EXPONENTIAL ASYMPTOTIC EXPANSIONS FOR NONLINEAR DIFFERENTIAL EQUATIONS*

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Abstract. An exponential method for obtaining asymptotic expansions of the solution of a linear differential equation was presented by Brull and Soler in [3]. We extend this method to the nonlinear differential equation of the type $L(u) + \sum_1^N f(x, t)\, u^n = 0$, where t is the small parameter. Three examples are used to illustrate the technique and to explain how uniformly valid expansions may be obtained.

1. Introduction. Asymptotic expansions of the solution of a differential equation are generally obtained in a power series of a small parameter in the differential equation, but Brull and Soler [3] introduced a method for obtaining expansions of linear differential equations by considering an exponential expansion $W(x) = W_0(x) \exp(p(x, t))$ where W_0 is the solution of the unperturbed problem, $p(x, t) = tp_1(x) + t^2p_2(x) + \cdots$, and t is the small parameter. One immediate advantage of an exponential expansion occurs in any problem in which it is practical to calculate only the first few terms. Because the exponential function converges faster than any power function, one would anticipate that an exponential expansion would require fewer terms for accuracy equivalent to a power series expansion. This improvement was verified in the original paper of Brull and Soler [3] by considering an equation in the theory of symmetrically loaded circular plates with variable flexural rigidity. Furthermore, an exponential expansion is a natural expansion for a large class of problems whose solutions involve exponentials in the unperturbed equation.

This paper serves the dual purpose of showing that the idea of an exponential asymptotic expansion can be extended to nonlinear differential equations of the type $L(u) + \sum_{n=0}^N f_n(x, t)u^n = 0$ (where L is a linear differential operator) and that such expansions can be applied to a variety of problems in mechanics. The equation considered includes any nonlinear differential equation where the nonlinearity enters by a power of u, the dependent variable. The method may also be extended to coupled nonlinear equations, as shown in example 3. Since the use of an exponential transformation in asymptotic analysis is now a familiar technique, it is anticipated that the nonlinear extensions presented here will stimulate interest in other, perhaps novel, applications. While the examples presented here are limited to problems which have known power series expansions, the results of the theory are equally valid for unsolved problems.

In Sec. 2 a theorem is proved to justify an exponential asymptotic expansion, and corollaries for generalizing this result are discussed. Since uniformly valid expansions have become important in recent years, we incorporate these techniques in Sec. 3 where examples of the theorem and its corollaries are presented. In all three examples a uni-

* Received June 6, 1978.

formly valid expansion is obtained by using variations of the Lindstedt-Poincaré or Krylov-Bogoliubov-Mitropolski (KBM) technique.

2. The general nonlinear equation. We begin by considering a general second-order differential equation $L(u) + \sum_{n=1}^{N} g_n(x, t)\, u^n = 0$, which can be put in the form

$$d^2u/dx^2 + \sum_{n=1}^{N} f_n(x, t)\, u^n = 0, \tag{1}$$

where $f_n(x, t) = f_{0n}(x) + F_n(x, t)$ and $F_n(x, 0) = 0$. t is a small parameter of the differential equation. We are interested in finding a convergent or asymptotic series solution of Eq. (1) in the form $u(x) = W(x) \exp(p(x, t))$.

THEOREM. If $f_n(x, t)$ is an analytic function ($n = 1, 2, \cdots, N$) in the parameter t on $[x_0, x_1)$, then the initial-value problem

$$d^2u/dx^2 + \sum_{n=1}^{N} f_n(x, t)\, u^n = 0 \tag{2}$$

$$u(x_0) = u_0\,, \; u'(x_0) = u_0' \tag{3}$$

has a solution

$$u(x) = w(x) \exp(p(x, t)) = w(x) \exp(tp_1(x) + t^2p_2(x) + \cdots)$$

which is either an asymptotic representation of u for small values of t or is a convergent series.

