NON-MONOTONIC SOLUTIONS OF THE FALKNER-SKAN BOUNDARY LAYER EQUATION*

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Abstract. We investigate the equation $f''' + ff'' + \beta(1 - f'^2) = 0$ together with boundary conditions $f(0) = f'(0) = 0$ and $f'(\infty) = 1$. Here β is negative. Previous results are summarized which describe solutions which satisfy $|f'| < 1$ for all $\eta \geq 0$. It is shown that there is a sequence $\{\beta_j\}_{j \in \mathbb{N}}$ of decreasing, negative values of β, and a corresponding sequence $\{f_j\}_{j \in \mathbb{N}}$ of solutions such that for each $j \in \mathbb{N}$ the equation $f_j' - 1 = 0$ has exactly j positive solutions and for some $\mu_j > 0$, $f_j' = 1 + o(\exp(-\mu_j \eta))$ as $\eta \to \infty$.

1. Introduction. For a fluid of constant density ρ and kinematic viscosity ν, a model for the first approximation to two-dimensional laminar flow in a boundary layer consists of the system

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \mathcal{U}(x) \frac{d\mathcal{U}}{dx} + \nu \frac{\partial^2 u}{\partial y^2} \tag{1}$$

$$u_x + v_y = 0. \tag{2}$$

Here x denotes distance measured along the bounding solid surface from some origin, y is distance normal to the surface, and u and v represent the components of the fluid velocity in the directions of increasing x and y, respectively. $\mathcal{U}(x)$ is the free stream velocity taken as a function of the single variable x. From Eq. (2) it follows that a stream function ψ satisfies

$$u = \partial\psi/\partial y, \qquad v = -\partial\psi/\partial x. \tag{3}$$

The boundary conditions at an impermeable wall with 'no slip' are

$$\psi = \partial\psi/\partial y = 0 \quad \text{at} \quad y = 0. \tag{4}$$

The boundary condition 'at infinity' is given by

$$\mathcal{Y}^N\left(\frac{u}{\mathcal{U}} - 1\right) \to 0 \quad \text{as} \quad y \to \infty \tag{5}$$

for each real N, where $\mathcal{Y} = y/\nu^{1/2}$. Note that the coordinate y does not tend to zero in the boundary layer. Rather the stretched coordinate $\mathcal{Y}$ approaches infinity as $\nu \to 0$, for nonzero y. Further discussion of condition (5) is given by Goldstein [2, pp. 34–35].

A class of solutions called similarity solutions are useful in obtaining information

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about the behavior of the fluid in the boundary layer. For convenience we let $\mathcal{U}(x) = cx^m$ ($c > 0$) and assume that ψ has the form

$$\psi = \nu^{1/2}\,\mathcal{U}(x)\,F(s)/(cs^{m-1})^{1/2}, \qquad s = y(\mathcal{U}/\nu x)^{1/2}. \tag{6}$$

The equation for F is the well-known equation derived by Falkner and Skan [1], namely

$$F''' + \frac{1}{2}(m+1)FF'' - mF'^2 + m = 0 \tag{7}$$

with boundary conditions

$$F(0) = F'(0) = 0, \tag{8}$$

$$s^N(F' - 1) \to 0 \quad \text{as} \quad s \to \infty \tag{9}$$

for each real N.

Goldstein [2] assumes that $m > -1$, sets

$$\eta = \left(\frac{1}{2}(m+1)\right)^{1/2} s, \qquad f(\eta) = \left(\frac{1}{2}(m+1)\right)^{1/2} F(s), \tag{10}$$

and transforms the problem (7)–(9) into

$$f''' + ff'' + \beta(1 - f'^2) = 0, \tag{11}$$

$$f(0) = f'(0) = 0, \tag{12}$$

$$\eta^N(f' - 1) \to 0 \quad \text{as} \quad \eta \to \infty, \qquad \text{for each real } N, \tag{13}$$

where

$$\beta = 2m/(m+1). \tag{14}$$

The problem (11)–(13) has received considerable attention in the literature and a fairly complete summary of results is given by Hartman [3, pp. 519–537]. All of these results describe the behavior of solutions of (11)–(13) which satisfy the additional restriction

$$-1 < f' < 1 \quad \text{for all} \quad \eta \geq 0. \tag{15}$$

In this paper we remove the restriction (15) and investigate the behavior of solutions of (7)–(9) which exhibit overshoot, i.e. $f' > 1$ for some $\eta > 0$.

In the next section we present a statement of our main results followed by a short discussion. Sec. 3 contains the proof of our theorem.

2. Statement of main results. In order to state our theorem properly we first need to summarize some known results about the related boundary-value problem

$$f''' + ff'' + \beta(1 - f'^2) = 0, \tag{16}$$

$$f(0) = C, \qquad f'(0) = 0, \qquad f'(\infty) = 1, \tag{17}$$

$$0 < f' < 1 \quad \text{on} \quad (0, \infty). \tag{18}$$

We assume throughout that $\beta < 0$. Hastings [6, p. 329] combines results obtained in [3], [5] and [7] into

THEOREM A. There is a decreasing function $C^*(\beta)$ defined on $(-\infty, 0)$ such that (16)–(17)–(18) has a solution if and only if $C \geq C^*(\beta)$. For sufficiently small $|\beta|$, $C^*(\beta) < 0$ but $C^*(\beta) \to \infty$ as $\beta \to -\infty$.

