [7, 8]. Anyone of the above complex vectors ($\bar{m}$, $\bar{n}$, $\bar{B}$, $\bar{C}$ etc.) is related to the corresponding even and odd real vectors, defined in [7, 8] for $m \geq 0$, as follows:

$$\bar{C}_{mn} = \bar{C}^{e}_{mn} + i\bar{C}^{0}_{mn}, \; \bar{C}_{-mn} = (-1)^m \frac{(n-m)!}{(n+m)!} [\bar{C}^{e}_{mn} - i\bar{C}^{2}_{mn}], \qquad (m \geq 0), \tag{7}$$

where $\bar{m}_{mn}$, $\bar{n}_{mn}$, $\bar{B}_{mn}$ etc. may be substituted for $\bar{C}_{mn}$. Relations among the surface vectors $\bar{P}$, $\bar{B}$, $\bar{C}$ and their orthogonal properties are as follows:

$$\hat{r} \times \bar{P}_{mn} = 0, \hat{r} \times \bar{B}_{mn} = -\bar{C}_{mn}, \hat{r} \times \bar{C}_{mn} = \bar{B}_{mn};$$

$$\bar{P}_{mn} \cdot \bar{B}_{m'n'} = \bar{P}_{mn} \cdot \bar{C}_{m'n'} = \bar{B}_{mn} \cdot \bar{C}_{mn} = 0; \tag{8}$$

$$\iint \bar{P}_{mn} \cdot \bar{P}_{mn}\, d\Omega = \iint \bar{B}_{mn} \cdot \bar{B}_{mn}\, d\Omega = \iint \bar{C}_{mn} \cdot \bar{C}_{mn}\, d\Omega$$

$$= \iint \bar{B}_{mn} \cdot \bar{C}_{m'n'}\, d\Omega = 0; \tag{9}$$

$$\iint \bar{P}_{mn} \cdot \bar{P}_{-m',n'}\, d\Omega = \iint \bar{B}_{mn} \cdot \bar{B}_{-m',n'}\, d\Omega = \iint \bar{C}_{mn} \cdot \bar{C}_{-m',n'}\, d\Omega$$

$$= \frac{4\pi}{2n+1} (-1)^m \delta_{mm'}\, \delta_{nn'}, \tag{10}$$

where $d\Omega = \sin\theta\, d\theta\, d\phi$ and the integration is over the ranges $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$; $n, n' = 1, 2, \cdots$; $m = -n, -n+1, \cdots 0, 1, \cdots n$; $m' = -n'+1, \cdots 0, 1, \cdots n'$; $\delta_{nn'} = 1$ for $n = n'$ and $\delta_{nn'} = 0$ for $n \neq n'$.

In expanding now $\bar{j}_1(q_1)$ and $\bar{j}_2(q_2)$ no loss of generality occurs if only one value of the index m, namely $m = M$, is considered. As in [1] this is due basically to the fact that with $O_1 O_2$ along the z axis the azimuthal angles remain the same ($\phi_1' = \phi_{12}'$, $\phi_2' = \phi_{21}'$) when referred to O_1 or O_2. Therefore:

$$\bar{j}_1(q_1) = \sum_{s=|M|}^{\infty} [s(s+1)]^{1/2} \left[\frac{A_{Ms}}{j_s^d(x_1)} \bar{B}_{Ms}(\theta_1', \phi_1') + \frac{F_{Ms}}{x_1 j_s(x_1)} \bar{C}_{Ms}(\theta_1', \phi_1')\right], \tag{11}$$

$$\bar{j}_2(q_2) = \sum_{s=|M|}^{\infty} [s(s+1)]^{1/2} \left[\frac{G_{Ms}}{j_s^d(x_2)} \bar{B}_{Ms}(\theta_2', \phi_2') + \frac{H_{Ms}}{x_2 j_s(x_2)} \bar{C}_{Ms}(\theta_2', \phi_2')\right], \tag{12}$$

where $x_1 = kR_1$, $x_2 = kR_2$. The more general case would simply involve a superposition (or further summation over $M = -s, -s+1, \cdots\cdot 0, 1, \cdots s$).

Use will also be made of the well-known expansion [7]:

$$\bar{\bar{G}}(P, Q) = \bar{\bar{I}} \frac{\exp(ikR)}{R} = ik \sum_{n=0}^{\infty} \frac{2n+1}{n(n+1)} \sum_{m=-n}^{\infty} (-n)^m [\bar{m}^{(1)}_{-m,n}(Q)\bar{m}^{(3)}_{mn}(P)$$

$$+ \bar{n}^{(1)}_{-m,n}(Q)\bar{n}^{(3)}_{mn}(P) + n(n+1)\, \bar{l}^{(1)}_{-m,n}(Q)\bar{l}^{(3)}_{mn}(P)], \qquad (r = OP > r' = OQ) \tag{13}$$

for the free space Green's dyadic in terms of the complex spherical eigenvectors. The superscript (1) is associated with $j_n(x)$, while (3) implies use of $h_n^{(1)}(x) \equiv h_n(x)$. On the basis of (7), written for the $\bar{m}$, $\bar{n}$, $\bar{l}$ vectors, it is easy to show that (13) is equivalent to the alternative formula involving the even and odd vectors [7]. In (13) $\bar{\bar{I}}$ is the identity dyadic ($\bar{A} \cdot \bar{\bar{I}} = \bar{\bar{I}} \cdot \bar{A} = \bar{A}$) and in case $r\ (= OP) < r'\ (= OQ)$ one simply interchanges the superscripts 1 and 3 of the spherical eigenvectors, or, equivalently, their arguments P and Q.

In evaluating the surface integrals of (1) and (2) it is convenient to consider initially the integral

$$\bar{L}(P) = \nabla \times \int_S \bar{j}(q)G(P, q)dS_q = \nabla \times \int_S \bar{j}(q) \cdot \bar{\bar{G}}(P, q)dS_q$$
$$= \int_S \bar{j}(q) \times \nabla' G(P, q)dS_q\,, \qquad (14)$$

where S is a spherical surface of radius a and center O, $q(a, \theta', \phi')$ is a point on S, $P(r, \theta, \phi)$ is a fixed point not on S ($r > a$ or $r < a$), $R = qP$ and $\bar{j}(q)$ is a vector tangential to S, which can be expanded in a series like (11) with the subscript 1 deleted ($x = ka$). Under these conditions it is well known [2, 5] that $\bar{L}(P)$ is an analytic vector function of $P(r, \theta, \phi)$. The last expression for $\bar{L}(P)$ can be deduced from the first and the relation $\nabla \times [\bar{j}(q)G(P, q)] = -\bar{j}(q) \times \nabla G = \bar{j}(q) \times \nabla' G$.

