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ON THE NUMERICAL SOLUTION OF SINGULAR INTEGRODIFFERENTIAL EQUATIONS*

BY N. I. IOAKIMIDIS AND P. S. THEOCARIS *(The National Technical University of Athens)*

Summary. A method of numerical solution of a sufficiently wide class of Cauchy-type singular integrodifferential equations along a straight finite interval is presented. This method consists of approximating the integrals in such an equation by using appropriate numerical integration rules and appropriately-selected collocation points and reducing such an equation to a system of linear algebraic equations. This technique constitutes a direct generalization of the corresponding methods of numerical solution of Cauchy-type singular integral equations and presents some advantages over the classical Multhopp method of numerical solution of Cauchy-type singular integrodifferential equations, to which it reduces in some special cases. An application to a specific equation is also made.

1. Introduction. The numerical solution of singular integral equations with kernels presenting Cauchy-type singularities (called in what follows simply singular integral equations) by reduction to systems of linear algebraic equations by approximating the integrals through the use of numerical integration rules and applying the resulting equations to appropriately selected collocation points has recently become the subject of intensive research. Kalandiya [1, 2] has used a variable transformation and further interpolation techniques for such a reduction, applicable to singular integral equations of the first kind along the interval $(-1, 1)$ reducible to Fredholm integral equations. Further, Erdogan and Gupta [3] succeeded in getting rid of the variable transformation, and thus in working along the real integration interval $(-1, 1)$. Theocaris and Ioakimidis [4] proved that the method of Erdogan and Gupta was in reality equivalent to the application of the Gauss-Chebyshev numerical integration rule to Cauchy-type principal value integrals. In the same work, they also developed a new technique for solving singular integral equations, based on the Lobatto-Chebyshev numerical integration rule.

Several more publications concerning wider or different classes of singular integral equations along a real integration interval have also recently appeared, together with the extension of almost every numerical integration rule for regular integrals to the case of Cauchy-type principal value integrals proposed by Ioakimidis and Theocaris [5]. A considerable number of these contributions were due to Theocaris and Ioakimidis, who in a series of publications (see e.g. [6–10]) have generalized the results of [1–4] to the most general cases of singular integral equations of the first or the second kind, with constant or variable coefficients, with regular of generalized Fredholm kernels (besides the Cauchy kernels), with weight functions presenting complex singularities at the endpoints of the integration intervals or pairs of such singularities, etc., so that any practical case of singular integral equations could be faced. Although any numerical integration rule can be used for the solution of singular integral equations (Gauss, Radau and Lobatto rules, rules

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with completely preassigned abscissae, etc.), particular attention was paid to Gaussian rules (and especially Lobatto-type rules) associated with the weight function $w(x) = (1 - x)^{\alpha}(1 + x)^{\beta}$, the corresponding Jacobi polynomials and the associated interval $(-1, 1)$. In almost all these cases, the existence of an appropriate number of collocation points inside the integration interval was proved. Of course, in the case of complex singularities α, β, by using a theorem of analytic continuation, the use of collocation points outside the integration interval was seen to be permissible [8]. [9] is a review of the results of Theocaris and Ioakimidis on the numerical solution of singular integral equations. Several more recent publications by the same authors are mentioned in [10].

Finally, in [11] the results of several of the above-mentioned publications have been derived by a similar method and also by using Jacobi-type numerical integration rules. The aim of this paper was also to make possible the use of an arbitrary number of preassigned abscissae lying outside the integration interval $(-1, 1)$. Unfortunately, the lack of practical interest and physical meaning in this case, together with the inability to determine an appropriate number of collocation points, restrict the practical usefulness of the results of [11].

It is the intention of this paper to show that the direct methods of numerical solution of singular integral equations can also be used for the numerical solution of singular integrodifferential equations. Such equations appear in several problems of mathematical physics and, particularly, in fluid mechanics problems [2, 12], as well as in several plane elasticity problems [2, 13, 14]. Unfortunately, to the authors' knowledge, only one method for the numerical solution of a special class of these equations has been proposed, by Multhopp [12]. The method of Multhopp is also described in [13] and [2], in the latter together with a proof of its convergence. This method has the disadvantages that it is not general enough, that it makes use of a variable transformation, exactly as made in [1] for singular integral equations, and also that in this method the evaluation of the singular integral (the only integral in the equation considered by Multhopp [2, 12, 13]) is evaluated by using interpolation techniques and not Gaussian integration rules.