Proof. We use $u' = du/dx$ and $u'' = d^2u/dx^2$. Substituting $u(x) = w(x)\, e^p$ into Eq. (1), we have

$$w''e^p + 2w'\, p'\, e^p + w\, p''\, e^p + w(p')^2\, e^p + \sum_{n=1}^{N} f_n(x, t)\, w^n\, e^{np} = 0.$$

Dividing by $e^p \neq 0$ yields

$$w'' + 2w'\, p' + w\, p'' + w(p')^2 + \sum_{n=1}^{N} f_n(x, t)\, w^n \exp((n-1)p) = 0. \tag{4}$$

Since we are interested in solutions valid for small t, we make the expansions

$$f_n(x, t) = f_{0n}(x) + \sum_{k=1}^{\infty} f_{kn}(x) t^k$$

and look for solutions of the form

$$u(x, t) = w(x) \exp\left(\sum_{k=1}^{\infty} p_k(x)\, t^k \right) .$$

With these expansions we have in Eq. (4)

$$\begin{aligned} w'' &+ 2w'(tp_1' + t^2p_2' + \cdots) + w(tp_1'' + t^2p_2'' + \cdots) \\ &+ w(tp_1' + t^2p_2' + \cdots)^2 + \sum_{n=1}^{N} f_{0n}(x)\, w^n \exp(n-1)p) \\ &+ \sum_{n=1}^{N} \sum_{k=1}^{\infty} f_{kn}(x)\, t^k\, w^n \exp((n-1)p) = 0. \end{aligned}$$

Collecting similar powers of t, we obtain a series of differential equations

$$t^0: \quad w'' + \sum_{n=1}^{N} f_{0n}(x)\, w^n = 0, \tag{5}$$

$$t^1: \quad wp_1'' + 2w'\, p_1' + \sum_{n=1}^{N} f_{0n}(x)\, w^n\, p_1\,(n-1) = -\sum_{n=1}^{N} f_{1n}(x)\, w^n, \tag{6}$$

$$t^2: \quad wp_2'' + 2w'\, p_2' + \sum_{n=1}^{N} f_{0n}(x)\, w^n\, p_2\,(n-1)$$

$$= -(p_1')^2 w^- \sum_{n=1}^{N} f_{0n}(p_1^2/2)\,(n-1)^2\, w^n - \sum_{n=1}^{N} (f_{2n} + (n-1)\, f_{1n} p_1) w^n, \tag{7}$$

and in general

$$t^k: \quad wp_k'' + 2w'\, p_k' + \sum_{n=1}^{N} f_{0n}(x)\, w^n\, P_k = -\sum_{n=1}^{k-1} p_n'\, p_{k-n}' w - \sum_{n=1}^{N} \sum_{j=1}^{k} f_{jn}\, P_{k-j}\, w^n \tag{8}$$

where P_k is the expansion of $\exp((n-1)p)$ in powers of t. Thus,

$$P_0 = 1, \qquad P_1 = (n-1)p_1,$$
$$P_2 = (n-1)p_2 + (n-1)^2 p_1^2/2,$$
$$P_3 = (n-1)^3 p_1^3/6 + (n-1)^2(2p_1p_2)/2 + (n-1)p_3,$$
$$P_4 = (n-1)^4 p_1^4/24 + (n-1)^3(3p_1^2p_2)/6 + (n-1)^2(2p_1p_3 + p_2^2)/2 + (n-1)p_4, \cdots.$$

Let $u_m(x, t)$ be the mth partial sum

$$u_m(x, t) = w(x) \exp\left(\sum_{n=1}^{m} p_n(x)\, t^n\right).$$

If we substitute $u_m(x, t)$ into Eq. (1), we have

$$u_m'' + \sum_{n=1}^{N} f_n\, u_m^n = \left(w'' + \sum_{n=1}^{N} f_{0n}\, w^n\right) e^p$$
$$+ e^p\left[\sum_{k=1}^{m}\left(wp_k'' + 2w'\, p_k' + \sum_{n=1}^{k-1} p_n'\, p_{k-n}' w + \sum_{n=1}^{N}\sum_{j=0}^{k} f_{jn}\, P_{k-j} w^n\right) t^k\right]$$
$$+ e^p\left[\sum_{k=m+1}^{m^2}\sum_{n=1}^{k-1} p_n'\, p_{k-n}' w t^k + \sum_{k=m+1}^{\infty}\sum_{j=0}^{k}\sum_{n=1}^{N} f_{jn}\, w^n \bar{P}_{k-j} t^k\right]$$
$$= 0 + N_m$$

