ON CONTINUOUS DEPENDENCE IN FINITE ELASTICITY*

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Abstract. We investigate the relation between stability and continuous dependence for a nonlinearly elastic body at equilibrium. We show that solutions of the governing equations that lie in a convex, stable set of deformations depend continuously on the body forces and the surface tractions. The definition of stability used is essentially due to Hadamard.

1. Introduction. In this note we consider the relation between stability and continuous dependence for a nonlinearly elastic body at equilibrium. We show that solutions of the governing equations that lie in a convex, uniformly Hadamard-stable set of deformations depend continuously on the body forces and surface tractions.

Gurtin and Spector [1] have shown that Hadamard stability is also sufficient for uniqueness of solutions of the governing equations. Thus uniqueness and continuous dependence follow from the same assumptions.

Hadamard stability of a set Ω of deformations is the requirement that for some $\alpha > 0$

$$\int_{\mathscr{B}} \nabla u \cdot A(\nabla f)\nabla u \geq \alpha \|\nabla u\|^2_{L^2(\mathscr{B})}$$

for every deformation f in Ω and every variation u. Here A is the elasticity tensor, the derivative of the stress response function with respect to the deformation gradient, while $\mathscr{B}$ is the region of space occupied by the body in a fixed reference configuration.

2. The response function. Stability. We consider a body $\mathscr{B}$ and we identify $\mathscr{B}$ with the properly regular[1] region of $\mathbb{R}^3$ it occupies in a fixed reference configuration. Further, we denote by $\mathscr{D}$ and $\mathscr{S}$ complementary subsets of the boundary $\partial\mathscr{B}$ with $\mathscr{D}$ *non-empty* and relatively open.

A **deformation** f (of $\mathscr{B}$) will be a member of the space

$$\text{Def} = \{f \in C^1(\mathscr{B}, \mathbb{R}^3): \det \nabla f > 0\},$$

while a **variation** u will be a smooth function $u: \mathscr{B} \to \mathbb{R}^3$ which satisfies

$$u = 0 \text{ on } \mathscr{D}.$$

Here det is the determinant and ∇ the gradient operator in $\mathbb{R}^3$. In particular, ∇f is the tensor field with components $(\nabla f)_{ij} = \partial f_i/\partial x_j$.

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[1] Cf. Fichera [2], p. 351. In particular, $\mathscr{B}$ is compact and has piecewise smooth (C^1) boundary.

We assume that the body is elastic with smooth response function S. Thus

$$S(\nabla f(x), x)$$

gives the (Piola-Kirchhoff) stress at any point $x \in \mathscr{B}$ when the body is deformed by f. Writing F for $\nabla f(x)$ we define the **elasticity tensor** $A(F, x)$: Lin $\to$ Lin by[2]

$$A(F, x) = \partial_F S(F, x). \tag{1}$$

For convenience we write $S(\nabla f)$ and $A(\nabla f)$ for the fields on $\mathscr{B}$ with values $S(\nabla f(x), x)$ and $A(\nabla f(x), x)$ respectively.

DEFINITION [1]. A set $\Omega \subset$ Def is (uniformly) **Hadamard-stable** or simply **H-stable** if for some $\alpha > 0$

$$\int_{\mathscr{B}} \nabla u \cdot A(\nabla f)\nabla u \geq \alpha \|\nabla u\|^2_{L^2(\mathscr{B})} \tag{2}$$

for all $f \in \Omega$ and variations u, in which case we write stab(Ω) for the largest α with this property.

In the above inequality

$$\|\nabla u\|^2_{L^2(\mathscr{B})} = \int_{\mathscr{B}} \nabla u \cdot \nabla u$$

where for any T, $V \in$ Lin,

$$V \cdot T = \sum_{i,j} V_{ij} T_{ij}.$$

We will also have occasion to use the L^2-norm of vector fields over $\mathscr{B}$ and $\mathscr{S}$;

$$\|u\|^2_{L^2(\mathscr{B})} = \int_{\mathscr{B}} u \cdot u, \qquad \|u\|^2_{L^2(\mathscr{S})} = \int_{\mathscr{S}} u \cdot u.$$

Remark. The above notion of stability is essentially due to Hadamard[3] [3, p. 252]. For a further discussion of stability cf. [1] and [4].