Although these disadvantages of Multhopp's method of solution of singular integrodifferential equations are not of critical importance, the authors think that it is worthwhile to show the applicability of the results of [3, 4] to the numerical solution of singular integrodifferential equations sufficiently more general than that considered by Multhopp. Perhaps the results of the present paper may also be found interesting by anyone wishing to solve singular integrodifferential equations more general than that considered here by applying the methods described in [6–10] for simple singular integral equations. The critical difficulty encountered in the present paper is the lack of appropriate collocation points along the integration interval. This difficulty will be faced by using the same collocation points used for the numerical solution of singular integral equations. This will become clear in what follows. Finally, a simple application of the method of the present paper will be also made.

2. Description of the method. We consider the singular integrodifferential equation

$$B(x)\varphi(x) + \int_{-1}^{1} \frac{\varphi'(t)}{t-x}\,dt + C(x)\int_{-1}^{1} \frac{\varphi(t)}{t-x}\,dt + \int_{-1}^{1} D(t,x)\varphi'(t)\,dt$$
$$+ \int_{-1}^{1} E(t,x)\varphi(t)dt = f(x), \qquad -1 < x < 1, \tag{1}$$

where $B(x)$, $C(x)$, $D(x)$, $E(x)$ and $f(x)$ are known functions along $(-1, 1)$ and $\varphi(x)$ is an unknown function to be determined. This equation is sufficiently more general than the Multhopp singular integrodifferential equation [2]

$$\frac{\Gamma(x)}{B(x)} - \frac{1}{2\pi} \int_{-1}^{1} \frac{\Gamma'(t)}{t - x} \, dt = f(x), \qquad -1 < x < 1, \tag{2}$$

where $B(x)$ and $f(x)$ are known functions and $\Gamma(x)$ the unknown function.

Evidently, if the known functions in Eq. (1) are regular along the interval $(-1, 1)$, the same will hold also for the unknown function $\varphi(x)$ and its derivative $\varphi'(x)$. As regards the behavior of $\varphi'(x)$ as $x \to +1$ or -1, this depends on the behaviors of the known functions near the same endpoints of the interval $[-1, 1]$. Of course, it is necessary that $\varphi'(x)$ does not present strong power singularities as $x \to \pm 1$. This is necessary for the existence of the first integral in Eq. (1) and will be assumed fulfilled. Then, near $x = \pm 1$, we will have:

$$\varphi'(x) = (1 - x)^{\alpha} \Phi_{\alpha}(x), \qquad \varphi'(x) = (1 + x)^{\beta} \Phi_{\beta}(x), \qquad \alpha, \beta > -1, \tag{3}$$

where α and β are constants and $\Phi_{\alpha}(x)$ and $\Phi_{\beta}(x)$ functions tending to finite limits as $x \to +1$ and -1 respectively. Furthermore, in order that the solution $\varphi(x)$ of Eq. (1) be uniquely determined, two conditions are necessary. Following the development of [2], we assume that

$$\varphi(1) = \varphi(-1) = 0. \tag{4}$$

Analogous conditions can also assure the uniqueness of the solution of Eq. (1), but here only conditions (4) will be considered.

Next, analogously to [2], we assume that $\alpha = \beta = -\frac{1}{2}$ in Eqs. (3) and we approximate $\varphi(x)$ by

$$\varphi(x) = w^*(x) g(x), \tag{5}$$

where

$$w^*(x) = (1 - x^2)^{1/2} \tag{6}$$

and $g(x)$ a new unknown function assumed regular along the whole interval $[-1, 1]$. If this assumption is not true, then the approximation of $\varphi(x)$ by the numerical solution to be obtained will not be very good near $x = \pm 1$. But because of Eqs. (4), this is not of much importance. Furthermore, in this paper we will prefer to replace $g(x)$ in Eq. (5) by

$$h(x) = (1 - x^2) g(x). \tag{7}$$

Then we have:

$$\varphi(x) = w(x) h(x), \tag{8}$$

where:

$$w(x) = (1 - x^2)^{-1/2}. \tag{8a}$$

Evidently, because of Eqs. (4, 8), it is clear that:

$$h(1) = h(-1) = 0. \tag{9}$$

Now, in order to get rid of the derivative of $\varphi(x)$ in Eq. (1), we apply the rule of integration by parts in the integrals containing $\varphi'(x)$. Thus, by defining

$$K(t, x) = D'(t, x) + E(t, x) \tag{10}$$

and taking into account Eqs. (4), we obtain the following equivalent form of Eq. (1):