Substitution of (13) into (14) and term-by-term integration yields, in case $r > a$, terms of the form

$$\bar{m}_{mn}^{(3)}(P) \int_S \bar{j}(q) \cdot \bar{m}_{-m,n}^{(1)}(q)d\Omega_q\,, \qquad \text{where } dS_q = a^2 d\Omega_q\,.$$

For $r < a$ the superscripts of the eigenvectors are interchanged. Noticing that ∇ operates on r, θ, ϕ of P, that $\nabla \times \bar{l} = 0$, $\nabla \times \bar{m} = k\bar{n}$, $\nabla \times \bar{n} = k\bar{m}$ and that $\bar{j} \cdot \bar{P}_{mn} = 0$ ($\bar{j}$ being transverse and $\bar{P}$ radial), one finds

$$\bar{L}(P) = ix^2 \sum_{n=0}^{\infty} \frac{2n+1}{[n(n+1)]^{1/2}} \sum_{m=-n}^{n} (-1)^m \Big[\bar{n}_{mn}^{(3)}(P) j_n(x) \int_S \bar{j}(\theta', \phi') \cdot \bar{C}_{-m,n}(\theta', \phi')\, d\Omega_q$$
$$+ \bar{m}_{mn}^{(3)}(P) \frac{j_n^d(x)}{x} \int_S \bar{j}(\theta', \phi') \cdot \bar{B}_{-m,n}(\theta', \phi')\, d\Omega_q\Big]\,, \qquad (15)$$

where $\bar{m}_{-m,n}^{(1)}(q)$ and $\bar{n}_{-m,n}^{(1)}(q)$ have been replaced in accordance with (4), (5). Substitution of (11) (with the subscript 1 deleted) for $\bar{j}(q)$ in (15) and use of the orthogonal properties (9), (10) yield:

$$\bar{L}(P) = 4\pi i x \sum_{s=|M|}^{\infty} [F_{Ms}\bar{n}_{Ms}^{(3)}(P) + A_{Ms}\bar{m}_{Ms}^{(3)}(P)], \qquad r > a. \qquad (16)$$

For $r < a$ a similar procedure yields:

$$\bar{L}(P) = 4\pi i x \sum_{s=|M|}^{\infty} \left[F_{Ms} \frac{h_s(x)}{j_s(x)} \bar{n}_{Ms}^{(1)}(P) + A_{Ms} \frac{h_s^d(x)}{j_s^d(x)} \bar{m}_{Ms}^{(1)}(P)\right] \cdot \qquad (17)$$

On the basis of (17) and (16) it is now possible to evaluate immediately the nonsingular integrals in (1) and (2), respectively:

$$\hat{n}_{p1} \times \int_{S2} [\bar{j}_2(q_2) \times \nabla' G(p_1, q_2)]dS_{q2} = 4\pi i x_2 \hat{n}_{p1} \times \sum_{n=|M|}^{\infty} \left[H_{Mn} \frac{h_n(x_2)}{j_n(x_2)} \bar{n}_{Mn}^{(1)}(R_{12}, \theta_{12}, \phi_{12})\right.$$
$$\left. + G_{Mn} \frac{h_n^d(x_2)}{j_n^d(x_2)} \bar{m}_{Mn}^{(1)}(R_{12}, \theta_{12}, \phi_{12})\right]\,, \qquad (18)$$

where (12) was used instead of (11) and the coordinates of p_1 $(R_{12}, \theta_{12}, \phi_{12})$ with respect to center O_2 have been explicitly shown. Also:

$$\hat{n}_{p2} \times \int_{S1} [\bar{j}_1(q_1) \times \nabla' G(p_2, q_1)] dS_{q1} = 4\pi i x_1 \hat{n}_{p2} \times \sum_{n=|M|}^{\infty} [F_{Mn} \bar{n}_{Mn}^{(3)} (R_{21}, \theta_{21}, \phi_{21}) + A_{Mn} \bar{m}_{Mn}^{(3)} (R_{21}, \theta_{21}, \phi_{21})] \cdot \tag{19}$$

In order to evaluate the singular surface integrals in (1), (2) it is convenient to return to (14) and cross-multiply it by $\hat{n}_p = \hat{r}_p$, where $p(a, \theta, \phi)$ is a point on S, on the same radius as $P(R, \theta, \phi)$, with $r > a$ or $r < a$. It is then possible to make use of the classical result [2, 4, 5]:

$$\hat{r}_p \times \int_S [\bar{j}(q) \times \nabla' G(p, q)] dS_q = \lim_{P \to p} [\hat{r}_p \times \bar{L}(P)] \mp 2\pi \bar{j}(p), \tag{20}$$

in which the upper/lower sign of the last term corresponds to P approaching p from the exterior/interior of S, i.e., for $r > a/r < a$. Before the limit is taken, it is possible, as before, to use (16) (for $r > a$) or (17) (for $r < a$) to obtain

$$\hat{r}_p \times \int_S [\bar{j}(q) \times \nabla' G(p, q)] dS_q = -2\pi \bar{j}(p) + 4\pi i x \lim_{P \to p} \sum_{s=|M|}^{\infty} [F_{Ms} \hat{r}_p \times \bar{n}_{Ms}^{(3)}(P) + A_{Ms} \hat{r}_p \times \bar{m}_{Ms}^{(3)}(P)]$$

$$= -2\pi \bar{j}(p) + 4\pi i x \sum_{s=|M|}^{\infty} [s(s+1)]^{1/2} \cdot \left[-F_{Ms} \frac{h_s^d(x)}{x} \bar{C}_{Ms}(\theta, \phi) + A_{Ms} h_s(x) \bar{B}_{Ms}(\theta, \phi) \right] \tag{21}$$

or

$$\hat{r}_p \times \int_S [\bar{j}(q) \times \nabla' G(p, q)] dS_q = +2\pi \bar{j}(p) + 4\pi i x \lim_{P \to p} \sum_{s=|M|}^{\infty} \left[F_{Ms} \frac{h_s(x)}{j_s(x)} \cdot \hat{r}_p \times \bar{n}_{Ms}^{(1)}(P) + A_{Ms} \frac{h_s^d(x)}{j_s^d(x)} \hat{r}_p \times \bar{m}_{Ms}^{(1)}(P) \right]$$

$$= 2\pi \bar{j}(p) + 4\pi i x \sum_{s=|M|}^{\infty} [s(s+1)]^{1/2} \left[-F_{Ms} \frac{h_s(x) j_s^d(x)}{x j_s(x)} \cdot \bar{C}_{Ms}(\theta, \phi) + A_{Ms} \frac{h_s^d(x)}{j_s^d(x)} j_s(x) \bar{B}_{Ms}(\theta, \phi) \right], \tag{22}$$

in which (4), (5) and (8) were also used. The two results for the singular surface integral are identical, if the relations

$$\bar{j}(p) = \sum_{s=|M|}^{\infty} [s(s+1)]^{1/2} \left[\frac{A_{Ms}}{j_s^d(x)} \bar{B}_{Ms}(\theta, \phi) + \frac{F_{Ms}}{x j_s(x)} \bar{C}_{Ms}(\theta, \phi) \right],$$

$$\frac{i}{x} = j_s(x) h_s^d(x) - h_s(x) j_s^d(x)$$

are also invoked.

