

ASYMPTOTIC EXPANSIONS OF EIGENVALUES AND EIGENFUNCTIONS OF RANDOM BOUNDARY-VALUE PROBLEMS*

BY

WILLIAM B. DAY

Auburn University, Auburn, Alabama

Abstract. An asymptotic procedure is developed for calculating the eigenvalues and eigenfunctions of linear boundary-value problems which may contain random coefficients in the operator. The corresponding asymptotic series for the solution of a second-order initial-value problem is shown to be convergent.

1. Introduction. This paper is concerned with the linear eigenvalue problem

$$Lu = \lambda u, \tag{1}$$

$$U_i(u) = 0, i = 1, 2, \dots, 2m \tag{2}$$

where $Lu = \sum_{k=0}^m (-1)^k (a_k u^{(k)})^{(k)}$ and the coefficients $a_k = a_k(x, \omega) = \sum_{i=0}^{m_k} b_{ki}(\omega)x^i$ are polynomials with random variable coefficients. The outcomes ω are elements of Ω in the underlying probability space (Ω, β, P) . We assume that $a_m > 0$ with probability one and that problem (1)–(2) is self-adjoint and positive definite. The boundary conditions $U_i(u)$ occur at the ends of the interval $[a, b]$ which we take without loss of generality to be $[0, 1]$ and are

$$U_i(u) = \sum_{j=0}^{2m-1} \alpha_{ij} u^{(j)}(0) = 0, i = 1, 2, \dots, k,$$

$$U_i(u) = \sum_{j=0}^{2m-1} \beta_{ij} u^{(j)}(1) = 0, i = k + 1, \dots, 2m.$$

For this general problem the eigenfunction u and eigenvalue λ will both be random and can be completely described by knowledge of all of their moments, assuming that all moments of the coefficients a_k are known. In practice one often settles for expressions of the first two moments, the mean and the variance. For the Gaussian process this is a complete description.

In the special case that $a_k(x, \omega) = a_k(\omega)$, for all k , the eigenvalue λ can absorb all the randomness of the problem and the eigenfunction u is deterministic. The eigenfunction u is not random in this case because the eigenvalue λ and the coefficients a_k combine in just the right manner to eliminate the randomness in u . This case was treated in a paper by Soong and Bogdanoff [14].

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The general eigenvalue problem has been formulated and studied in different ways by Boyce [2, 3, 4, 5, 6, 9], Goodwin [6, 9], Haines [10], Purkert and vom Scheidt [12] and others. In the survey paper of Boyce [3], an asymptotic method is mentioned briefly for determining the eigenvalues and eigenfunctions of a randomly perturbed problem. In particular, if the operator L can be written as a sum of a non-random operator L_1 and a random perturbation εL_2 , then one may assume asymptotic expansions for u and λ as

$$u = \sum_{n=0}^{\infty} u_n \varepsilon^n, \quad \lambda = \sum_{n=0}^{\infty} \lambda_n \varepsilon^n.$$

We expand on this asymptotic approach and show how to determine the eigenvalues and eigenfunctions asymptotically for a larger class of problems than may be supposed.

2. Asymptotic formulation. The difficulty in handling random eigenvalue problems results from trying to separate the moments of the eigenvalue and the moments of the eigenfunction. The methods available are generally classified as honest or dishonest, and each has its limitations. Honest methods have an inherent problem of closure (solving an infinite number of coupled equations). Dishonest methods assume that the expectation of the product λu is the product of the expectations of λ and u . Therefore, it is necessary in both cases to make a priori assumptions that cannot always be justified a posteriori or are even false. A different approach to this problem is presented in this paper by considering asymptotic expansions of the eigenvalue and eigenfunction in terms of the moments of the coefficients.

Since the moments of the eigenfunction u and eigenvalue λ are determined solely by the moments of the coefficients b_{ki} , the form of the differential equation and the boundary conditions, our *ansatz* for u and λ should reflect this dependence on the moments of the coefficients b_{ki} explicitly.