where

$$\bar{P}_j = P_j, \qquad j = 0, 1, \cdots, m,$$
$$\bar{P}_{m+1} = P_m - (n-1)p_{m+1},$$
$$\bar{P}_{m+2} = P_m - (n-1)p_{m+2} - (n-1)^2(2p_1p_{m+1})/2,$$
$$\cdots$$

and

$$N_m = e^p\left[\sum_{k=m+1}^{m^2}\sum_{n=1}^{k-1} p_n'\, p_{k-n}'\, w\, t^k + \sum_{k=m+1}^{\infty}\sum_{j=0}^{k}\sum_{n=1}^{N} f_{jn}\, w^n\, \bar{P}_{n-j}\, t^k\right].$$

Then we have

$$(1/t)^m N_m = e^p \left[\sum_{k=1}^{m^2-m} \sum_{n=1}^{k+m-1} p_n' \, p_{k+m-n}' \, w \, t^k + \sum_{k=1}^{\infty} \sum_{j=0}^{k+m} \sum_{n=1}^{N} f_{jn} \, w^n \, \bar{P}_{k+m-j} \, t^k \right]. \tag{9}$$

Finally,

$$\lim_{\substack{t \to 0 \\ m \text{ fixed}}} (1/t)^m \, N_m = 0. \tag{10}$$

Thus, our differential equation is satisfied asymptotically for small t by $w(x) \exp(p(x))$. Should the series be convergent for a particular differential equation, then the series solution is exact rather than asymptotic.

In spite of the formidable sequence of equations (5)–(8) presented in the theorem, it is often the case that Eq. (8) reduces to

$$wp_k'' + w'(2p_k') = \text{known terms}$$

which can be easily solved once w is known by multiplying by w to get

$$(w^2 \, p_k')' = \text{known terms.}$$

There are several corollaries that are immediate extensions of the theorem. First, one may replace d^2u/dx^2 in Eq. (1) by any linear operator or a finite sum of linear operators. While this may increase the difficulty of solving the series of differential equations (5)–(8), no new complications in the proof arise. A second modification can be easily proved by allowing the summation in Eq. (1) to begin at zero rather than one. This is tantamount to allowing the equation to contain a forcing function which may or may not depend on the small parameter t. Thirdly, one could allow terms of the form $\sum_{m=1}^{M} g_m(x, t) \, (u')^m$ to be included in the theorem or the other two corollaries. These extensions will be examined in the following examples as well as the question of uniformly valid expansions.

3. Examples. *Example 1. The Duffing equation.* The standard techniques available for obtaining uniformly valid expansions can be used in conjuction with an asymptotic exponential expansion. We illustrate the exponential expansion here by obtaining a uniformly valid expansion of the Duffing equation, $u'' + u + t\,u^3 = 0$, with the Lindstedt-Poincaré method. We expand the independent variable s in the small parameter t as

$$x = s(1 + t\,w_1 + t^2\,w_2 + \cdots)$$

and the solution u as $u = W\,e^p$ where $p = t\,p_1(x) + t^2\,p_2(x) + \cdots$. Substituting these expansions into the Duffing equation, we have

$$W_{ss}\,e^p + 2W_s\,p_s\,e^p + W\,p_{ss}\,e^p + W\,(p_s)^2\,e^p + (1 + t\,w_1 + t^2\,w_2 + \cdots)^2\,(W\,e^p + t\,W^3\,e^{3p}) = 0 \tag{11}$$

or, on division by e^p,

$$W_{ss} + 2W_s\,p_s + W\,p_{ss} + W\,(p_s)^2 + (1 + 2t\,w_1 + t^2(2w_2 + w_1^2) + \cdots)(W + tW^3\,e^{2p}) = 0. \tag{12}$$

Collecting like powers of t, our first three equations are

$$t^0: \quad W_{ss} + W = 0,$$

$$t^1: \quad W p_{1,ss} + 2\, W_s p_{1,s} = -W^3 - 2w_1\, W, \tag{13}$$

$$t^2: \quad W p_{2,ss} + 2\, W_s p_{2,s} = -(p_{1,s})^2\, W - 2\, p_1\, W^3 - (2w_2 + w_1^2)\, W - 2w_1\, W^3.$$