It is now possible to substitute back into the integral equations (1) and (2). Recalling from Fig. 1 that $\hat{n}_{p_1} = -\hat{r}_{p_1}$, $\hat{n}_{p_2} = \hat{r}_{p_2}$, it is convenient to use (22) in (1) and (21) in (2). Together with (18), (19) and (12) instead of (11), wherever $\bar{j}_2(q_2)$ is involved, one finally gets:

$$x_1 \sum_{s=|M|}^{\infty} [s(s+1)]^{1/2}\left[-F_{Ms}\frac{h_s(x_1)j_s^d(x_1)}{x_1 j_s(x_1)}\bar{C}_{Ms}(\theta_1, \phi_1) + A_{Ms}\frac{h_s^d(x_1)}{j_s^d(x_1)}j_s(x_1)\bar{B}_{Ms}(\theta_1, \phi_1)\right] + x_2\hat{r}_{p_1}$$

$$\times \sum_{n=|M|}^{\infty} \left[H_{Mn}\frac{h_n(x_2)}{j_n(x_2)}\bar{n}_{Mn}^{(1)}(R_{12}, \theta_{12}, \phi_{12}) + G_{Mn}\frac{h_n^d(x_2)}{j_n^d(x_2)}\bar{m}_{Mn}^{(1)}(R_{12}, \theta_{12}, \phi_{12})\right] = 0, \qquad (23)$$

$$x_1\hat{r}_{p_2} \times \sum_{n=|M|}^{\infty} \left[F_{Mn}\bar{n}_{Mn}^{(3)}(R_{21}, \theta_{21}, \phi_{21}) + A_{Mn}\bar{m}_{Mn}^{(3)}(R_{21}, \theta_{21}, \phi_{21})\right] + \sum_{s=|M|}^{\infty} [s(s+1)]^{1/2}$$

$$[-H_{Ms}h_s^d(x_2)\bar{C}_{Ms}(\theta_2, \phi_2) + G_{Ms}x_2h_s(x_2)\bar{B}_{Ms}(\theta_2, \phi_2)] = 0\cdot \qquad (24)$$

The second/first term in (23)/(24) contains spherical eigenvectors with respect to origin O_2/O_1. They may be expanded into sums of spherical eigenvectors with respect to O_1/O_2 (like the other term of the equation) using the well-known translation al addition theorems of Cruzan [6]:

$$\bar{m}_{Mn}^{(1)}(R_{12}, \theta_{12}, \phi_{12}) = \sum_{s=1}^{\infty}\sum_{\mu=-s}^{s} [A_{\mu s}^{Mn}\bar{m}_{\mu s}^{(1)}(R_1, \theta_1, \phi_1) + B_{\mu s}^{Mn}\bar{n}_{\mu s}^{(1)}(R_1, \theta_1, \phi_1)], \qquad (25)$$

$$\bar{n}_{Mn}^{(1)}(R_{12}, \theta_{12}, \phi_{12}) = \sum_{s=1}^{\infty}\sum_{\mu=-s}^{s} [A_{\mu s}^{Mn}\bar{n}_{\mu s}^{(1)}(R_1, \theta_1, \phi_1) + B_{\mu s}^{Mn}\bar{m}_{\mu s}^{(1)}(R_1, \theta_1, \phi_1)], \qquad (26)$$

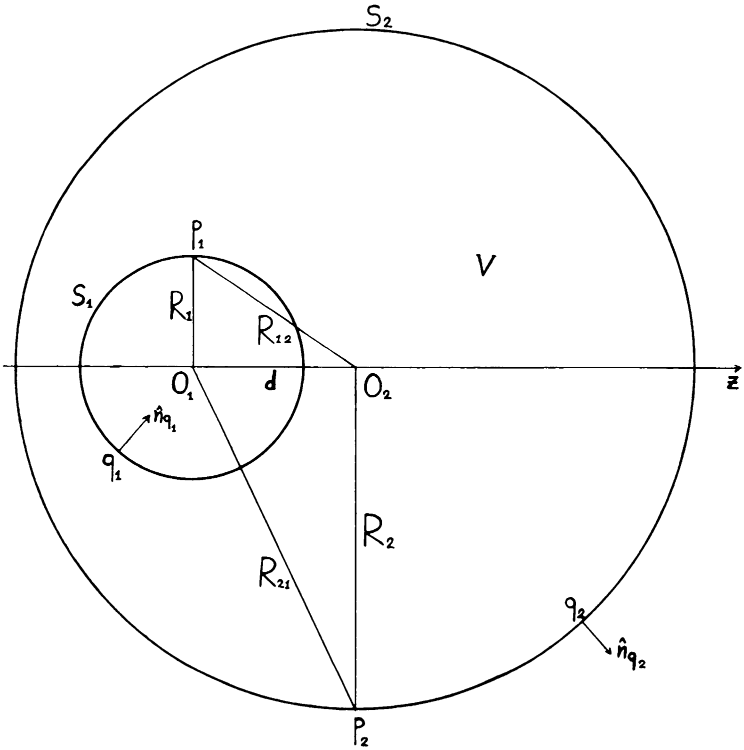

FIG. 1. The geometry of the cavity.

$$\bar{m}^{(3)}_{Mn}(R_{21}, \theta_{21}, \phi_{21}) = \sum_{s=1}^{\infty} \sum_{\mu=-s}^{s} [C^{Mn}_{\mu s}\bar{m}^{(3)}_{\mu s}(R_2, \theta_2, \phi_2)$$

$$+ D^{Mn}_{\mu s}\bar{n}^{(3)}_{\mu s}(R_2, \theta_2, \phi_2)], \qquad (R_2 > d), \tag{27}$$

$$\bar{n}^{(3)}_{Mn}(R_{21}, \theta_{21}, \phi_{21}) = \sum_{s=1}^{\infty} \sum_{\mu=-s}^{s} [C^{Mn}_{\mu s}\bar{n}^{(3)}_{\mu s}(R_2, \theta_2, \phi_2)$$

$$+ D^{Mn}_{\mu s}\bar{m}^{(3)}_{\mu s}(R_2, \theta_2, \phi_2)], \qquad (R_2 > d), \tag{28}$$

with

$$A^{Mn}_{\mu s} = (-1)^{\mu} \sum_{p} a(M, n|-\mu, s|p)a(n, s, p)j_p(kd)\, P^{M-\mu}_{p}(\cos\theta_0) \exp(i(M-\mu)\phi_0), \tag{29}$$

$$B^{Mn}_{\mu s} = (-1)^{\mu} \sum_{p} a(M, n|-\mu, s|p+1, p)b(n, s, p+1)j_{p+1}(kd)\, P^{M-\mu}_{p+1}(\cos\theta_0)$$

$$\cdot \exp(i(M-\mu)\phi_0), \tag{30}$$

$$C^{Mn}_{\mu s} = (-1)^{\mu} \sum_{p} a(M, n|-\mu, s|p)a(n, s, p)j_p(kd)\, P^{M-\mu}_{p}(\cos\theta_0') \exp(i(M-\mu)\phi_0'), \tag{31}$$

$$D^{Mn}_{\mu s} = (-1)^{\mu} \sum_{p} a(M, n|-\mu, s|p+1, p)b(n, s, p+1)j_{p+1}(kd)\, P^{M-\mu}_{p+1}(\cos\theta_0')$$

$$\cdot \exp(i(M-\mu)\phi_0'), \tag{32}$$

where d, θ_0, ϕ_0 are the coordinates of O_1 with respect to O_2, d, θ_0', ϕ_0' those of O_2 with respect to O_1 and the summation index p varies from $|n-s|$ to $n+s$ by steps of 2. Finally, the symbols $a(M, n|-\mu, s|p)$, $a(M, n|-\mu, s|p+1, p)$, $a(n, s, p)$ and $b(n, s, p)$ are defined in the Appendix, where certain particular values of them are also given. The expressions for $B^{Mn}_{\mu s}$ and $D^{Mn}_{\mu s}$, Eqs. (30) and (32), differ by a minus sign from Cruzan's values [6], as discussed in the Appendix.