Furthermore, if $C > C^*(\beta)$ then the solution of (16)–(17)–(18) is not unique. In fact, for each $\beta < 0$ there is a non-negative function $\gamma(\cdot) = \gamma_\beta(\cdot)$ such that a solution f of (16) satisfies (17) and (18) if and only if $C \geq C^*(\beta)$ and $0 \leq f''(0) \leq \gamma(C)$. The function γ is continuous and strictly increasing on $(C^*(\beta), \infty)$, and $\gamma(C^*(\beta)) = 0$.

Hastings [6] further notes that if $f''(0) = \gamma(C)$ then

$$1 - f'(\eta) \sim d_0\eta^{-1-2\beta} \exp\left(-\frac{\eta^2}{2} - d_1\eta\right) \tag{19}$$

for some $d_1 \in R$ and $d_0 > 0$. He then proves

THEOREM B (Hastings [6]). Suppose $|\beta|$ is small enough to insure that $C^*(\beta) < 0$. Then (16)–(17) has a family of solutions f such than

$$-1 < f' < 1 \quad \text{on} \quad (0, \infty). \tag{20}$$

In fact, there is a $\delta = \delta_\beta < 0$ such that if $f(0) = f'(0) = 0$ then $f'(\infty) = 1$, and (20) holds if and only if $\delta \leq f''(0) \leq \gamma_\beta(0)$. Also, if $\delta < f''(0) < \gamma_\beta(0)$ then

$$1 - f'(\eta) \sim C_0\eta^{2\beta}$$

for some $C_0 > 0$ while if $f''(0) = \delta$ or $f''(0) = \gamma_\beta(0)$ then (19) holds. Thus there is a unique solution f_* of (16)–(17) such that $f_*''(0) < 0$ and $f_*' \to 1$ exponentially.

We are now prepared to state our main result. From Theorem B it follows that there are values $\tilde{\alpha} < 0$ and $\tilde{\beta} < 0$ such that if $\beta = \tilde{\beta}$ and $f''(0) = \tilde{\alpha}$ then the solution of the initial-value problem (11)–(13) satisfies the conditions (19) and (20). In addition, there exist values $\hat{\alpha} > 0$ and $\hat{\beta} < 0$ for which that same conclusion holds. In our theorem we let $\alpha = \tilde{\alpha}$ for simplicity although the same result will be true if $\alpha = \hat{\alpha}$.

THEOREM. There is a decreasing sequence $\{\beta_j\}_{j\in\mathbb{N}}$ of negative numbers such that for each $j \in \mathbb{N}$, if $\beta = \beta_j$ then the solution of the problem

$$f''' + ff'' + \beta(1 - f'^2) = 0, \tag{21}$$

$$f(0) = f'(0) = 0, \qquad f''(0) = \tilde{\alpha} \tag{22}$$

satisfies the boundary condition

$$1 - f' \sim \delta_j\eta^{-1-2\beta} \exp\left(-\frac{\eta^2}{2} - \rho_j\eta\right) \tag{23}$$

as $\eta \to \infty$, for some δ_j and ρ_j.

Furthermore, for each $j \in \mathbb{N}$ there are exactly j positive, distinct values of η for which $f^1 - 1 = 0$.

Discussion of results. In Fig. 1 we sketch the graphs of f' vs. η for β_1, β_2, β_3. A slight controversy has developed over the significance of solutions which exhibit overshoot, i.e. $f' > 1$ for some $\eta > 0$. Since $f' = u/\mathcal{U}$ it follows that $u > \mathcal{U}$ whenever $f' > 1$. One might argue that the maximum principle predicts that u may never exceed its value on the boundary of the boundary layer, and thus $u \leq \mathcal{U}$. However, the maximum principle would require each function in the equation to be bounded. This condition is violated since we are assuming that $\mathcal{U}(x) = cx^m$ where $c > 0$ and $m < 0$, and this function becomes unbounded as $x \to 0^+$.

The requirement that $f' < 1$ for all $\eta \geq 0$ appears to stem from a remark by Stewartson [9] who claims that solutions which exhibit overshoot must be rejected on physical grounds. He fails to discuss what these physical grounds are. As noted by Libby [8], this remark by Stewartson has been carried forth in the literature and solutions with overshoot