$$B(x)\varphi(x) + \frac{d}{dx}\int_{-1}^{1} \frac{\varphi(t)}{t-x}\,dt + C(x)\int_{-1}^{1} \frac{\varphi(t)}{t-x}\,dt$$

$$+\int_{-1}^{1} K(t, x)\varphi(t)dt = f(x), \qquad -1 < x < 1. \tag{11}$$

Now we consider the Gauss-Chebyshev numerical integration rule [3] with n abscissae:

$$\int_{-1}^{1} w(t)h(t)dt \simeq A \sum_{i=1}^{n} h(t_i), \tag{12}$$

where the weights A are given by

$$A = \pi/n \tag{13}$$

and the abscissae t_i by:

$$t_i = \cos\frac{(i-0.5)\pi}{n}, \qquad i = 1, 2, \cdots, n. \tag{14}$$

Similarly, we have to use the Lobatto-Chebyshev numerical integration rule [4] with $(n + 1)$ abscissae. This rule, for an integrand $h(t)$ satisfying Eqs. (9), takes the form

$$\int_{-1}^{1} w(t)h(t)dt \simeq A \sum_{i=1}^{n-l} h(y_i), \tag{15}$$

where the abscissae y_i are given now by:

$$y_i = \cos(i\pi/n), \qquad i = 1, 2, \cdots, n-1. \tag{16}$$

By taking further into account the development in [3–5], we can write Eqs. (12) and (15) as

$$\int_{-1}^{1} w(t)\frac{h(t)}{t-x}\,dt \simeq A \sum_{i=1}^{n} \frac{h(t_i)}{t_i - x} + K_{nG}(x)h(x), \tag{17}$$

and

$$\int_{-1}^{1} w(t)\frac{h(t)}{t-x}\,dt \simeq A \sum_{i=1}^{n-1} \frac{h(y_i)}{y_i - x} + K_{nL}(x)h(x), \tag{18}$$

respectively, where the functions $K_{nG}(x)$ and $K_{nL}(x)$ are determined by:

$$K_{nG}(x) = \frac{\pi U_{n-1}(x)}{T_n(x)}, \qquad K_{nL}(x) = \frac{\pi T_n(x)}{(x^2-1)U_{n-1}(x)}, \tag{19}$$

$T_n(x)$ and $U_n(x)$ denoting the Chebyshev polynomials of degree n of the first and the second kind respectively.

Furthermore, by differentiating Eqs. (17) and (18) with respect to x, we obtain:

$$\frac{d}{dx}\int_{-1}^{1} w(t)\frac{h(t)}{t-x}\,dt \simeq A \sum_{i=1}^{n} \frac{h(t_i)}{(t_i - x)^2} + K_{nG}'(x)\varphi(x) + K_{nG}(x)\varphi'(x), \tag{20}$$

$$\frac{d}{dx}\int_{-1}^{1} w(t)\,\frac{h(t)}{t-x}\,dt \simeq A \sum_{i=1}^{n} \frac{h(y_i)}{(y_i - x)^2} + K_{nL}'(x)\varphi(x) + K_{nL}(x)\varphi'(x), \tag{21}$$

where

$$K_{nG}'(x) = \frac{\pi[xU_{n-1}(x)T_n(x) - n]}{(1-x^2)T_n^2(x)}, \qquad K_{nL}'(x) = \frac{\pi[(1-x)U_{n-1}(x)T_n(x) - n]}{(1-x^2)U_{n-1}^2(x)} \tag{22}$$

as clearly seen from Eqs. (19). We can also note that for the points y_i determined by Eqs. (16), we have:

$$K_{nG}(y_i) = 0, \qquad K_{nG}'(y_i) = -\pi n/(1 - y_i^2). \tag{23}$$

Similarly, for the points t_i determined by Eqs. (14), we have:

$$K_{nL}(t_i) = 0, \qquad K_{nL}'(t_i) = \pi n/(1 - t_i^2). \tag{24}$$

The first of Eqs. (23) and (24) were already known [3, 4]. The second of these equations is derived here for the first time.