With $O_1 O_2$ along the z axis one has $\theta_0 = \pi$, $\theta_0' = 0$. Therefore, $P^{M-\mu}_{p}(\cos\pi) = P^{M-\mu}_{p}(\cos 0) = 0$ for $\mu \neq M$ and $P_p(\cos\pi) = (-1)^p = (-1)^{n+s}$, $P_p(\cos 0) = 1$. As a result the expressions (29)–(32) simplify to

$$A^{Mn}_{Ms} = (-1)^{n+s}C^{Mn}_{Ms} = (-1)^{M+n+s} \sum_{p=|n-s|}^{n+s} a(M, n|-M, s|p)a(n, s, p)j_p(kd), \tag{33}$$

$$B^{Mn}_{Ms} = (-1)^{n+s+1} D^{Mn}_{Ms} = (-1)^{M+n+s+1}$$

$$\cdot \sum_{p=|n-s|}^{n+s} a(M, n|-M, s|p+1, p)b(n, s, p+1)j_{p+1}(kd), \tag{34}$$

while in (25)–(28) μ takes only the value $\mu = M$, forcing s to start from $s \geq |M|$. Substituting in (23), (24), interchanging the summations over s and n and using the relations

$$\hat{r} \times \bar{m}_{Ms} = [s(s+1)]^{1/2} z_s(kr)\bar{B}_{Ms}, \; \hat{r} \times \bar{n}_{Ms} = -[s(s+1)]^{1/2} \frac{z^d_s(kr)}{kr} \bar{C}_{Ms},$$

one finally obtains:

$$x_1 \sum_{s=|M|}^{\infty} [s(s+1)]^{1/2} \left[- F_{Ms} \frac{h_s(x_1) j_s^d(x_1)}{x_1 j_s(x_1)} \bar{C}_{Ms}(\theta_1, \phi_1) + A_{Ms} \frac{h_s^d(x_1)}{j_s^d(x_1)} j_s(x_1) \bar{B}_{Ms}(\theta_1, \phi_1) \right]$$

$$+ x_2 \sum_{s=|M|}^{\infty} [s(s+1)]^{1/2} \sum_{n=|M|}^{\infty} \left\{ H_{Mn} \frac{h_n(x_2)}{j_n(x_2)} \left[-A_{Ms}^{Mn} \frac{j_s^d(x_1)}{x_1} \bar{C}_{Ms}(\theta_1, \phi_1) \right.\right.$$

$$\left. + B_{Ms}^{Mn} j_s(x_1) \bar{B}_{Ms}(\theta_1, \phi_1) \right] + G_{Mn} \frac{h_n^d(x_2)}{j_n^d(x_2)} \left[A_{Ms}^{Mn} j_s(x_1) \bar{B}_{Ms}(\theta_1, \phi_1) \right.$$

$$\left.\left. - B_{Ms}^{Mn} \frac{j_s^d(x_1)}{x_1} \bar{C}_{Ms}(\theta_1, \phi_1) \right] \right\} = 0, \tag{35}$$

$$x_1 \sum_{s=|M|}^{\infty} [s(s+1)]^{1/2} \sum_{n=|M|}^{\infty} \left\{ F_{Mn} \left[-C_{Ms}^{Mn} \frac{h_s^d(x_2)}{x_2} \bar{C}_{Ms}(\theta_2, \phi_2) + D_{Ms}^{Mn} h_s(x_2) \bar{B}_{Ms}(\theta_2, \phi_2) \right] \right.$$

$$\left. + A_{Mn} \left[C_{Mn}^{Mn} h_s(x_2) \bar{B}_{Ms}(\theta_2, \phi_2) - D_{Ms}^{Mn} \frac{h_s^d(x_2)}{x_2} \bar{C}_{Ms}(\theta_2, \phi_2) \right] \right\} + \sum_{s=|M|}^{\infty} [s(s+1)]^{1/2}$$

$$[-H_{Ms} h_s^d(x_2) \bar{C}_{Ms}(\theta_2, \phi_2) + G_{Ms} x_2 h_s(x_2) \bar{B}_{Ms}(\theta_2, \phi_2)] = 0. \tag{36}$$

Invoking the orthogonal properties of the $\bar{B}_{Ms}$ and $\bar{C}_{Ms}$ vectors over the spherical surfaces S_1 and S_2, one finally gets four sets of homogeneous linear equations for the expansion coefficients A_{Ms}, F_{Ms}, G_{Ms}, H_{Ms} of $\bar{j}_1(p_1), \bar{j}_2(p_2)$. The last two, originating from (36), express G_{Ms}, H_{Ms} in terms of A_{Ms}, F_{Ms}. When substituted back into the first two equations, originating from (35), they produce two sets of homogeneous linear equations for the determination of A_{Ms}, F_{Ms}. The final formulas are:

$$G_{Ms} = -\frac{x_1}{x_2} \sum_{n=|M|}^{\infty} (F_{Mn} D_{Ms}^{Mn} + A_{Mn} C_{Ms}^{Mn}), \quad H_{Ms} = -\frac{x_1}{x_2} \sum_{n=|M|}^{\infty} (F_{Mn} C_{Ms}^{Mn} + A_{Mn} D_{Ms}^{Mn}), \tag{37}$$

$$A_{Ms} \frac{h_s^d(x_1)}{j_s^d(x_1)} - \sum_{\nu=|M|}^{\infty} A_{M\nu} \sum_{n=|M|}^{\infty} \left[\frac{h_n(x_2)}{j_n(x_2)} B_{Ms}^{Mn} D_{Mn}^{M\nu} + \frac{h_n^d(x_2)}{j_n^d(x_2)} A_{Ms}^{Mn} C_{Mn}^{M\nu} \right] - \sum_{\nu=|M|}^{\infty} F_{M\nu}$$

$$\cdot \sum_{n=|M|}^{\infty} \left[\frac{h_n(x_2)}{j_n(x_2)} B_{Ms}^{Mn} C_{Mn}^{M\nu} + \frac{h_n^d(x_2)}{j_n^d(x_2)} A_{Ms}^{Mn} D_{Mn}^{M\nu} \right] = 0, \tag{38}$$

$$F_{Ms} \frac{h_s(x_1)}{j_s(x_1)} - \sum_{\nu=|M|}^{\infty} A_{M\nu} \sum_{n=|M|}^{\infty} \left[\frac{h_n(x_2)}{j_n(x_2)} A_{Ms}^{Mn} D_{Mn}^{M\nu} + \frac{h_n^d(x_2)}{j_n^d(x_2)} B_{Ms}^{Mn} C_{Mn}^{M\nu} \right] - \sum_{\nu=|M|}^{\infty} F_{M\nu}$$

$$\cdot \sum_{n=|M|}^{\infty} \left[\frac{h_n(x_2)}{j_n(x_2)} A_{Ms}^{Mn} C_{Mn}^{M\nu} + \frac{h_n^d(x_2)}{j_n^d(x_2)} B_{Ms}^{Mn} D_{Mn}^{M\nu} \right] = 0 \cdot \tag{39}$$

Setting the determinant of the coefficients equal to 0 provides the equation from which the resonant values $k_{ns} = \omega_{ns}/c$ are determined. For general values of kd one can procede from here by numerical methods only. The complications and uncertainties of such an approach are discussed fully in [1].