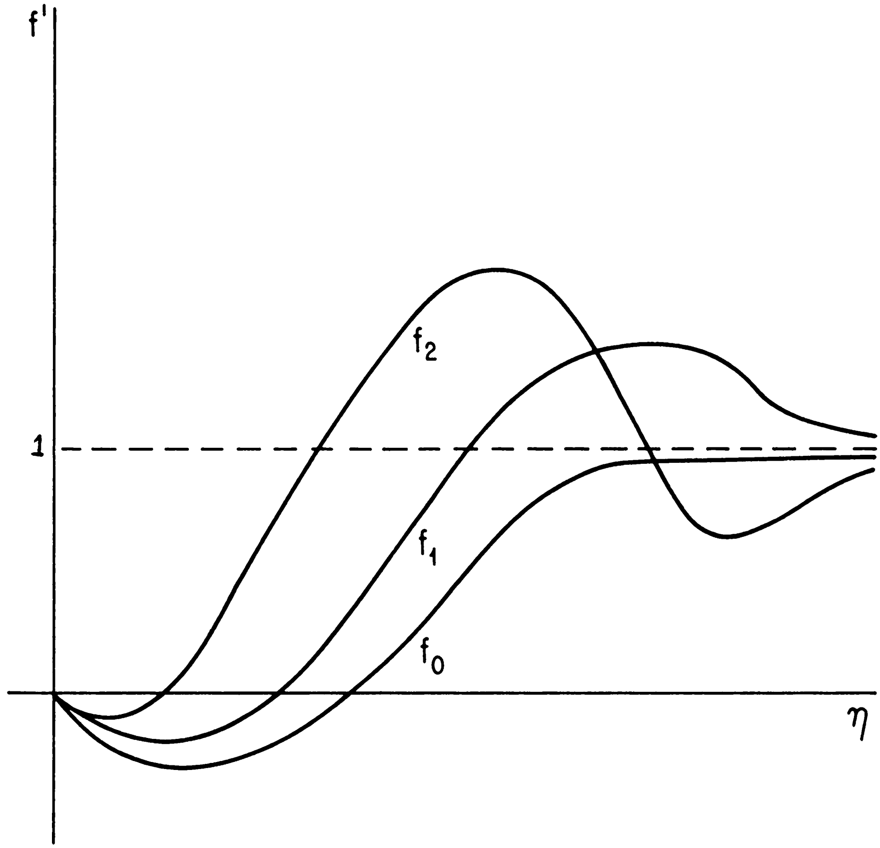

FIG. 1. The graphs of f' vs. η are given for the values β_0, β_1 and β_2.

appear to have been "banned in perpetuity." Libby [8] then suggests a possible mechanism responsible for overshoot.

On a purely mathematical basis the solutions described in our theorem should be of value since their description contributes to a more complete knowledge of the model of Falkner and Skan.

Outline of proof. We prove our theorem in several parts. First we consider the original problem posed by Falkner and Skan and we show that at $m = -1$ the steady-state solution $(1, 0)$ in the (F', F'') plane is surrounded by a connected family of closed orbits. Thus since $\beta = 2m/(m + 1)$ it follows from continuity that as $m \to -1^+$ the solution f of (11)–(12) spirals around $(1, 0)$ in the (f', f'') plane an arbitrarily large number of times.

Using several technical lemmas and a shooting method we construct a sequence $\{\beta_j\}_{j \in \mathbb{N}}$ of negative numbers such that for each $j \in \mathbb{N}$, if $\beta = \beta_j$ and f_j is the solution of (11)–(12) then $f_j' - 1 = 0$ has exactly j positive solutions. Furthermore, if $\beta < \beta_j$ then $f' - 1 = 0$ has more than j positive solutions.

A comparison technique shows that $f_i' \to 1$ exponentially as $\eta \to \infty$ for each $i \in \mathbb{N}$.

3. Proof of theorem. We first investigate (7)–(8), the problem originally posed by Falkner and Skan. For the special case $m = -1$, (7)–(8) becomes

$$F''' + F'^2 - 1 = 0, \qquad F(0) = F'(0) = 0. \tag{24}$$

Integrating, we obtain

$$\frac{(F'')^2}{2} + \frac{(F')^3}{3} - F' = C$$

where C is an arbitrary constant. It then follows that

$$F'' = (2(C + F' - F'^3/3))^{1/2}. \tag{25}$$

An elementary analysis shows that if $-2/3 < C < 2/3$ then Eq. (25) represents the trajectory of a periodic solution whose orbit surrounds the equilibrium point (1, 0) in the F'' vs. F' phase plane (see Fig. 2). When $C = 2/3$ we obtain a homoclinic orbit with $\lim_{s\to\pm\infty}(F', F'') = (-1, 0)$.

From the analysis presented above we are led to

LEMMA 1. Let $j \in \mathbb{N}$ and consider the initial-value problem (21)–(22). If $-\beta > 0$ is sufficiently large then the equation $f' - 1 = 0$ has at least j solutions, each non-degenerate, on the interval $0 < \eta < \infty$.

Proof. From our preceding analysis we note that there are values $d_1 < 0$, $d_2 > 0$ such that if $F'(0) = 0$ and $d_1 < F''(0) < d_2$ then the resultant solution represents a closed orbit surrounding the steady-state solution (1, 0) in the F' vs. F'' plane. Thus F' and F'' are periodic. For a given $j \in \mathbb{N}$ it then follows from continuity of solutions with respect to