Suppose the operator L has only one coefficient $b(\omega)$. Then we may expect u or λ to be described by a series of the form

$$\sum_{i=0}^{\infty} c_i b^i$$

so that the n th moment of u or λ would be expressed as

$$\left\langle \left(\sum_{i=0}^{\infty} c_i b^i \right)^n \right\rangle;$$

i.e., in terms of the moments of b . (We use $\langle . \rangle$ to denote expectation.) But this must be true for each coefficient b_{ki} in the operator L ; therefore, we anticipate that u and λ can be written as

$$u = \sum_{n_0=0}^{\infty} \cdots \sum_{n_N=0}^{\infty} u_{n_0, \dots, n_N} (b_{00})^{n_0} \cdots (b_{m m_m})^{n_N}, \quad (3)$$

$$\lambda = \sum_{n_0=0}^{\infty} \cdots \sum_{n_N=0}^{\infty} \lambda_{n_0, \dots, n_N} (b_{00})^{n_0} \cdots (b_{m m_m})^{n_N}, \quad (4)$$

where the coefficients u_{n_0, \dots, n_N} depend on x and individually satisfy the boundary conditions. The coefficients $\lambda_{n_0, \dots, n_N}$ are constants which are determined by the boundary

conditions. With such expressions, the calculations of the moments of u and λ become straightforward. For example,

$$\langle u \rangle = \sum_{n_0=0}^{\infty} \cdots \sum_{n_N=0}^{\infty} u_{n_0, \dots, n_N} \langle (b_{00})^{n_0} \cdots (b_{m m_m})^{n_N} \rangle$$

and if the b_{ij} 's are independent

$$\langle u \rangle = \sum_{n_0=0}^{\infty} \cdots \sum_{n_N=0}^{\infty} u_{n_0, \dots, n_N} \langle b_{00}^{n_0} \rangle \cdots \langle b_{m m_m}^{n_N} \rangle. \quad (5)$$

We can see why such a series expansion seems plausible by examining a related deterministic second-order initial-value problem. Consider $y'' + p(x)y = 0$, $y(0) = A$, $y'(0) = B$. If y_1 is the solution satisfying $y(0) = 1$ and $y'(0) = 0$ and y_2 is the solution satisfying $y(0) = 0$ and $y'(0) = 1$, then any solution can be written as a linear combination of y_1 and y_2 . If p is a polynomial, then y_1 (and equivalently y_2) can be written as an analytic power series in x . Since this power series is analytic and uniformly convergent for all x , then any rearrangement is also analytic. Indeed, a power series solution obtained by any means must be convergent and must be the same solution found by elementary methods. Hence, if $p(x) = p_0 + p_1 x + \cdots + p_m x^m$, then we can represent y_1 (or y_2) as

$$y_1 = \sum_{k_0=0}^{\infty} \cdots \sum_{k_m=0}^{\infty} c_{k_0, \dots, k_m}(x) (p_0)^{k_0} \cdots (p_m)^{k_m}. \quad (6)$$

On substituting (6) into $y'' + p(x)y = 0$ and setting like powers of p_i to zero, we find initially

$$c''_{0, \dots, 0} = 0$$

or

$$c_{0, \dots, 0} = ax + b$$

which, with the initial conditions, implies $c_{0, \dots, 0} = 1$. Equations for higher-order coefficients may then be obtained in terms of lower-order coefficients and ultimately in terms of $c_{0, \dots, 0}$. They will all be higher powers of x . In fact, the powers of x can be seen to increase (because of two integrations) as $(2 + i)$, $i = 0, 1, \dots, m$. In general,

$$c_{k_0, k_1, \dots, k_m} = qx^{2k_0 + 3k_1 + \cdots + (m+2)k_m} \quad (7)$$

for some constant q .