The solution of Eq. (13) is $W = a\cos(s + s_0)$, the solution of the unperturbed equation. After multiplication by W, Eq. (14) becomes

$$W^2\, p_{1,ss} + 2W_s\, p_{1,s} = -W^4 - 2w_1\, W^2$$

or

$$(W^2\, p_{1,s})_{,s} = -W^4 - 2w_1\, W^2.$$

Then

$$W^2\, p_{1,s} = \int(-a^4\cos^4(v) - 2w_1\, a^2\cos^2(v))\, dv$$

where $v = s + s_0$. Thus,

$$W^2\, p_{1,s} = -a^4(3v/8 + (\sin 2v)/4 + (\sin 4v)/32)$$
$$-2w_1a^2(v/2 + (\sin 2v)/4) + a^2K_1$$

where K_1 is a constant of integration. Dividing by W^2 and integrating a second time, we have

$$p_1 = (-3a^2/8 - w_1)\, v\tan v - (a^2/8)\sin^2 v$$
$$+ K_1\tan v + a^2K_2\,. \tag{14}$$

We can eliminate the secular term $v\tan v$ by choosing $w_1 = -3a^2/8$. Since we are presenting a typical solution only, we may also choose $K_1 = K_2 = 0$, although it should be noted that to compare our final answer with the usual Lindstedt-Poincaré expansion (see Nayfeh [4, p. 176]), it would be necessary to use $K_1 = 0$ and $K_2 = 1/32$.

Substituting W and p_1 into Eq. (13) produces

$$W p_{2,ss} + 2W_s\, p_{2,s} = -p_{1,s}^2\, W - 2p_1\, W^3 - (2w_2 + w_1^2)\, W - 2w_1\, W^3$$

or

$$(W^2\, p_{2,s})_{,s} = -p_{1,s}^2\, W^2 - 2p_1\, W^4 - (2w_2 + w_1^2)\, W^2 - 2w_1\, W^4. \tag{15}$$

Again, integrating twice and using $K_1 = K_2 = 0$, we have

$$p_2 = (57a^4/256 - w_2)\, v\tan v + (a^4/128)\sin^4 v + (21a^4/256)\sin^2 v. \tag{16}$$

We eliminate the secular term $v\tan v$ by setting $w_2 = 57\, a^4/256$. This leaves our solution u as

$$u = (a\cos v)\exp\,(t(-\sin^2 v)(a^2/8) + t^2((a^4\sin^4 v/128 + (21a^4\sin^2 v)/256) + O(t^3)) \tag{17}$$

where $v = s + s_0 = x(1 + 3a^2t/8 - 21a^4t^2/256) + s_0 + O(t^3)$. This expansion is in full agreement through $O(t^2)$ with the standard expansion.

Example 2. Nonlinear response of elastic plates to pulse excitations. A second example is provided by the problem of the nonlinear response of elastic plates to pulse excitations as described in Bauer [1]. The case of a rectangular or a circular plate, which is simply supported or clamped in, will produce the nonhomogeneous Duffing equation

$$u'' + m^2(u + t\,u^3) = P(t), \tag{18}$$

$$u(0) = 0 = u'(0). \tag{19}$$

We consider here Bauer's case where $P(t)$ is a step function P_0 and use his transformation

$$u(x) = g(x) + q(x) \tag{20}$$

where $g(x)$ is the particular solution of the nonhomogeneous linear equation

$$g'' + m^2 g = P_0 \tag{21}$$

and $q(x)$ satisfies

$$q'' + m^2 q + tm^2(g + q)^3 = 0, \tag{22}$$

$$q(0) = -g(0) = P_0, \qquad q'(0) = -g'(0) = 0. \tag{23}$$

Problem (22)–(23) includes the nonhomogeneous term tm^2g^3 as described in the second corollary.