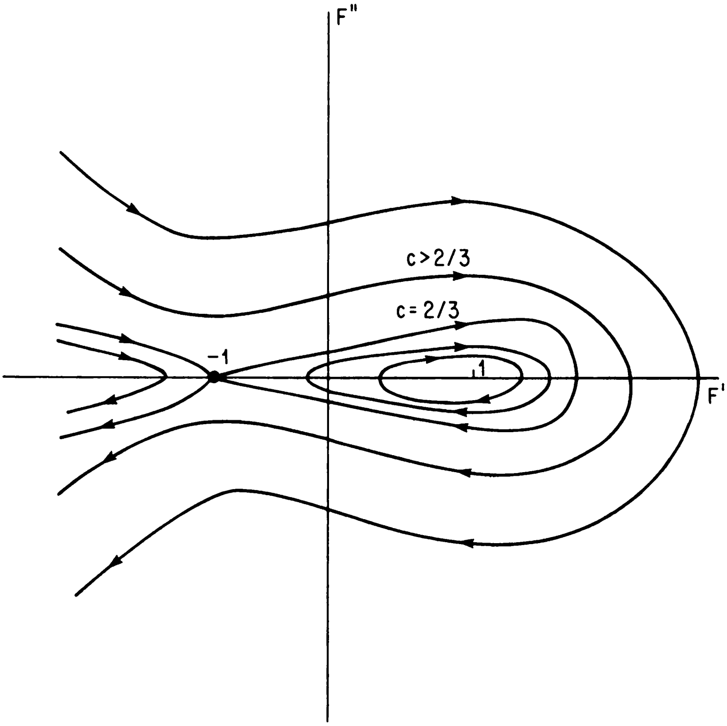

FIG. 2. For the special case $m = -1$ the original Falkner-Skan equations may be integrated once. In the resulting F' vs. F'' phase plane there is a homoclinic orbit leading to and from the equilibrium point (−1, 0), and surrounding the steady state solution (1, 0). Within the homoclinic orbit lies a family of connected closed orbits surrounding the steady-state solution (1, 0). These closed orbits represent solutions of Eq. (24) which are periodic in F' and F''.

parameters that if $F(0) = 0$, $F'(0) = 0$, $d_1/2 \le F''(0) \le d_2/2$ and $|m + 1| > 0$ is sufficiently small then there are at least j positive values of s for which $F' - 1 = 0$. The lemma follows from these observations and the transformation given in (10) and (14).

Next we need a few technical lemmas necessary for the completion of our theorem. Following Hartman [3, p. 523], we introduce new variables. Along a solution $f(\eta)$ for which $f'(\eta) > 0$ (or $f'(\eta) < 0$) let $z(f) = f'^2$. Then $\dot{z} = 2f''$ and $z^{1/2}\ddot{z} = 2f'''$ where $\cdot = d/df$ and $' = d/d\eta$. Eq. (21) becomes

$$z^{1/2}\ddot{z} + f\dot{z} + 2\beta(1 - z) = 0. \tag{26}$$

If $0 \le z \le 1$ let $w = 1 - z$ and $r = \dot{w}/w$. Then w and r satisfy the equations

$$(1 - w)^{1/2}\ddot{w} + f\dot{w} - 2\beta w = 0 \tag{27}$$

and

$$\dot{r} = -r^2 + (2\beta - fr)/(1 - w)^{1/2}. \tag{28}$$

If $z \le 1$ we set $w = z - 1$ and $r = \dot{w}/w$. Then w and r satisfy the equations

$$(1 + w)^{1/2}\ddot{w} + f\dot{w} - 2\beta w = 0 \tag{27'}$$

and

$$\dot{r} = -r^2 + 2\beta - fr/(1 + w)^{1/2}. \tag{28'}$$

A Ricatti equation related to (28) and (28′) is

$$\dot{s} = -s^2 + (2\beta - fs)/\alpha \tag{29}$$

where α is a constant. Set $s = \dot{v}/v$ and transform Eq. (29) into

$$\ddot{v} + \frac{f}{\alpha}\dot{v} - \frac{2\beta}{\alpha}v = 0. \tag{30}$$

Let $t = f^2/2$. Then Eq. (30) becomes

$$\frac{d^2v}{dt^2} + \left(\frac{1}{2t} + \frac{1}{\alpha}\right)\frac{dv}{dt} - \frac{\beta}{\alpha t}v = 0. \tag{31}$$

An application of Theorem 17.4 of Hartman [3, p. 317] to Eq. (31) and subsequently to Eq. (30) leads to

LEMMA 2. Let $\alpha > 0$ and $\beta < 0$ be given. Then Eq. (30) has a solution $\bar{v}(f) \ne 0$ for all large f such that $\dot{\bar{v}}/\bar{v} \sim -f/\alpha$ as $f \to \infty$.

LEMMA 3. Let $\bar{\beta} < 0$, $M > 1$ and $\rho \in (0, 1)$ be given. There are values $\bar{f} > 0$, $\bar{\delta} > 0$ such that if $|\beta - \bar{\beta}| < \bar{\delta}$ then

(i) if $\rho < z < 1$ and $-1/M < -\dot{z}/(1 - z) \le 0$ for some $f_1 > \bar{f}$ then $\dot{z} > 0$ and $\rho < z < 1$ for all $f > f_1$, *or*

(ii) if $1 < z < M$ and $-1/M < \dot{z}/(z - 1) \le 0$ for some $f_1 > \bar{f}$ then $\dot{z} < 0$ and $1 < z < M$ for all $f > f_1$.

Proof. Let $\alpha = M$, $\beta = \bar{\beta}$ in Lemma 2. Then Eq. (30) has a solution $\bar{v}(f) \ne 0$ for all large f such that $\dot{\bar{v}}/v \sim -f/M$ as $f \to +\infty$. Choose $\bar{f} > 0$ such that

$$\bar{v} \ne 0 \quad \text{and} \quad \dot{\bar{v}}/\bar{v} < -1/M \tag{32}$$

for all $f > \bar{f}$.