Even though the series solution (6) of the initial-value problem is convergent, it is generally too optimistic to expect the series (3) and (4) for u and λ to be convergent. However, if one identifies each b_{ik} as a perturbation, then the series (3) and (4) are multiple asymptotic expansions as $b_{00}, \dots, b_{m m_m} \rightarrow 0$. The accuracy of such expansions, of course, depends on the "size" of the coefficients b_{ij} , but we show in our examples that good results can generally be obtained by using only the zero- and first-order approximations even when the coefficients b_{ij} are not necessarily small.

While this method has been developed primarily to allow one to obtain easily the moments of u and λ in the case of random coefficients, it is seen that the method will also

provide asymptotic results for the deterministic problem. Since these results are asymptotic results, it may be impossible for a given problem (fixed b_{ij}) to calculate u and λ to a prescribed degree of accuracy; nevertheless, we can obtain reasonable approximations for λ and concrete expressions for u , the latter of which is often omitted by other eigenvalue approximation schemes (for example, [7, 8, 11]).

We shall prove here that these asymptotic results are valid for the eigenvalue problem

$$-y'' + (b_0 + b_1x + b_2x^2)y = \lambda y, \quad y(0) = 0 = y(1).$$

Definition. A series $\sum_{n_1=0}^\infty \sum_{n_2=0}^\infty a_{n_1, n_2} b_1^{n_1} b_2^{n_2}$ is an asymptotic series for y if and only if

$$y = \sum_{n_1=0}^{m_1} \sum_{n_2=0}^{m_2} a_{n_1, n_2} b_1^{n_1} b_2^{n_2} + o(\|(b_1^{m_1}, b_2^{m_2})\|)$$

as $\|(b_1, b_2)\| = (b_1^2 + b_2^2)^{1/2} \rightarrow 0$.

THEOREM. The eigenvalue problem

$$-y'' + (b_0 + b_1x + b_2x^2)y - \lambda y = 0, \quad y(0) = 0 = y(1)$$

has eigenfunctions

$$y(x) = \sum_{n_1=0}^\infty \sum_{n_2=0}^\infty y_{n_1, n_2}(x) b_1^{n_1} b_2^{n_2}$$

and eigenvalues

$$\lambda = \sum_{n_1=0}^\infty \sum_{n_2=0}^\infty \lambda_{n_1, n_2} b_1^{n_1} b_2^{n_2}$$

which are either asymptotic representations of y and λ for small values of b_1 and b_2 or are convergent series.

Proof. On substituting our expansions for y and λ into the differential equation, we find the general equation for $y_{k, i}$ and $\lambda_{k, i}$ to be

$$y''_{k, i} + (\lambda_{0,0} - b_0)y_{k, i} = - \left[\sum_{p=1}^i \lambda_{0, p} y_{k, i-p} + \sum_{j=1}^k \sum_{p=0}^i \lambda_{j, p} y_{k-j, i-p} + x y_{k-1, i} + x^2 y_{k, i-1} \right]$$

for $k \geq 1, i \geq 1$. We also require $y_{k, i}(0) = 0 = y_{k, i}(1)$ for $k \geq 0, i \geq 0$.

Let U_{m_1, m_2} and δ_{m_1, m_2} be the partial sums

$$U_{m_1, m_2} = \sum_{n_1=0}^{m_1} \sum_{n_2=0}^{m_2} y_{n_1, n_2} b_1^{n_1} b_2^{n_2}, \quad \delta_{m_1, m_2} = \sum_{n_1=0}^{m_1} \sum_{n_2=0}^{m_2} \lambda_{n_1, n_2} b_1^{n_1} b_2^{n_2}.$$