We again use the Lindstedt-Poincaré method to obtain a uniformly valid expansion. With $x = s(1 + tw_1 + t^2w_2 + \cdots)$ and $q(x) = W(x)\exp(p(x, t))$ where $p(x, t) = tp_1(x) + t^2p_2(x) + \cdots$, we have, on substituting q and x into Eq. (22), the following equations for like powers of t:

$$t^0: \qquad W_{ss} + m^2 W = 0 \tag{24}$$

$$t^1: \qquad Wp_{1,ss} + W_s p_{1,s} = -m^2g^3 - 3m^2g^2W - 3m^2gW^2 - m^2W^3 - 2w_1m^2W. \tag{25}$$

The solution of (24) which satisfies the initial conditions (23) is

$$W(s) = -(P_0/m^2)\cos ms. \tag{26}$$

Using expression (26) in Eq. (25), we have

$$(\cos^2 r\, \dot{p}_{1,r})_{,r} = -(2w^2\cos^2 r + a^2\cos^4 r + 3ga\cos^3 r + 3g^2\cos^2 r + g^3(\cos r)/a)$$

where $r = ms$, $a = -P_0/m^2 = -g$. Integrating twice, we find

$$\begin{aligned} p_1 = (1/m)[&(-mw_1 - 3ma^2/8 - 3P_0^2/2m^3)(r\tan r) \\ &+ (-9aP_0/4m - P_0^3/am^5 - P_0a/4m)\sec r + (P_0a/m)(\cos r) \\ &- (a^2m/8)\sin^2 r + K_2 + K_1\tan r]. \end{aligned}$$

To eliminate the secular term $r\tan r$, we choose

$$-mw_1 - (3ma^2/8) - (3P_0^2/2m^3) = 0$$

or

$$w_1 = -15P_0^4/8m^4. \tag{27}$$

Finally, to satisfy the initial conditions we need $K_1 = 0$ and $K_2 = -2P_0^2/m^3$. Thus

$$\begin{aligned} q(x) = (-P_0/m^2)\cos ms\exp(t(&-2P_0^2/m^4 + (3P_0^2/m^4)\sec ms \\ &- (P_0^2/m^4)\cos ms - (P_0^2/8m^4)\sin^2 ms) + O(t^2)) \end{aligned} \tag{28}$$

where

$$s = x(1 + 15P_0^2t/8m^4 + O(t^2)). \tag{29}$$

This expansion is in total agreement with Bauer's expansion through terms of $O(t)$.

Example 3. Transient coupled thermoelastic problems. Our third example is the problem of a mechanical disturbance propagating in a semi-infinite medium. The exact coupled thermoelastic solution has been obtained by Boley and Toline [2] using Fourier transforms. An approximate solution using standard perturbations techniques was found by Soler and Brull [5]. The equations of interest are

$$d^2Y/dx^2 + Y = tH, \tag{30}$$

$$dH/dx + H = -dY/dx, \tag{31}$$

$$Y(0) = 1, \qquad Y'(0) = 0 = H(0), \tag{32}$$

and are derived in [5].

The coupled system (30)–(32) can be treated as one third-order equation in the small parameter t and solved as suggested by the first corollary.

Since the unperturbed system would have solutions corresponding to $a(\cos(x + x_0))$ for Y and $c\,e^{-x}$ for H, we will use the KBM method here to find a uniformly valid asymptotic expansion. Thus, we use

$$Y = a \cos s \exp(tp_1 + O(t^2)), \tag{33}$$

$$H = c\,e^{-s} \exp(tq_1 + O(t^2)), \tag{34}$$

$$da/dx = tA_1 + t^2A_2 + \cdots, \tag{35}$$

$$ds/dx = 1 + tB_1 + t^2B_2 + \cdots, \tag{36}$$

$$dc/dx = tC_1 + t^2C_2 + \cdots, \tag{37}$$

where $A_i = A_i(a, c)$, $B_i = B_i(a, c)$ and $C_i = C_i(a, c)$, $i = 1, 2, \cdots$. The terms A_i, B_i, and C_i will be chosen to eliminate secular terms and thereby produce a uniformly valid expansion.