To prove part (i) we assume that there is a value $f_1 > \bar{f}$ for which $0 < w(f_1) < 1 - \rho$ and $-1/M < \dot{w}(f_1)/w(f_1) \leq 0$. If $\dot{w}(f_1) = 0$ then $\ddot{w}(f_1) < 0$. Thus $\dot{w} < 0$ is an interval to the right of f_1. If $\dot{w}(\hat{f}) = 0$ for some first $\hat{f} > f_1$ before $w = 0$ then $w(\hat{f}) > 0$ and $\ddot{w}(\hat{f}) \leq 0$. However, from Eq. (27) it follows that $\ddot{w}(\hat{f}) = 2\beta\, w(\hat{f})/(1 - w(\hat{f}))^{1/2} < 0$, a contradiction. Therefore $\dot{w} < 0$ for $f > f_1$ as long as $w > 0$. Next, from (32) we note that

$$r(f_1) = \dot{w}(f_1)/w(f_1) > \dot{v}(f_1)/\bar{v}(f_1) = s(f_1). \tag{33}$$

Since Eq. (30) is linear we may assume that $\bar{v}(f_1) = w(f_1)$ and therefore $\dot{\bar{v}}(f_1) < \dot{w}(f_1) < 0$. If $w = 0$ for some $f > f_1$ then there must be a first $\hat{f} > f_1$ for which $w = v$ and thus $\dot{w}(\hat{f}) \leq \dot{\bar{v}}(\hat{f})$, and $r(\hat{f}) \leq s(\hat{f})$. This, together with (33), implies that there is a first $f_2 > f_1$ for which $r(f_2) = s(f_2)$; hence

$$r'(f_2) - s'(f_2) \leq 0\cdot \tag{34}$$

Note that $0 < w(f_2) < 1 - \rho$ and $\dot{w}(f_2) < 0$. From Eqs. (28) and (29) we obtain

$$r'(f_2) - s'(f_2) = 2\left(\frac{\beta}{(1 - w(f_2))^{1/2}} - \frac{\bar{\beta}}{M}\right) + f_2 s(f_2)\left(\frac{1}{M} - \frac{1}{(1 - w(f_2))^{1/2}}\right).$$

Since $f_2 > \bar{f}$, $M > 1$, $s(f_2) \leq -1/M$ and $0 < w(f_2) < 1 - \rho$ it follows that

$$r'(f_2) - s'(f_2) > 2\left(\frac{\beta}{\rho^{1/2}} - \frac{\bar{\beta}}{M}\right) - \frac{\bar{f}}{M}\left(\frac{1}{M} - 1\right) > 0$$

for $\bar{f} > 0$ sufficiently large, contradicting (34). We conclude, therefore, that $\dot{w} < 0$ and $0 < w < 1 - \rho$ for all $f > f_1$. The proof of part (ii) is similar to that of part (i) and is omitted.

LEMMA 4. Let $\beta < 0$ be given. If $f'(\hat{\hat{\eta}}) = -1$ and $f''(\hat{\hat{\eta}}) < 0$ for some $\hat{\hat{\eta}} > 0$ then $f' < -1$ for all $\eta > \hat{\hat{\eta}}$.

Proof. Suppose that there is a first $\eta_0 > \hat{\hat{\eta}}$ for which $f''(\eta_0) = 0$. Then $f'(\eta_0) < -1$ and

$$f'''(\eta_0) \geq 0. \tag{35}$$

However, from Eq. (11) it follows that $f'''(\eta_0) = -\beta(1 - f'^2(\eta_0)) < 0$, contradicting (35).

LEMMA 5. Let $\beta < 0$ be given. If $f'(\hat{\eta}) = 1$ or $f'(\hat{\eta}) = -1$ for some $\hat{\eta} > 0$ then $f''(\hat{\eta}) \neq 0$.

Proof. As proved in Hartman [3, p. 521], Eq. (11) has the trivial solution $f(\eta) = a + \eta$, $f'(\eta) \equiv 1$, $f''(\eta) \equiv 0$ for each a. Thus if $f'(\hat{\eta}) = 1$ and $f''(\hat{\eta}) = 0$ then the uniqueness of solutions is violated. A similar violation of uniqueness arises if we assume that $f(\hat{\eta}) = -1$ and $f''(\hat{\eta}) = 0$ for some $\hat{\eta} > 0$.

LEMMA 6. Let $\beta < 0$ be given. If $0 < f'(\hat{\eta}) < 1$ and $f''(\hat{\eta}) < 0$ for some $\hat{\eta} > 0$ then there exists $\eta_0 > 0$ such that either