If we substitute U_{m_1, m_2} and δ_{m_1, m_2} into our original equation, we have

$$\begin{aligned} & - \sum_{n_1=0}^{m_1} \sum_{n_2=0}^{m_2} y''_{n_1, n_2} b_1^{n_1} b_2^{n_2} \\ & + \sum_{n_1=0}^{m_1} \sum_{n_2=0}^{m_2} x y_{n_1, n_2} b_1^{n_1+1} b_2^{n_2} + \sum_{n_1=0}^{m_1} \sum_{n_2=0}^{m_2} x^2 y_{n_1, n_2} b_1^{n_1} b_2^{n_2+1} \\ & - \left(\sum_{n_1=0}^{m_1} \sum_{n_2=0}^{m_2} \lambda_{n_1, n_2} b_1^{n_1} b_2^{n_2} - b_0 \right) \left(\sum_{n_1=0}^{m_1} \sum_{n_2=0}^{m_2} y_{n_1, n_2} b_1^{n_1} b_2^{n_2} \right) \end{aligned}$$

$$\begin{aligned}
&= - \left[\sum_{n_1=0}^{m_1} \sum_{n_2=0}^{m_2} \left(y''_{n_1, n_2} + (\lambda_{0,0} - b_0) y_{n_1 n_2} \right. \right. \\
&\quad + \sum_{p=1}^{n_2} \lambda_{0,p} y_{n_1, n_2-p} + \sum_{j=1}^{n_1} \sum_{p=0}^{n_2} \lambda_{j,p} y_{n_1-j, n_2-p} \left. \right) b_1^{n_1} b_2^{n_2} \\
&\quad + \sum_{n_1=1}^{m_1} \sum_{n_2=0}^{m_2} x y_{n_1-1, n_2} b_1^{n_1} b_2^{n_2} + \sum_{n_1=0}^{m_1} \sum_{n_2=0}^{m_2} x^2 y_{n_1, n_2-1} b_1^{n_1} b_2^{n_2} \left. \right] \\
&\quad - \left[\sum_{i=1}^{m_2} \lambda_{0,i} \sum_{k=0}^{m_1} \sum_{j=0}^{i-1} y_{k, m_2-j} b_1^k b_2^{m_2+i-j} + \sum_{q=1}^{m_1} \sum_{i=0}^{m_2} \lambda_{q,i} \left(\sum_{p=0}^{m_2-i} \sum_{r=0}^q y_{m_1-r, p} b_1^{q+m_1-r} b_2^{i+p} \right. \right. \\
&\quad \left. \left. + \sum_{k=0}^{m_1} \sum_{j=0}^{i-1} y_{k, m_2-j} b_1^{q+k} b_2^{m_2+i-j} - \sum_{j=0}^{m_2} x y_{m_1, j} b_1^{m_1+1} b_2^j - \sum_{j=0}^{m_1} x^2 y_{j, m_2} b_1^j b_2^{m_2+1} \right) \right] \\
&= 0 - [R_{m_1, m_2}].
\end{aligned}$$

Finally, we note that $R_{m_1, m_2} / \|(b_1^{m_1}, b_2^{m_2})\|$ is composed of terms in the form $b_1^{m_1+t} / \|(b_1^{m_1}, b_2^{m_2})\|$ or $b_2^{m_2+t} / \|(b_1^{m_1}, b_2^{m_2})\|$, where t is a nonnegative integer. But $b_1^{m_1} / \|(b_1^{m_1}, b_2^{m_2})\| \leq 1$ and $b_2^{m_2} / \|(b_1^{m_1}, b_2^{m_2})\| \leq 1$, so that $b_1^{m_1+t} / \|(b_1^{m_1}, b_2^{m_2})\| \leq b_1^t \rightarrow 0$ as $\|(b_1, b_2)\| \rightarrow 0$. Similarly, $b_2^{m_2+t} / \|(b_1^{m_1}, b_2^{m_2})\| \leq b_2^t \rightarrow 0$ as $\|(b_1, b_2)\| \rightarrow 0$. Hence, $R_{m_1, m_2} / \|(b_1^{m_1}, b_2^{m_2})\| \rightarrow 0$ as $\|(b_1, b_2)\| \rightarrow 0$. Thus, our eigenvalue problem is satisfied asymptotically for small b_1 and b_2 by our expansions of y and λ . Should the series be convergent for a particular problem, then the series solutions are exact rather than asymptotic.

The extension to the general eigenvalue problem $Ly = \lambda y$ and general homogeneous boundary conditions is clear.