Gurtin and Spector [1] have shown that there is a neighborhood of the reference configuration which is uniformly H-stable if either:

(a) the reference configuration is positive and natural[4]; or

(b) $\mathscr{D} = \partial\mathscr{B}$ and the reference configuration is homogeneous and strongly elliptic.

3. The mixed problem. Continuous dependence. The mixed problem (with dead loading) consists in finding a deformation f that satisfies:

(i) *the equation of virtual work*

$$\int_{\mathscr{B}} S(\nabla f) \cdot \nabla u = \int_{\mathscr{S}} s \cdot u + \int_{\mathscr{B}} b \cdot u \tag{3}$$

for all variations u; and

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[2] We use Lin to denote the space of linear transformations from $\mathbb{R}^3$ into $\mathbb{R}^3$.

[3] Hadamard stability is a static "criterion" for stability and its precise relation to dynamic stability is unclear. Eq. (2) cannot hold for all deformations, since global uniqueness would then follow.

[4] A natural configuration whose elasticity tensor is positive definite when restricted to symmetric tensors.

(ii) *the displacement boundary condition*

$$f = d \text{ on } \mathscr{D}. \tag{4}$$

Here, $d \in C^0(\mathscr{D}, \mathbb{R}^3)$ is the *surface deformation*, $s \in L^2(\mathscr{S}, \mathbb{R}^3)$ the *surface traction*, and $b \in L^2(\mathscr{B}, \mathbb{R}^3)$ the *body force*. The triplet (d, s, b) will be referred to as the **data**, while a deformation $f \in \text{Def}$ that satisfies (3) and (4) will be called a **solution** corresponding to the data (d, s, b).

Remark. The traction problem ($\mathscr{S} = \partial\mathscr{B}$, $\mathscr{D} = \emptyset$) is excluded from our consideration since stab(Ω) is, in general, zero when $\mathscr{D}$ is empty. For a partial resolution of this problem cf. [5].

Remark. If s and b are continuous and f is a C^2 solution corresponding to the data (d, s, b) then the divergence theorem can be used to show that (3) and (4) are equivalent to

$$\operatorname{div} S(\nabla f) + b = 0 \text{ in } \mathscr{B},$$

$$f = d \text{ on } \mathscr{D}, \qquad S(\nabla f)n = s \text{ on } \mathscr{S}.$$

Note that the definition of stability is independent of the data although it does depend on $\mathscr{D}$, the domain of d.

Our main result is the following

THEOREM. Let Ω be convex[5] and H-stable. Then there exists a constant $\lambda > 0$, which depends only on stab(Ω) and $\mathscr{B}$, such that if f and $\hat{f}$ are solutions that lie in Ω and correspond to data (d, s, b) and $(d, \hat{s}, \hat{b})$ respectively, then

$$\|\hat{f} - f\|_{L^2(\mathscr{B})} \leq \lambda(\|\hat{s} - s\|_{L^2(\mathscr{S})} + \|\hat{b} - b\|_{L^2(\mathscr{B})}).$$

Remark. A direct consequence of our theorem is uniqueness: there is at most one solution corresponding to (d, s, b) in Ω. On the other hand, there may be additional solutions which do not lie in Ω and *may* therefore not depend continuously on the data. (For uniqueness under weaker hypotheses, cf. [1].)