$$f''(\eta_0) = 0 \text{ and } 0 \leq f'(\eta_0) < 1$$

or

$$f''(\eta_0) < 0 \text{ and } f'(\eta_0) = 0.$$

Proof. If neither of the conclusions holds then it follows that $f'' < 0$ for all $\eta > \hat{\eta}$ and $a \equiv \lim_{\eta\to\infty} f'(\eta)$ exists with $0 \leq a < 1$. If a > 1 then $\lim_{\eta\to\infty} f(\eta) = +\infty$, hence $f''' = -\beta(1 - f'^2) - ff'' \geq (\beta/2)(1 - a^2) > 0$ for all large η, which implies that $f'' = 0$ for some large $\eta > 0$. Next, consider the case that $\lim_{\eta\to\infty} f'(\eta) = 0$. If $\overline{\lim} f'' < 0$ then $\lim_{\eta\to\infty} f' = -\infty$. Therefore $\overline{\lim}_{\eta\to\infty} f'' = 0$. Suppose that $\underline{\lim}_{\eta\to\infty} f'' < 0$. Then there must be a value $\bar{\eta} > 0$ for which $f''(\bar\eta)$ attains a relative maximum. That is $f'''(\bar\eta) = 0$ and $f^{(4)}(\bar\eta) \le 0$. However, from Eq. (12) we obtain $f^{(4)}(\bar\eta) = (2\cdot\beta - 1)f'(\bar\eta)f''(\bar\eta) > 0$, a contradiction. Therefore $\lim_{\eta\to\infty} f''(\eta) = 0$. Thus, from Eq. (11) it follows that $f''' \ge -\beta/2 > 0$ for all large η and therefore $f'' = 0$ for some $\eta > 0$.

We now proceed with the proof of our theorem. Recall that if $f''(0) = \tilde\alpha$ and $\beta = \tilde\beta$ then the solution of (21)-(22) which we denote by $\tilde f$ satisfies $-1 < \tilde f' < 1$ for all $\eta > 0$, and $\tilde f' \to 1$ exponentially as $\eta \to \infty$. Therefore the set

$$H_0 = \{\beta < 0 \mid \text{the solution of (21)-(22) satisfies } f' < 1 \text{ for all } \eta > 0\}$$

is non-empty. From Lemma 1 it follows that H_0 is bounded below. Therefore

$$\beta_0 \equiv \inf H_0$$

is a well-defined, finite, negative number, and $\beta_0 \le \tilde\beta_{\alpha_0}$. If we set $\beta = \beta_0$ and let f_0 denote the solution of (21)-(22) then we obtain

LEMMA 7. $-1 < f_0' < 1$ for all $\eta > 0$, and $\lim_{\eta\to\infty} f_0' = 1$.

Proof. If $f_0' = -1$ for some first $\hat\eta > 0$ then $f_0''(\hat\eta) \le 0$ and Lemmas 4 and 5 imply that $f_0' < -1$ for all $\eta > \hat\eta$. But then from continuity it follows that if $\beta < \beta_0$ and $|\beta - \beta_0|$ is sufficiently small then $f' < 0$ for all $\eta > 0$, contradicting the definition of β_0. Furthermore, the uniqueness of solutions, together with Lemma 5, prevent the possibility that $f_0' = -1$, $f_0'' = 0$ at finite $\eta > 0$. Similar arguments show that (f_0', f_0'') cannot intersect the line segment $f' = 1$, $f'' \ge 0$ at finite $\eta > 0$. Thus $-1 < f_0' < 1$ for all $\eta > 0$.

Next we need to show that $\lim_{\eta\to\infty} f_0' = 1$. If $\beta_0 = \tilde\beta$, $\lim_{\eta\to\infty} f_0' = 1$. Suppose that $\beta_0 < \tilde\beta$. Let $\tilde\eta > 0$ be the first value for which $-1 < \tilde f'(\tilde\eta) < 0$ and $\tilde f''(\tilde\eta) = 0$. We wish to show that there is a value $\eta \in (0, \tilde\eta)$ for which $-1 < f_0' < 0$ and $f_0'' = 0$. Consider the function $h = f_0'' - \tilde f''$. Then $h(0) = 0$ and, from Eq. (11), $h'(0) = \tilde\beta - \beta_0 > 0$. Thus $f_0'' > \tilde f''$ on an interval to the right of 0. Also, since $f_0(0) = \tilde f(0) = f_0'(0) = \tilde f'(0) = 0$ then $f_0 > \tilde f$ and $f_0' > \tilde f$ as long as $f_0'' > \tilde f''$. If $f_0'' = \tilde f''$ for some first $\hat\eta \in (0, \tilde\eta)$ then $h(\hat\eta) = 0$ and

$$h'(\hat\eta) \le 0. \tag{36}$$

From Eq. (12) it follows that, at $\eta = \hat\eta$,

$$h'(\hat\eta) = \tilde f''(\tilde f - f_0) + (\tilde\beta - \beta_0)(1 - \tilde f'^2) + \beta_0(f_0' - \tilde f')(f_0' + \tilde f') > 0, \tag{37}$$

contradicting (36). Thus there must be a first $\eta^* \in (0, \tilde\eta)$ for which $-1 < f_0'(\eta^*) < 0$ and $f_0''(\eta^*) = 0$.