3. Examples. As noted in Sec. 2, our asymptotic expansions are equally valid for deterministic problems; therefore, to avoid initially the complication of random functions and to compare our expansions with known results, our first example is a deterministic second-order boundary-value problem.

Example 1. Consider

$$y'' + [\lambda - (1 + 9x - 24x^2 + 16x^3)]y = 0, \quad y(0) = 0 = y(1). \quad (8)$$

Here the coefficients 1, 9, 24 and 16 are not small parameters with limiting values of zero as required by an asymptotic analysis; nevertheless, we shall show here that accurate estimates for y and λ are obtained by considering problem (8) as

$$y'' + [\lambda - (e_0 + e_1 x + e_2 x^2 + e_3 x^3)]y = 0$$

and using the asymptotic series

$$\begin{aligned}
y &\sim \sum_{k_0=0}^{\infty} \cdots \sum_{k_3=0}^{\infty} c_{k_0, \dots, k_3}(x) (e_0)^{k_0} (e_1)^{k_1} (e_2)^{k_2} (e_3)^{k_3}, \\
\lambda &\sim \sum_{k_0=0}^{\infty} \cdots \sum_{k_3=0}^{\infty} d_{k_0, \dots, k_3} (e_0)^{k_0} (e_1)^{k_1} (e_2)^{k_2} (e_3)^{k_3}
\end{aligned}$$

before replacing e_0, e_1, e_2 and e_3 by 1, 9, 24 and 16, respectively. The constant term (one in this case) can always be grouped with λ so that we may assume

$$y \sim \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \sum_{k_3=0}^{\infty} y_{k_1 k_2 k_3} (9)^{k_1} (24)^{k_2} (16)^{k_3},$$

$$\lambda \sim \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \sum_{k_3=0}^{\infty} \lambda_{k_1 k_2 k_3} (9)^{k_1} (24)^{k_2} (16)^{k_3}.$$

We shall calculate terms through a first-order approximation only; i.e., y_{000} , y_{100} , y_{010} and y_{001} , with corresponding λ terms. Our four problems are these:

$$y_{000}: y''_{000} + (\lambda_{000} - 1)y_{000} = 0, \quad y_{000}(0) = 0 = y_{000}(1) \quad (9)$$

$$y_{100}: y''_{100} + (\lambda_{000} - 1)y_{100} = (x - \lambda_{100})y_{000}, \quad y_{100}(0) = 0 = y_{100}(1) \quad (10)$$

$$y_{010}: y''_{010} + (\lambda_{000} - 1)y_{010} = (-x^2 - \lambda_{010})y_{000}, \quad y_{010}(0) = 0 = y_{010}(1) \quad (11)$$

$$y_{001}: y''_{001} + (\lambda_{000} - 1)y_{001} = (x^3 - \lambda_{001})y_{000}, \quad y_{001}(0) = 0 = y_{001}(1). \quad (12)$$

Problem (9) has the solution

$$y_{000} = \sin k\pi x, \quad \lambda_{000} = 1 + k^2\pi^2$$

for the k th eigenvalue. Using these zeroth-order approximations in the three first-order problems (10)–(12), we find

$$y_{100} = (x/4\pi^2)\sin \pi x + (x/4\pi - x^2/4\pi)\cos \pi x, \quad \lambda_{100} = \frac{1}{2},$$

$$y_{010} = (-x^2/4\pi^2)\sin \pi x + (-x/6\pi + x^3/6\pi)\cos \pi x, \quad \lambda_{010} = -\frac{1}{3} + \frac{1}{2\pi^2}$$

$$y_{001} = \left(\frac{-3}{8\pi^4} + x^3/4\pi^2\right)\sin \pi x + \left[\left(\frac{1}{8\pi} - \frac{3}{8\pi^2}\right)x + 3x^2/8\pi^3 - x^4/8\pi\right]\cos \pi x$$