For small values of kd, however, an analytical solution is possible. In particular, for $kd = 0$ it is obvious from (33), (34) and the fact that $j_n(0) = 0, n \neq 0, j_0(0) = 1$ that $B_{Ms}^{Mn} = D_{Ms}^{Mn} = 0$ and $A_{Ms}^{Mn} = C_{Ms}^{Mn} = \delta_{ns}$, where $\delta_{ns} = 0$ for $n \neq s$ and $\delta_{ss} = 1$. The last result follows from (25)–(28) if it is noticed that for $d = 0$ there is no translation and that the eigenvectors $\bar{m}, \bar{n}$

form a complete and orthogonal set. It follows from (37)–(39) that

$$G_{Ms} = -(x_1/x_2)A_{Ms}\,,\; H_{Ms} = -(x_1/x_2)F_{Ms}\,(s = |M|, |M| + 1, \cdots), \tag{40}$$

$$\frac{h_s^d(x_1)}{j_s^d(x_1)} - \frac{h_s^d(x_2)}{j_s^d(x_2)} = 0 \text{ (electric modes)};\; \frac{h_s(x_1)}{j_s(x_1)} - \frac{h_s(x_2)}{j_s(x_2)} = 0 \text{ (magnetic modes)}. \tag{41}$$

The resonant frequencies $\omega_{sn}(0)$ are found from the solution of Eqs. (41) ($s = |M|, |M| + 1, \cdots; n = 1, 2, \cdots$) and are independent of M (apart from the restriction $s \geq |M|$). Moreover, there is no coupling between electric and magnetic modes. All these results are well known from separation of variables.

For $kd \neq 0$, but small, it is observed from (33), (34) and the relations

$$j_n(kd) \simeq \frac{2^n n!}{(2n+1)!}(kd)^n[1 + O(k^2d^2)] \qquad \text{for } n \neq 0, \qquad j_0(kd) \simeq 1 - \frac{(kd)^2}{6} + O(k^4d^4) \tag{42}$$

that for $n \neq s$ and $n = s$

$$A_{Ms}^{Mn}\,,\, C_{Ms}^{Mn} \sim (kd)^{|n-s|}\,[1 + O(k^2d^2)];\; B_{Ms}^{Mn}\,,\, D_{Ms}^{Mn} \sim (kd)^{|n-s|+1}[1 + O(k^2d^2)]. \tag{43}$$

These results, as in [1], simplify Eqs. (37)–(39) a great deal, but not to the extent achieved in [1]. The reason is that for $M \neq 0$ the coefficients A_{Ms} remain coupled to the F_{Ms} (an $\vec{m}$ or $\vec{n}$ mode is expanded under translation into both $\vec{m}$ and $\vec{n}$ modes, unlike the scalar case, Eqs. (20) or (21), (22) in [1]). Omitting details, it is found that in the determinant $D(a_{ns})$ of the coefficients of A_{Ms}, F_{Ms} terms of order kd that must be retained are found not only along the diagonal (a_{ss}) and the ones next to it ($a_{s,s+1}$; $a_{s+1,s}$), as in [1], but, in addition, along the next two diagonals ($a_{s,s+2}$; $a_{s+2,s}$). The evaluation of $D(a_{ns})$ in such a case can still be carried out, as will be shown in a forthcoming publication on the exterior (scattering) problem. However, here these difficulties can be avoided if one restricts the examination to the case $M = 0$. It is to be remembered here that the axial symmetry of the configuration makes this the most important case, since azimuthal dependence ($M \neq 0$) may be introduced only by an initial non-axisymmetric impressed field that generates the oscillations inside the cavity, not by the configuration. For $M = 0$ the $\vec{m}$ and $\vec{n}$ modes become completely decoupled. Indeed, from (34) and (A.5) of the Appendix (and for unrestricted values of kd) one has:

$$a(0, n|0, s|p + 1, p) = 0; \qquad B_{0s}^{0n} = D_{0s}^{0n} = 0. \tag{44}$$

These values imply that in (25)–(28) the $\vec{m}/\vec{n}$ modes are expanded under translation into $\vec{m}/\vec{n}$ modes only, while (37)–(39) simplify to:

$$G_{0s} = -\frac{x_1}{x_2}\sum_{n=1}^{\infty} C_{0s}^{0n} A_{0n}\,; \qquad H_{0s} = -\frac{x_1}{x_2}\sum_{n=1}^{\infty} C_{0s}^{0n} F_{0n}\,(s = 1, 2, \cdots), \tag{45}$$

$$A_{0s}\frac{h_s^d(x_1)}{j_s^d(x_2)} - \sum_{\nu=1}^{\infty} A_{0\nu}\sum_{n=1}^{\infty}\frac{h_n^d(x_2)}{j_n^d(x_2)} A_{0s}^{0n} C_{0n}^{0\nu} = 0, \tag{46}$$

$$F_{0s}\frac{h_s(x_1)}{j_s(x_1)} - \sum_{\nu=1}^{\infty} F_{0\nu}\sum_{n=1}^{\infty}\frac{h_n(x_2)}{j_n(x_2)} A_{0s}^{0n} C_{0n}^{0\nu} = 0. \tag{47}$$

In these equations the indices s, n start from $n, s = 1$ (not 0), since $\vec{m}_{00} = \vec{n}_{00} \equiv 0$, as seen from (4), (5). Both (46) and (47) are similar to (29) or (67) in [1] and can be put into the forms

$$\sum_{\nu=1}^{\infty} a_{s\nu} A_{0\nu} = 0, \qquad \sum_{\nu=1}^{\infty} f_{s\nu} F_{0\nu} = 0; \qquad \nu, s = 1, 2, \cdots, \tag{48}$$

$$a_{ss} = \frac{h_s^d(x_1)}{j_s^d(x_1)} - \sum_{n=1}^{\infty} \frac{h_n^d(x_2)}{j_n^d(x_2)} A_{0s}^{0n} C_{0n}^{0s}, \qquad a_{s\nu} = - \sum_{n=1}^{\infty} \frac{h_n^d(x_2)}{j_n^d(x_2)} A_{0s}^{0n} C_{0n}^{0\nu} \; (s \neq \nu), \tag{49}$$

$$f_{ss} = \frac{h_s(x_1)}{j_s(x_1)} - \sum_{n=1}^{\infty} \frac{h_n(x_2)}{j_n(x_2)} A_{0s}^{0n} C_{0n}^{0s}, \qquad f_{s\nu} = - \sum_{n=1}^{\infty} \frac{h_n(x_2)}{j_n(x_2)} A_{0s}^{0n} C_{0n}^{0\nu} \; (s \neq \nu), \tag{50}$$

similar to (30)–(32) in [1]. From here on it is obviously possible to treat only the second case for the coefficients $f_{s\nu}$ (magnetic modes), and obtain results for the $a_{s\nu}$ (electric modes) by mere substitution of $z_n(x_1)$, $z_n(x_2)$ by $z_n^d(x_1)$, $z_n^d(x_2)$, respectively, z_n being either j_n or h_n.

For small kd, reference to (33), (42), (43) shows that

$$A_{0s}^{0n} C_{0n}^{0\nu} \sim (kd)^{|s-n|+|\nu-n|}[1 + O(k^2d^2)], \tag{51}$$

implying that

$$f_{ss} = C_{ss} + C''_{ss}(kd)^2 + 0(k^4d^4); \qquad f_{s\nu} = C'_{s\nu}(kd)^{|s-\nu|}[1 + O(k^2d^2)], \qquad (\nu \neq s), \tag{52}$$

in which the C's are independent of kd. From this point on the procedure follows steps identical to [1], leading to an explicit evaluation of the determinant $D(f_{s\nu})$ to order $(kd)^2$ (Eq. (41) in [1]) and to a closed-form expression for the coefficients $g_{\nu s}$ in the relation $\omega_{\nu s}(d) = \omega_{\nu s}(0)\cdot[1 + g_{\nu s}(kd)^2]$ for the resonant frequencies. The latter (Eq. (55) in [1]) is repeated here for convenience:

$$g_{\nu s} = \left[x_1 \frac{dC_{\nu\nu}(x_1)}{dx_1}\right]^{-1} \left[\frac{C'_{\nu+1,\nu}C'_{\nu,\nu+1}}{C_{\nu+1,\nu+1}} + \frac{C'_{\nu,\nu-1}C'_{\nu-1,\nu}}{C_{\nu-1,\nu-1}} - C''_{\nu\nu}\right] \quad (\nu, s = 1, 2, \cdots), \tag{53}$$

where, in differentiating $dC_{\nu\nu}(x_1, x_2)/dx_1$, one writes $x_2 = rx_1$ with $r = x_2/x_1 = R_2/R_1 =$ constant.