Next, it follows from Eq. (11) that $f''' > 0$ as long as $-1 < f' < 0$, $f'' \ge 0$ and $f < 0$. Therefore since $-1 < f_0'(\eta^*) < 0$, $f_0''(\eta^*) = 0$ and $f_0(\eta^*) < 0$ we conclude that there is a first $\bar\eta > \eta^*$ for which $f_0'(\bar\eta) = 0$ and $f_0''(\bar\eta) > 0$. Since $f_0' < 1$ for all $\eta > 0$ and $f''' > 0$ along the line segment $f'' = 0$, $0 < f' < 1$ then it must be the case that $0 < f_0' < 1$, $f_0'' > 0$ for all $\eta > \bar\eta$. Hence a $\equiv \lim_{\eta\to\infty} f_0'$ exists and $0 < a \le 1$. We eliminate the possibility that $a < 1$. Let $z(f)$ denote the solution of Eq. (26) such that

$$z(f_0(\bar\eta)) = 0 \text{ and } \dot z(f_0(\bar\eta)) = 2f_0''(\bar\eta) > 0. \tag{38}$$

Since $a > 0$ then $\lim_{\eta\to\infty} f_0(\eta) = \infty$. Also, $f_0'' > 0$ for all $\eta > \bar\eta$ which implies that $\dot z > 0$ for all $f > f_0(\bar\eta)$. Furthermore, $\lim_{f\to\infty} z(f) = a^2 < 1$ since $\lim_{\eta\to\infty} f_0' = a$. Thus from continuity of solutions with respect to parameters, and Lemma 3, part (i), it follows that if $\beta < \beta_0$ and $|\beta - \beta_0|$ is sufficiently small then the solution f of (21)-(22) must satisfy $-1 < f' < 1$ for all $\eta > 0$, violating the definition of β_0. Therefore the assumption that $a < 1$ leads to a contradiction.

LEMMA 8. For each $\beta < \beta_0$ there is a first $\eta_0 = \eta_0(\beta) > 0$ for which $f'(\eta_0) = 1$ and $f''(\eta_0) > 0$.

Proof. As in the proof of Lemma 7, it is easy to show that there is a first $\eta^* > 0$ for which $-1 < f'(\eta^*) < 0$ and $f''(\eta^*) = 0$. Then, from Eq. (11), $f'' > 0$ for $\eta > \eta^*$ as long as $f' < 1$. If $f' < 1$ for all $\eta > \eta^*$ then the definition of β_0 is violated. Therefore there is a first $\eta_0 > 0$ for which $f'(\eta_0) = 1$ and $f''(\eta_0) \geq 0$. Lemma 5 implies that $f''(\eta_0) > 0$.

LEMMA 9. Let $\beta < \beta_0$ be given and suppose that $f'(\hat{\eta}) \geq 1$, $f''(\hat{\eta}) > 0$ for some $\hat{\eta} > 0$. Then $f'' = 0$ for some first $\tilde{\eta} > \hat{\eta}$.

Proof. If $f'' > 0$ for all $\eta > \hat{\eta}$ then $f' > 1$ for all $\eta > \hat{\eta}$ and there is a value $\bar{\eta} \geq \hat{\eta}$ such that $f > 0$ for all $\eta > \bar{\eta}$. Let $m = f'(\bar{\eta} + 1) > 1$. Then, from Eq. (11) we obtain $f''' < -\beta(1 - m^2) < 0$ for all $\eta > \bar{\eta} + 1$ and $\lim_{\eta\to\infty} f'' = -\infty$, a contradiction.

For each $\beta < \beta_0$ we define $\eta^0 = \eta^0(\beta) > \eta_0$ to be the first value past η_0 for which $f''(\eta^0) = 0$, and $f'(\eta^0) > 1$.

LEMMA 10. If $\beta < \beta_0$ and $|\beta - \beta_0|$ is sufficiently small then $1 < f'(\eta^0) < \sqrt{2}$.

Proof. From Lemma 9 and continuity we can choose $\beta_0 - \beta > 0$ sufficiently small so that

$$f(\eta_0) > \frac{1}{\sqrt{2} - 1}, \qquad f'(\eta_0) = 1, \qquad 0 < f''(\eta_0) < 1. \tag{39}$$

Define $h = f'' - (f'/(1 - \sqrt{2}) - \sqrt{2}/(1 - \sqrt{2})$. Note that the straight line $f'' = f'/(1 - \sqrt{2}) - \sqrt{2}/(1 - \sqrt{2})$ passes through the points $(1, 1)$ and $(\sqrt{2}, 0)$. From (39) we obtain $h(\eta_0) < 0$. If $f'(\eta^0) > \sqrt{2}$ then there is a first $\bar{\eta} \in (\eta_0, \eta^0)$ for which $h(\bar{\eta}) = 0$ and

$$h'(\bar{\eta}) \geq 0. \tag{40}$$

From Eq. (11) it follows that $f(\bar{\eta}) > f(\eta_0)$, $f''(\bar{\eta}) \geq 0$ and therefore $h'(\bar{\eta}) = f''(\bar{\eta})(1/\sqrt{2} - 1) - f(\bar{\eta})) - \beta(1 - f'^2(\bar{\eta})) < 0$, contradicting (40) for sufficiently small $\beta_0 - \beta > 0$.

LEMMA 11. If $\beta_0 - \beta > 0$ is sufficiently small then $f' > 1$ for all $\eta > \eta_0$.

Proof. Let $M > 2$, $\bar{\beta} = \beta_0$ and let $\bar{f}$, $\bar{\delta}$ satisfy Lemma 3, part (ii). From continuity we choose $\beta_0 - \beta > 0$ sufficiently small such that $|\beta_0 - \beta| < \bar{\delta}$ and

$$f(\eta^0) > \bar{f}, \qquad 1 < f'(\eta^0) < \sqrt{2}, \qquad f''(\eta^0) = 0. \tag{41}$$

Thus from Lemma 3, part (ii), it follows that $z > 1$ for all $f > f(\eta^0)$ and therefore $f' > 1$ for all $\eta > \eta^0$.