$$\lambda_{001} = \frac{1}{4} - \frac{3}{4\pi^2}$$

for the first eigenvalue $\lambda^{(1)}$ and eigenfunction $y^{(1)}$ ($k = 1$). Thus

$$\lambda^{(1)} \sim (1 + \pi^2) + \left(\frac{1}{2}\right)(9) + \left(-\frac{1}{3} + \frac{1}{2\pi^2}\right)(24) + \left(\frac{1}{4} - \frac{3}{4\pi^2}\right)(16) \quad (13)$$

$$\lambda^{(1)} \sim 11.370.$$

This numerical value for $\lambda^{(1)}$ is correct to two-decimal accuracy according to [8]. This seems somewhat surprising since the coefficients 9, 24 and 16 are not small perturbations. The function $p(x)$, however, shows small variation on the interval $[0, 1]$; indeed, $1 \leq p(x) \leq 2$. Other examples yield nice results for different boundary conditions and more variation for $p(x)$ on $[0, 1]$. Improved values of λ may be found by including second order approximations.

Example 2. Consider a shaft of uniform cross-section which is rotating around its axis with angular frequency s . The equation of equilibrium for the lateral deflection w of the shaft is

$$EI(d^4w/dx^4) = ps^2w. \quad (14)$$

This is identical with the equation for the modes of vibration of a beam of uniform cross-section. We consider p , the mass per unit length, to have the following non-

constant random behavior:

$$p = p_0 + \delta_1 f_1(x) + \delta_2 f_2(x), \quad f_1(x) = x(x-1), \quad f_2(x) = x(x-\frac{1}{2})^2(x-1), \quad (15)$$

where δ_1 and δ_2 are random variables with known moments. A physical interpretation of our expression for p is that p is approximately a constant, p_0 , but contains two "impurities" which are random. The first impurity, f_1 , behaves in a quadratic fashion (zero at $x = 0$ and 1 and a maximum at $x = \frac{1}{2}$). The second impurity, f_2 , behaves in a quartic fashion. We write Eq. (14) as

$$(d^4w/dx^4) - [\lambda + p(x)]w = 0$$

where $\lambda = p_0 s^2/EI$ and

$$\begin{aligned} p(x) &= (s^2/EI)[-(\delta_1 - \delta_2/4)x + (\delta_1 + 1.25\delta_2)x^2 - 2\delta_2 x^3 + \delta_2 x^4] \\ &= \varepsilon_1 x + \varepsilon_2 x^2 + \varepsilon_3 x^3 + \varepsilon_4 x^4. \end{aligned} \quad (16)$$

For the boundary conditions $w(0) = w''(0) = 0 = w(1) = w''(1)$, we can expand w and λ asymptotically as

$$w(x) \sim \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} \sum_{n_4=0}^{\infty} w_{n_1 n_2 n_3 n_4} \varepsilon_1^{n_1} \varepsilon_2^{n_2} \varepsilon_3^{n_3} \varepsilon_4^{n_4}$$

and

$$\lambda \sim \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} \sum_{n_4=0}^{\infty} \lambda_{n_1 n_2 n_3 n_4} \varepsilon_1^{n_1} \varepsilon_2^{n_2} \varepsilon_3^{n_3} \varepsilon_4^{n_4}.$$

In this problem we shall concentrate on finding a first-order asymptotic expression for λ only. In the usual manner, we have the following zero- and first-order problems:

$$\begin{aligned} w_{0000}^{iv} - \lambda_{0000} w_{0000} &= 0, \\ w_{1000}^{iv} - \lambda_{0000} w_{1000} &= (\lambda_{1000} + x) w_{0000}, \\ w_{0100}^{iv} - \lambda_{0000} w_{0100} &= (\lambda_{0100} + x^2) w_{0000}, \\ w_{0010}^{iv} - \lambda_{0000} w_{0010} &= (\lambda_{0010} + x^3) w_{0000}, \\ w_{0001}^{iv} - \lambda_{0000} w_{0001} &= (\lambda_{0001} + x^4) w_{0000}, \end{aligned}$$