Proof of the theorem. Let $\Omega \subset \text{Def}$ be convex and H-stable. Let $u = \hat{f} - f$. Then $\hat{f} = f$ on $\mathscr{D}$, so that $u = 0$ on $\mathscr{D}$ and u is a variation; hence (3) yields

$$\int_{\mathscr{B}} \nabla u \cdot [S(\nabla \hat{f}) - S(\nabla f)] = \int_{\mathscr{S}} u \cdot (\hat{s} - s) + \int_{\mathscr{B}} u \cdot (\hat{b} - b). \tag{5}$$

Consider the function $g\colon [0, 1] \to \mathbb{R}$ defined by

$$g(\sigma) = \int_{\mathscr{B}} \nabla u \cdot S(\nabla f + \sigma \nabla u).$$

By (1)

$$g'(\sigma) = \int_{\mathscr{B}} \nabla u \cdot A(\nabla f + \sigma \nabla u) \nabla u.$$

Hence, by the mean value theorem, $g(1) - g(0) = g'(\xi)$ at some $\xi \in (0, 1)$; so that

$$\int_{\mathscr{B}} \nabla u \cdot [S(\nabla \hat{f}) - S(\nabla f)] = \int_{\mathscr{B}} \nabla u \cdot A(\nabla f + \xi \nabla u) \nabla u. \tag{6}$$

[5] Convexity here is with respect to the linear structure in $C^1(\mathscr{B}, \mathbb{R}^3)$.

Since Ω is convex, $(f+\xi u)\in\Omega$ and therefore (2) gives us

$$\operatorname{stab}(\Omega)\|\nabla u\|^2_{L^2(\mathscr{B})}\le\int_{\mathscr{B}}\nabla u\cdot A(\nabla f+\xi\,\nabla u)\,\nabla u. \tag{7}$$

Next, by the Cauchy–Schwarz inequality,

$$\begin{aligned}\int_{\mathscr{B}} u\cdot(\hat b-b)&\le\|u\|_{L^2(\mathscr{B})}\|\hat b-b\|_{L^2(\mathscr{B})},\\ \int_{\mathscr{S}} u\cdot(\hat s-s)&\le\|u\|_{L^2(\mathscr{S})}\|\hat s-s\|_{L^2(\mathscr{S})}.\end{aligned} \tag{8}$$

Finally, the desired result follows from (5), (6), (7), (8) and the standard inequalities[5]

$$\|u\|_{L^2(\mathscr{B})}\le\lambda_1\|\nabla u\|_{L^2(\mathscr{B})},$$
$$\|u\|^2_{L^2(\mathscr{S})}\le\lambda_1(\|u\|^2_{L^2(\mathscr{B})}+\|\nabla u\|^2_{L^2(\mathscr{B})}),$$

where $\lambda_1>0$ depends only on $\mathscr{B}$.

Remark. It is clear from the proof of the theorem that the L^2-norm of $(\hat f-f)$ can be replaced by the H^1-norm:

$$\|u\|^2_{H^1(\mathscr{B})}=\|u\|^2_{L^2(\mathscr{B})}+\|\nabla u\|^2_{L^2(\mathscr{B})}.$$

In addition, if we replace (8) by[7]

$$\int_{\mathscr{B}} u\cdot(\hat b-b)\le\|u\|_{H^1(\mathscr{B})}\|\hat b-b\|_{H^{-1}(\mathscr{B})},$$

$$\int_{\mathscr{S}} u\cdot(\hat s-s)\le\|u\|_{H^{1/2}(\mathscr{S})}\|\hat s-s\|_{H^{-1/2}(\mathscr{S})},$$

and use the trace theorem (cf. e.g. [6, p. 277])

$$\|u\|_{H^{1/2}(\mathscr{S})}\le\lambda_2\|u\|_{H^1(\mathscr{B})},$$

we arrive at the stronger result

$$\|\hat f-f\|_{H^1(\mathscr{B})}\le\lambda(\|\hat s-s\|_{H^{-1/2}(\mathscr{S})}+\|\hat b-b\|_{H^{-1}(\mathscr{B})}).$$

References

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[6] R. A. Adams, *Sobolev spaces*, New York, Academic Press (1975)

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[6] The first is the Poincaré inequality, cf., e.g., Fichera [2, p. 274, footnote 17], whose proof with minor modification applies in the present circumstance. The second is the trace theorem, cf., e.g., Fichera [2, p. 353], Adams [6, p. 113].

[7] The negative and fractional norms are defined in the usual manner. Cf., e.g., [6].