Define the set $H_1 = \{\beta < \beta_0 |\ f' - 1 = 0$ has exactly one positive solution$\}$. From Lemmas 1 and 11 it follows that H_1 is non-empty and bounded below and therefore the value $\beta_1 \equiv \inf H_1$ is a well-defined, finite negative number. Let f_1 denote the solution of the problem (21)-(22) for $\beta = \beta_1$.

LEMMA 12. $f_1' > 1$ for all $\eta > \eta_0$ and $\lim_{\eta\to\infty} f_1' = 1$.

Proof. If $f_1' = 1$ for some first $\eta_1 > \eta_0$ then $f_1''(\eta_1) \leq 0$ and Lemma 5 implies that $f_1''(\eta_1) < 0$. Thus from continuity with respect to parameters it follows that if $\beta - \beta_1 > 0$ is sufficiently small then $f' = 1$ for some first $\eta_1 = \eta_1(\beta) > 0$, contradicting the definition of β_1. Therefore $f_1' > 1$ for all $\eta > \eta_0$.

Recall that $f_1''(\eta^0) = 0$ for some first $\eta^0(\beta_1) > \eta_0$. Using Eq. (11), we can easily show that $f_1'' < 0$ and $f_1' > 1$ for all $\eta > \eta^0$ and therefore the value $a = \lim_{\eta\to\infty} f_1'$ exists and $a \geq 1$. If we assume that $a > 1$ then, from Lemma 3, part (ii) it follows that if $\beta_1 - \beta > 0$ is sufficiently small then $f' > 1$ for all $\eta > \eta_0$, where f denotes the solution of (21)–(22). This contradicts the definition of β_1. Therefore $a = 1$, completing the proof of Lemma 12.

We have thus far defined the values β_0 and β_1, and have investigated the properties of the corresponding solutions f_0 and f_1, respectively. For each $i \in \mathbb{N}$ define

$$H_i = \{\beta < \beta_{i-1} \,|\, f' - 1 = 0 \text{ has exactly } i \text{ positive solutions}\}$$

and

$$\beta_i = \inf H_i$$

for each $i \in \mathbb{N}$. A repetition of the methods of proof used in Lemmas 2 through 12 and an application of the principle of mathematical induction leads to

LEMMA 13. For each $i \in N$ the set H_i is non-empty, bounded below and β_i is a well-defined negative number. In addition, if $\beta = \beta_i$ in Eq. (11) then the solution f_i of the problem (21)-(22) satisfies the following:

(1) $f_i' - 1 = 0$ has exactly i positive solutions the largest of which we denote by η_i;
(b) $f_i' > 1$ for all $\eta > \eta_i$ if i is odd;
(c) $f_i' < 1$ for all $\eta > \eta_i$ if i is even,
(d) $\lim_{\eta \to \infty} f_i' = 1$.

From the same procedure as that employed by Hartman [3, pp. 534–536] it follows that for each $i \in \mathbb{N}$ either

$$1 - f_i' \sim C_0 \eta^{-1-2\beta_i} \exp\left(-\frac{1}{2}\eta^2 - C_1\eta\right), \; f_i'' \sim \eta(1 - f_i') \tag{42}$$

as $\eta \to \infty$ where $C_0 > 0$, C_1 are constants, *or*

$$1 - f_i' \sim C_0 \eta^{2\beta_i}, \qquad f_i'' \sim -2\beta_i\, C_0 \eta^{-1+2\beta_i} \tag{43}$$

as $\eta \to \infty$ where $C_0 > 0$ is a constant.

LEMMA 14. For each $i \in \mathbb{N}$ the conditions of (42) are satisfied.

Proof. The proof is similar to that of Theorem 9.2 in Hartman [3, p. 536]. We first consider odd values of i. Choose $M > 1$ and let $z_i(f_i) = f_i'^2$, $w_i = z_i - 1$ and $r_i = \dot{w}_i/w_i$. From Lemma 2 it follows that if $\beta = \beta_i$ and $\alpha = M$ then there is a solution v_i of Eq. (30) such that $s_i = \dot{v}_i/v_i \sim -f/M$ as $f \to \infty$.

Then for large f, $r_i \leq s_i$. For suppose that $r_i > s_i$ for some large $\hat{f}$. Then $r(\hat{f}) > s_i(\hat{f})$ if $r(f) = -\dot{z}/(1 - z)$ belongs to a solution of (21)–(22) where $|\beta - \beta_i|$ is sufficiently small. But then, as in the proof of part (ii) of Lemma 3, it follows that $r > s_i$ for all $f \geq \hat{f}$ for sufficiently small $|\beta - \beta_i|$, which leads to a contradiction of the definition of β_i.

Hence $r_i \leq s_i$ for all large f and so $z_i - 1 \leq cv_i$ for all large f and some constant $c > 0$. Since $\ln(v_i) \sim -f^2/2M$ as $f \to \infty$ it follows that $1 - f_i'$ cannot satisfy (43) and therefore (42) must hold. The proof for i an even nonnegative integer follows similarly and is omitted.

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