There remain the explicit expressions for the constants C. They are obtained from (33), (42), (51) and (52):

$$f_{nn} = C_{nn} + C''_{nn}(kd)^2 + O(k^4d^4) = \frac{h_n(x_1)}{j_n(x_1)} - \frac{h_n(x_2)}{j_n(x_2)} A_{0n}^{0n} C_{0n}^{0n} - \frac{h_{n-1}(x_2)}{j_{n-1}(x_2)} A_{0n}^{0,n-1} C_{0,n-1}^{0n}$$

$$- \frac{h_{n+1}(x_2)}{j_{n+1}(x_2)} A_{0n}^{0,n+1} C_{0,n+1}^{0n} + O(k^4d^4), \tag{54}$$

$$f_{n,n+1} = - \frac{h_n(x_2)}{j_n(x_2)} A_{0n}^{0n} C_{0n}^{0,n+1} - \frac{h_{n+1}(x_2)}{j_{n+1}(x_2)} A_{0n}^{0,n+1} C_{0,n+1}^{0,n+1} + O(k^3d^3)$$

$$= C'_{n,n+1}(kd) \quad [1 + O(k^2d^2)], \tag{55}$$

$$f_{n+1,n} = - \frac{h_n(x_2)}{j_n(x_2)} A_{0,n+1}^{0n} C_{0n}^{0n} - \frac{h_{n+1}(x_2)}{j_{n+1}(x_2)} A_{0,n+1}^{0,n+1} C_{0,n+1}^{0n} + O(k^3d^3)$$

$$= C'_{n+1,n}(kd)[1 + O(k^2d^2)]. \tag{56}$$

Using from the Appendix formulas for $a(0, n|0, s|p)a(n, s, p)$ one gets:

$$A_{0n}^{0n} C_{0n}^{0n} = [a(0, n|0, n|0)a(n, n, 0)\,(1 - k^2d^2/6)$$

$$+ a(0, n|0, n|1)a(n, n, 2)(kd)^2/15 + O(k^4d^4)]^2$$

$$= 1 - \frac{(kd)^2}{3}\left[1 + \frac{2(n^2+n-3)}{(2n+3)(2n-1)}\right] + O(k^4d^4), \tag{57}$$

$$A_{0n}^{0,n-1} C_{0,n-1}^{0n} = [-a(0, n-1|0, n|1)a(n-1, n, 1)(kd/3) + O(k^3d^3)][a(0, n|0, n-1|1)$$

$$\cdot a(n, n-1, 1)(kd/3) + O(k^3d^3)] = \frac{n^2-1}{4n^2-1}(kd)^2 + O(k^4d^4), \tag{58}$$

$$A_{0n}^{0,n+1} C_{0,n+1}^{0n} = [-a(0, n+1|0, n|1)a(n+1, n, 1)(kd/3) + O(k^3d^3)][a(0, n|0, n+1|1)$$

$$\cdot a(n, n+1, 1)(kd/3) + O(k^3d^3)] = \frac{n(n+2)}{(2n+1)(2n+3)}(kd)^2 + O(k^4d^4)\cdot \tag{59}$$

Therefore:

$$C_{nn} = \frac{h_n(x_1)}{j_n(x_1)} - \frac{h_n(x_2)}{j_n(x_2)} = i\left[\frac{n_n(x_1)}{j_n(x_1)} - \frac{n_n(x_2)}{j_n(x_2)}\right] = f_{nn}(0), \tag{60}$$

$$C''_{nn} = \frac{(2n+3)(2n-1) + 2n^2 + 2n - 6}{3(2n+3)(2n-1)}\frac{h_n(x_2)}{j_n(x_2)} - \frac{n^2-1}{4n^2-1}\frac{h_{n-1}(x_2)}{j_{n-1}(x_2)}$$

$$- \frac{n(n+2)}{(2n+1)(2n+3)}\frac{h_{n+1}(x_2)}{j_{n+1}(x_2)}$$

$$= i\left\{\left[\frac{n^2-1}{4n^2-1} + \frac{n(n+2)}{(2n+1)(2n+3)}\right]\frac{n_n(x_2)}{j_n(x_2)} - \frac{n^2-1}{4n^2-1}\frac{n_{n-1}(x_2)}{j_{n-1}(x_2)}\right. \tag{61}$$

$$\left. - \frac{n(n+2)}{(2n+1)(2n+3)}\frac{n_{n+1}(x_2)}{j_{n+1}(x_2)}\right\}.$$

Also, following similar steps:

$$C'_{n,n+1} = \frac{n+2}{2n+3}\left[\frac{h_{n+1}(x_2)}{j_{n+1}(x_2)} - \frac{h_n(x_2)}{j_n(x_2)}\right] = i\,\frac{n+2}{2n+3}\left[\frac{n_{n+1}(x_2)}{j_{n+1}(x_2)} - \frac{n_n(x_2)}{j_n(x_2)}\right], \tag{62}$$

$$C'_{n+1,n} = \frac{n}{2n+1}\left[\frac{h_{n+1}(x_2)}{j_{n+1}(x_2)} - \frac{h_n(x_2)}{j_n(x_2)}\right] = i\,\frac{n}{2n+1}\left[\frac{n_{n+1}(x_2)}{j_{n+1}(x_2)} - \frac{n_n(x_2)}{j_n(x_2)}\right]. \tag{63}$$

As in [1], one obtains the remarkable result that all the C's turn out to be imaginary. This means that all elements $f_{\nu s}$ of the determinant $D(f_{\nu s})$, at least to order $(kd)^2$, are imaginary, and therefore that the $\omega_{ns}(d)$ are real, as predicted by theory for all cavities. This is a very convincing check on all relations and results obtained in this paper.

For the electric modes one simply replaces $j_n(x)$ and $n_n(x)$ in (60)–(63) by $j_n^d(x) = [xj_n(x)]'$ and $n_n^d(x) = [xn_n(x)]'$. Comparing with the results of [1] for $M = 0$ one observes that the $\omega_{ns}(d)$ for the magnetic modes are no longer the same with those of the Dirichlet scalar case, as happens to be true for concentric spheres ($d = 0$).

Numerical results. The roots $(x_1)_{\nu s}$ ($\nu = 1, 2, \cdots; s = 1, 2, \cdots$) of (41) were obtained by a modified Newton-Raphson method, called the Regula-Falsi method. In the case of the magnetic modes these roots are the same as those of the Dirichlet scalar problem and have been checked against tabulated values, as explained in [1]. For $M = 0$ the roots $(x_1)_{\nu s}$

for the magnetic modes and the corresponding $g_{\nu s}$ from (53) are given in Tables I-IV for four values of the ratio $r = x_2/x_1 = R_2/R_1$, namely, $r = 1.2$–$1.35 - 1.5$–2.0. Similar values for the electric modes and for $M = 0$ are found in Tables V-VIII.