where all solutions w_{ijkl} must satisfy the boundary conditions. We have initially

$$w_{0000}^{(n)} = \sin n\pi x, \quad \lambda_{0000}^{(n)} = (n\pi)^4$$

for $n = 1, 2, \dots$. For the first eigenvalue, $\lambda^{(1)} = \lambda$, we have the asymptotic estimate

$$\lambda \sim \lambda_{0000} + \lambda_{1000} \varepsilon_1 + \lambda_{0100} \varepsilon_2 + \lambda_{0010} \varepsilon_3 + \lambda_{0001} \varepsilon_4$$

and we find the first-order λ -approximations by the requirements

$$\begin{aligned} ((\lambda_{1000} + x)\sin \pi x, \sin \pi x) &= 0, & ((\lambda_{0100} + x^2)\sin \pi x, \sin \pi x) &= 0, \\ ((\lambda_{0010} + x^3)\sin \pi x, \sin \pi x) &= 0, & ((\lambda_{0001} + x^4)\sin \pi x, \sin \pi x) &= 0, \end{aligned}$$

where $(f(x), g(x)) = \int_0^1 f(x)g(x) dx$; hence,

$$\begin{aligned}\lambda_{1000} &= -2 \int_0^1 x \sin^2 \pi x dx, & \lambda_{0100} &= -2 \int_0^1 x^2 \sin^2 \pi x dx, \\ \lambda_{0010} &= -2 \int_0^1 x^3 \sin^2 \pi x dx, & \lambda_{0001} &= -2 \int_0^1 x^4 \sin^2 \pi x dx,\end{aligned}$$

since $\int_0^1 \sin^2 \pi x dx = \frac{1}{2}$. Thus, we have

$$\begin{aligned}\lambda_{1000} &= -\frac{1}{2}, & \lambda_{0100} &= -\frac{1}{3} + \frac{1}{2\pi^2}, \\ \lambda_{0010} &= -\frac{1}{4} + \frac{3}{4\pi^2}, & \lambda_{0001} &= -\frac{1}{5} + 1/\pi^2 - \frac{3}{2\pi^4},\end{aligned}$$

so that

$$\lambda \sim \pi^4 + \varepsilon_1\left(-\frac{1}{2}\right) + \varepsilon_2\left(-\frac{1}{3} + \frac{1}{2\pi^2}\right) + \varepsilon_3\left(-\frac{1}{4} + \frac{3}{4\pi^2}\right) + \varepsilon_4\left(-\frac{1}{5} + 1/\pi^2 - \frac{3}{2\pi^4}\right).$$

To complete the description of the eigenvalue, one must compute the moments of λ . But this is straightforward once the moments of ε_i are known. These are found from the expression (16) in terms of δ_1 and δ_2 .

4. Remarks. We note that, from a practical standpoint, any analytic function can be approximated by a finite series of the form $\sum_{n=0}^N a_n x^n$, and the approximation can be made arbitrarily accurate; hence, more general coefficients a_k than polynomials may be allowed. Also one may allow coefficients a_k which are continuous but only piecewise differentiable by writing an asymptotic expansion of y for each piece and requiring y and y' to be continuous at the nondifferentiable points of a_k .

There are two directions for extending this method of asymptotic expansions to more general eigenvalue problems. These generalizations will be considered in a subsequent publication.

In the first case, if the coefficients a_k can be represented by a finite (or possibly infinite) Fourier series, then we anticipate that one could obtain asymptotic expansions for the solution and eigenvalue as described in this paper. Indeed, we have seen that for polynomial coefficients, the solution is another polynomial times a sinusoidal; hence, it appears reasonable that the solution of a problem with Fourier-series coefficients would itself be a Fourier series. We should also be able to consider coefficients composed of sums of Fourier series but for which the coefficients may be only piecewise differentiable.

Finally, the extension to nonlinear boundary value problems would present a fruitful conclusion. Here also, the application of this method seems to be immediate, but comparisons with bifurcation results need to be examined.

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