Now, by approximating the integrals in Eq. (11) by using the Gauss-Chebyshev numerical integration rule for regular integrals, Eq. (12), Cauchy-type principal value integrals, Eq. (17), and their derivatives, Eq. (20), and applying the resulting approximate equation to the collocation points y_k, we obtain because of Eqs. (8) and (23):

$$\left[w(y_k)B(y_k) - \frac{\pi n}{1 - y_k^2}\right] h(y_k) + A \sum_{i=1}^{n} \left[\frac{1}{(t_i - y_k)^2} + \frac{C(y_k)}{t_i - y_k} + K(t_i\,, u_k)\right]$$
$$\times\, h(t_i) = f(y_k), \qquad k = 1, 2, \cdots, n-1. \tag{25}$$

In a quite analogous manner, by applying the Lobatto-Chebyshev numerical integration rule, Eqs. (15, 18, 21), to the integrals of Eq. (11) and applying the resulting approximate equation to the collocation points t_i, Eqs. (14), we obtain because of Eqs. (8) and (24):

$$\left[w(t_i)B(t_i) - \frac{\pi n}{1 - t_i^2}\right] h(t_i) + A \sum_{i=1}^{n-1} \left[\frac{1}{(y_k - t_i)^2} + \frac{C(t_i)}{y_k - t_i} + K(y_k\,, t_i)\right]$$
$$\times\, h(y_k) = f(t_i), \qquad i = 1, 2, \cdots, n. \tag{26}$$

Eqs. (25) and (26) constitute a system of $(2n - 1)$ linear algebraic equations with an equal number of unknowns, the values of the unknown function $h(x)$ at the points:

$$x_i = \cos(i\pi/2n), \qquad i = 1, 2, \cdots, 2n-1, \tag{27}$$

that is at both sets of points t_i and y_i. By solving this system of equations we can determine the values of $h(x_i)$ at these points and, further, by using interpolation techniques [15], along the whole interval $[-1, 1]$.

In another way of thinking, the form of the system of equations (25) and (26) permits us to reduce it to a system of only n linear algebraic equations with the unknowns the values of $h(x)$ at the abscissae t_i, or to a system of $(n - 1)$ such equations with the unknowns the values of $h(x)$ at the abscissae y_k. To do so, we have just to replace the values of $h(y_k)$, directly available from Eqs. (25) in terms of $h(t_i)$, in Eqs. (26) or, inversely, the values of $h(t_i)$, directly available from Eqs. (26) in terms of $h(y_k)$ in Eqs. (25). Thus, we obtain a system of only n or $(n - 1)$ linear algebraic equations. In the special case when $C(x) \equiv K(t, x) \equiv 0$, Eq. (11) can also be solved by applying Multhopp's method with n points and by reduction to a system of n linear algebraic equations. Both our method with n abscissae and

TABLE 1.

Numerical results for the solution of Eq. (28) by using Multhopp's method (with n = 9,19 and 29) and the method of this paper (with n = 5, 10 and 15) for several points along the interval (−1, 1).

		n	
Multhopp's method	9	19	29
Present method	5	10	15
x		$\varphi(x)$	
0.000000	0.697024	0.697061	0.697062
0.309017	0.678098	0.678111	0.678112
0.587785	0.615907	0.615979	0.615981
0.809017	0.494775	0.494803	0.494806
0.951057	0.291296	0.291745	0.291750

Multhopp's method with $(2n - 1)$ abscissae can be seen to use in this special case the same abscissae and to produce the same numerical results. Yet we think that in general our method, free from variable transformations, using directly Gaussian high-accuracy numerical integration rules, applicable to a more general class of equations than Multhopp's method, and probably applicable to more complicated cases than that considered in this paper, presents in some cases advantages over Multhopp's method for the numerical solution of singular integrodifferential equations, exactly as the methods of [3, 4] present advantages over the analogous method of [1] for solving singular integral equations.

Finally, we mention that Eqs. (25) and (26) can also be written directly with unknowns the values of the function $g(x)$, Eq. (5), at the abscissae t_i and y_k if Eq. (7) is taken into account.

3. An application. As an application we consider the following simple singular integrodifferential equation:

$$\varphi(x) - \frac{1}{2\pi}\int_{-1}^{1} \frac{\varphi'(t)}{t - x}\, dt = 1, \qquad -1 < x < 1. \tag{28}$$

This equation was solved numerically both by Multhopp's method [2] and by the method of this paper for several values of n. In Table 1 the numerical results obtained by Multhopp's method for n = 9, 19 and 29 are presented for several values of x. The same results were also obtained by the method proposed in this paper for n = 5, 10 and 15 respectively. In this special case it was seen that these two sets of results were identical although the programs used for their derivation were completely different. Evidently, in more complicated cases, Multhopp's method is not applicable any more.

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