One general observation is the fact that the $g_{\nu s}$ are almost independent of the order s of the resonant frequency; in other words, the percentage change of $\omega_{\nu s}(d)$ from $\omega_{\nu s}(0)$ is almost the same for all s (same ν), particularly for $s \geq 2$. As far as the complications of a direct numerical evaluation of the roots of the equation $D(f_{\nu s}) = 0$ are concerned, the reader is referred to the detailed discussion of this aspect in [1].

Reference to [1] should also be made concerning the possible generalization of the present approach to exterior problems (scattering by an eccentrically coated sphere is at present under investigation), as well as to other geometrical configurations.

Recently, cylindrical geometries were considered, in particular eccentric waveguides and eccentrically coated waveguides. Our results were found in excellent agreement with those of [9], in which the cutoff frequencies of the lower TM and TE modes were obtained numerically and experimentally for eccentric, perfectly conducting waveguides. For small kd agreement to four decimals was obtained. Even for the largest kd considered in [9] the agreement extended to the first two decimals, an indication that the restriction $kd \ll 1$ of the present method is not really as severe as it may appear. This is further corroborated by the fact that the $g_{\nu s}$ get smaller quickly with increasing ratio r, as seen from the tables. Besides, when r is small and both $g_{\nu s}$ and x_1 become relatively large, kd gets necessarily small due to the physical restriction $d \leq R_2 - R_1$ or $kd \leq (r - 1)x_1$. This application to waveguides will be the subject of a forthcoming paper.

Appendix. The symbols $a(M, n| -\mu, s|p)$, $a(n, s, p)$ etc. appearing in (29)-(32) are defined as follows [6]:

$$a(m, n|\mu, s|p) = (-1)^{m+\mu}(2p + 1)\left[\frac{(n + m)!(s + \mu)!(p - m - \mu)!}{(n - m)!(s - \mu)!(p + m + \mu)!}\right]^{1/2}$$

$$\cdot\begin{bmatrix} n & s & p \\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix} n & s & p \\ m & \mu & -m-\mu\end{bmatrix}, \tag{A.1}$$

$$a(m, n|\mu, s|p + 1, q) = (-1)^{m+\mu}(2p + 3)\left[\frac{(n + m)!(s + \mu)!(p + 1 - m - \mu)!}{(n - m)!(s - \mu)!(p + 1 + m + \mu)!}\right]^{1/2}$$

$$\cdot\begin{bmatrix} n & s & q \\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix} n & s & p+1 \\ m & \mu & -m-\mu\end{bmatrix}, \tag{A.2}$$

$$a(n, s, p) = i^{p+s-n}[2s(s + 1)(2s + 1) + (s + 1)(n - s + p + 1)(n + s - p)$$
$$- s(s - n + p + 1)(n + s + p + 2)]/[2s(s + 1)], \tag{A.3}$$

$$b(n, s, q) = i^{s+q-n}[q^2 - (s - n)^2]^{1/2}[(s + n + 1)^2 - q^2]^{1/2}(2s + 1)/[2s(s + 1)], \tag{A.4}$$

where

$$\begin{bmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3\end{bmatrix}$$

is the Wigner $3 - j$ symbol. Some useful properties of the latter are: the symbol is 0 unless $m_1 + m_2 + m_3 = 0$, unless the triangular condition $|j_1 - j_2| \leq j_3 \leq j_1 + j_2$ is satisfied

Magnetic modes (M = 0)

TABLE I. $r = x_2/x_1 = 1.2$.

		s = 1	s = 2	s = 3	s = 4	s = 5	s = 6
	ν = 1	15.76063	31.44237	47.14150	62.84508	78.55033	94.2565
	2	15.86550	31.49526	47.17682	62.87154	78.57154	94.27419
$(x_1)_{\nu s}$	3	16.02151	31.57444	47.22976	62.91128	78.60335	94.3007
	4	16.22717	31.67971	47.30025	62.96423	78.64574	94.33604
	5	16.48061	31.81082	47.38822	63.03036	78.69869	94.38019
	ν = 1	−2.94460	−2.98393	−2.98868	−2.98745	−2.98846	−2.97993
	2	.7531856	.7275187	.7258693	.7285967	.7340752	.7403467
$g_{\nu s}$	3	.5301020	.5099873	.5074118	.5089340	.5115610	.5158321
	4	.3564039	.3393728	.3365035	.3368490	.3381674	.3404283
	5	.2524817	.2378851	.2350669	.2348713	.2360829	.2375913

TABLE II. $r = x_2/x_1 = 1.35$.

		s = 1	s = 2	s = 3	s = 4	s = 5	s = 6
	ν = 1	9.057011	17.99305	26.95539	35.92452	44.89639	53.86962
	2	9.216926	18.07494	27.01022	35.96571	44.92936	53.89711
$(x_1)_{\nu s}$	3	9.451597	18.19702	27.09224	36.02739	44.97876	53.93831
	4	9.755591	18.35861	27.20123	36.10947	45.04456	53.99319
	5	10.12253	18.55865	27.33688	36.21183	45.12667	54.06172
		s = 1	s = 2	s = 3	s = 4	s = 5	s = 6
	ν = 1	−1.04854	−1.08817	−1.09551	−1.09844	−1.09953	−1.10042
	2	.2976899	.2714140	.2665779	.2650669	.2642967	.2632253
$g_{\nu s}$	3	.2090797	.1917095	.1874811	.1858838	.1851030	.1847804
	4	.1406013	.1285457	.1249803	.1235807	.1229980	.1225309
	5	.0987714	.0909096	.0878274	.0865854	.0860460	.0856145

TABLE III. $r = x_2/x_1 = 1.5$.

		s = 1	s = 2	s = 3	s = 4	s = 5	s = 6
	ν = 1	6.38580	12.61895	18.88476	25.15918	31.43708	37.71673
	2	6.58613	12.72351	18.95504	25.21203	31.47942	37.75204
$(x_1)_{\nu s}$	3	6.87535	12.87882	19.05999	25.29113	31.54283	37.80495
	4	7.24246	13.08311	19.19908	25.39622	31.62720	37.87539
	5	7.67580	13.33421	19.37160	25.52701	31.73236	37.96326
	ν = 1	−.548217	−.585778	−.593381	−.596055	−.597151	−.597566
	2	.1751128	.1529077	.1477128	.1460435	.1453082	.1451701
$g_{\nu s}$	3	.1207802	.1077643	.1038657	.1023786	.1018547	.1015902
	4	.0795457	.0726922	.0695374	.0682722	.0676965	.0674194
	5	.0541408	.0516278	.0491016	.0479806	.0474355	.0471801

TABLE IV. $r = x_2/x_1 = 2.0$.

		s = 1	s = 2	s = 3	s = 4	s = 5	s = 6
	$\nu = 1$	3.28600	6.36067	9.47718	12.60587	15.73963	18.87597
	2	3.55578	6.51306	9.58126	12.68459	15.80286	18.92877
$(x_1)_{\nu s}$	3	3.92252	6.73556	9.73553	12.80189	15.89729	19.00774
	4	4.35839	7.02183	9.93792	12.95687	16.02247	19.11260
	5	4.84099	7.36467	10.18591	13.14838	16.17778	19.24300
	$\nu = 1$	−.154770	−.185607	−.193237	−.196091	−.197428	−.198144
	2	.0701890	.0573415	.0524788	.0505131	.0495400	.0490610
$g_{\nu s}$	3	.0429004	.0402039	.0371142	.0356451	.0348920	.0344651
	4	.0247884	.0270546	.0252153	.0240656	.0234315	.0230618
	5	.0144915	.0188940	.0180254	.0171499	.0166155	.0162764