Using the change of variables (35)–(37), we can substitute Eq. (33) and (34) into Eq. (30) and separate corresponding powers of t to obtain these first two equations:

$$t^0: \quad W_{0,ss} + W_0 = 0 \tag{38}$$

$$t^1: \quad 2A_1W_{0,as} + 2W_{0,s}\,p_{1,s} + W_0\,p_{1,ss} + 2b_1W_{0,ss} + 2C_1W_{0,sc} = H_0\,. \tag{39}$$

Similar substitutions into Eq. (31) yield

$$t^0: \quad H_{0,s} + H_0 = -W_{0,s} \tag{40}$$

$$t^1: \quad A_1H_{0,a} + B_1H_{0,s} + h_0\,q_{1,s} + C_1\,H_{0,c} = -(A_1W_{0,a} + B_1W_{0,s} + C_1H_{0,c} + W_0\,p_{1,s} + W_{0,s}(p_1 - q_1)). \tag{41}$$

If we begin by solving (38) and (40), we obtain the expected unperturbed solutions $W_0 = a \cos s$ and $H_{0,s} + H_0 = a \sin s$ or $H_0 = c\,e^{-s} + (a/2)(\sin s - \cos s)$. Using these expressions in (39), we have

$$a \cos s\,(p_{1,ss}) + 2(-a \sin s)\,p_{1,s} = c\,e^{-s} - (a/2)(\sin s - \cos s) - 2A_1(-\sin s) - 2B_1(-a \cos s).$$

Multiplying by $\cos(s)$ and integrating twice, we find

$$ap_1 = (c\,e^{-s}/2)\sec s + (a/4 + A_1)(\tan s - s) + (aB_1 - a/4)(s \tan s).$$

We can eliminate the secular terms s and $s \tan s$ by choosing

$$A_1 = -a/4, \qquad B_1 = 1/4. \tag{42}$$

Thus,

$$p_1 = (c/2a)\, e^{-s} \sec s. \tag{43}$$

Finally, we make the appropriate substitutions into Eq. (41) to obtain

$$H_0 q_{1,s} + (H_{0,s} + H_0)\, q_1 = e^{-s}(-C_1 + 3c/4) + (a/4)\sin s.$$

Thus,

$$(H_0\, e^s\, q_1)_{,s} = (3c/4 - C_1) + (a/4)\, \sin s\, e^s$$

or

$$H_0\, e^s\, q_1 = (3c/4 - C_1)\, s + (a/4)(\sin s - \cos s)\, e^s.$$

Here, we choose $C_1 = 3c/4$ to eliminate the term involving s and obtain

$$q_1 = (a/4)(\sin s - \cos s)/H_0\,. \tag{44}$$

Assembling all these results, we have

$$a = a_0 \exp(-tx/4), \qquad s = (1 + t/4)x + x_0\,, \qquad c = c_0 \exp(3tx/4),$$

$$\begin{aligned} Y &= a_0 \exp(-tx/4) \cos(\alpha_0 + (1 + t/4)x) \\ &\quad \cdot \exp(t(c_0/2a_0) \exp(x + 5xt/4 + x_0) \sec(x_0 + (1 + xt/4)) + O(t^2)), \\ H_0 &= c_0 \exp(xt/2 - t + x_0) + (a_0/2) \exp(-xt/4) \\ &\quad \cdot (\sin(x_0 + (1 + t/4)x) - \cos(x_0 + (1 + t/4)x)), \\ H &= H_0 \exp(t(a_0/4) \exp(-xt/4)(\sin(x_0 + (1 + t/4)) \\ &\quad - \cos(x_0 + (1 + t/4)x)/H_0 + O(t^2)). \end{aligned}$$

Through terms of $O(t)$, this is in agreement with the expansion of [5] where we satisfy the initial conditions by choosing

$$x_0 = \arctan \frac{-t(1 - t/8 + t^2/16)}{2(1 + t/4)(1 + t^2/4)}\,, \qquad c_0 = (1/2) \exp(x_0)(1 + t/2)/(1 + t^2/2),$$

$$a_0^2 = \frac{(t/2)^2(1 - t/8 - t^2/16)^2}{(1 + t/4)^2(1 + t^2/4)^2} + \frac{(1 - t/4 + t^2/8)^2}{(1 + t^2/4)^2}\,.$$

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