Electric Modes (M = 0)

TABLE V. $r = x_2/x_1 = 1.2$.

		s = 1	s = 2	s = 3	s = 4	s = 5	s = 6
	$\nu = 1$	15.76117	31.44230	47.14152	62.84504	78.55033	94.25651
	2	15.86712	31.49547	47.17688	62.87157	78.57155	94.27419
$(x_1)_{\nu s}$	3	16.02478	31.57485	47.22988	62.91133	78.60337	94.30071
	4	16.23269	31.68040	47.30045	62.96431	78.64578	94.33606
	5	16.48902	31.81185	47.38852	63.03048	78.69876	94.38023
		s = 1	s = 2	s = 3	s = 4	s = 5	s = 6
	$\nu = 1$	−2.91148	−2.97116	−2.97599	−2.96962	−2.95778	−2.94446
	2	.7471076	.7255597	.7258737	.7286365	.7364067	.7448191
$g_{\nu s}$	3	.5260365	.5085578	.5066550	.5082459	.5113348	.5153637
	4	.3540509	.3384358	.3363047	.3370900	.3385343	.3411405
	5	.2511180	.2373111	.2348577	.2349251	.2356447	.2372174

TABLE VI. $r = x_2/x_1 = 1.35$.

		s = 1	s = 2	s = 3	s = 4	s = 5	s = 6
	$\nu = 1$	9.059273	17.99332	26.95547	35.92456	44.89641	53.86963
	2	9.223809	18.07577	27.01047	35.96581	44.92941	53.89714
$(x_1)_{\nu s}$	3	9.465746	18.19879	27.09276	36.02761	44.97888	53.93837
	4	9.779913	18.36159	27.20211	36.10984	45.04475	53.99330
	5	10.16033	18.56318	27.33821	36.21238	45.12695	54.06189
	$\nu = 1$	−1.01467	−1.07988	−1.09213	−1.09625	−1.09828	−1.09950
	2	.2935301	.2707237	.2663895	.2646275	.2638187	.2637762
$g_{\nu s}$	3	.2067111	.1905825	.1868240	.1856129	.1849674	.1844813
	4	.1398599	.1278972	.1246793	.1233498	.1228392	.1226116
	5	.0989798	.0905222	.0876058	.0864837	.0858591	.0855934

TABLE VII. $r = x_2/x_1 = 1.5$.

		s = 1	s = 2	s = 3	s = 4	s = 5	s = 6
	$\nu = 1$	6.39111	12.69165	18.88497	25.15926	31.43712	37.71676
	2	6.60254	12.72563	18.95567	25.21230	31.47956	37.75212
$(x_1)_{\nu s}$	3	6.90955	12.88312	19.06126	25.29166	31.54311	37.80511
	4	7.30225	13.09045	19.20122	25.39712	31.62766	37.87565
	5	7.77013	13.34553	19.37485	25.52837	31.73305	37.96366
	$\nu = 1$	−.511572	−.577275	−.589400	−.593255	−.594819	−.595524
	2	.1730606	.1514145	.1469667	.1455730	.1450011	.1450366
$g_{\nu s}$	3	.1203757	.1068492	.1033312	.1020863	.1015466	.1013812
	4	.0805281	.0722090	.0692174	.0680568	.0675526	.0673707
	5	.0557589	.0514180	.0489064	.0478255	.0473364	.0470939

TABLE VIII. $r = x_2/x_1 = 2.0$.

		s = 1	s = 2	s = 3	s = 4	s = 5	s = 6
	$\nu = 1$	3.30922	6.36402	9.47820	12.60631	15.73986	18.87610
	2	3.63013	6.52362	9.58440	12.68592	15.80353	18.92917
$(x_1)_{\nu s}$	3	4.08033	6.75823	9.74207	12.80460	15.89867	19.00853
	4	2.95404	7.06326	9.94941	12.96154	16.02481	19.11394
	5	3.53114	7.43401	10.2043	13.15567	16.18139	19.24505
	$\nu = 1$	−.109733	−.175373	−.188952	−.193740	−.195908	−.196986
	2	.0739110	.0569686	.0519261	.0500943	.0492965	.0488182
$g_{\nu s}$	3	.0441847	.0405678	.0368313	.0353987	.0346946	.0343293
	4	−.013230	.0281633	.0251838	.0239423	.0233305	.0229799
	5	−.009227	.0205987	.0182036	.0171116	.0165553	.0162232

cyclically by all indices j_1, j_2, j_3 and unless $|m_n| \leq j_n$. Moreover,

$$\begin{bmatrix} j_1 & j_2 & j_3 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

unless $j_1 + j_2 + j_3 =$ even.

It is this last property that makes

$$a(0, n|0, s|p + 1, p) = 0, \tag{A.5}$$

since it contains, according to (A.2), the factors

$$\begin{bmatrix} n & s & p \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} n & s & p+1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The symbols $a(m, n|\mu, s|p)$ appear also in the expansion

$$P_n^m P_s^\mu = \sum_p a(m, n|\mu, s|p)P_p^{m+\mu} \qquad (p = n + s, n + s - 2, \cdots |n - s|). \tag{A.6}$$

In particular, for $m = \mu = 0$:

$$a(0, n|0, s|p) = (2p+1)\frac{(n+s-p)!(n+p-s)!(s+p-n)!}{(n+s+p+1)!} \cdot \left\{\frac{[(n+s+p)/2]!}{[(n+s-p)/2]![(n+p-s)/2]![(s+p-n)/2]!}\right\}^2, \quad (A.7)$$

an equation that has been used extensively in the derivation of (57)–(59).

Finally, an independent check of the minus sign correction in Cruzan's formulas for $B_{\mu s}^{mn}$, $D_{\mu s}^{mn}$ (Eqs. (30) and (32) in this paper) can be provided by evaluating the rectangular components of the $\bar{m}_{mn}$ vector and comparing them with those of the $\bar{m}_{e/omn}$ given in [7]. Indeed, the rectangular components of $\bar{m}_{mn}$ are easily obtained if one starts by applying (25) for $R_1 = 0$ and $R_{12} = d$. Then:

$$\bar{m}_{mn}^{(1)}(d, \theta_0, \phi_0) = \sum_{\mu=-1,0,1} B_{\mu 1}^{mn}(d, \theta_0, \phi_0)\bar{n}_{\mu 1}^{(1)}(0), \quad (A.8)$$

a result based on the fact that

$$\begin{gathered} \bar{n}_{-1,1}^{(1)}(0) = -(\hat{x} - i\hat{y})/3; \quad \bar{n}_{01}^{(1)}(0) = 2\hat{z}/3; \quad \bar{n}_{11}^{(1)}(0) = 2(\hat{x} + i\hat{y})/3 \\ \bar{m}_{\mu s}^{(1)}(0) = 0 \text{ (all } \mu, s); \quad \bar{n}_{\mu s}^{(1)}(0) = 0 \text{ (unless } s = 1, \mu = -1, 0, 1). \end{gathered} \quad (A.9)$$

These results follow easily from (4), (5) and the well known relations among spherical ($\hat{r}$, $\hat{\theta}$, $\hat{\phi}$) and rectangular ($\hat{x}$, $\hat{y}$, $\hat{z}$) unit vectors. Substitution of (A.9) into (A.8) yields the rectangular components of $\bar{m}_{mn}^{(1)}$ in terms of the symbol $B_{\mu 1}^{mn}$ ($\mu = -1, 0, 1$). This, as explained above, verifies the necessity of the sign correction in (30) and (32).

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