

ON THREE-DIMENSIONAL GENERALIZATIONS OF THE BOUSSINESQ AND KORTEWEG–DE VRIES EQUATIONS*

By

E. INFELD

Institute of Nuclear Research, Warsaw

Abstract. Three-dimensional generalizations of two different forms of the Boussinesq equation are derived. They are investigated for stability of slowly varying nonlinear wavetrains. The results obtained are then compared with the stability properties following from the full water wave equations. Agreement is found to be good for $h_0 k_0$ (depth times wavenumber) of order one. This is very satisfactory, as the Boussinesq equations are only supposed to be valid for small $h_0 k_0$. In particular, one version of the Boussinesq equation is found to yield instability with respect to *one-dimensional* perturbations for $h_0 k_0 > 1.5$ (as against 1.36 for the full equations). Finally, a similar comparison is performed for the three-dimensional Korteweg–de Vries equation.

1. Introduction. In this paper some model equations for water waves will be derived. They generalize existing model equations (Boussinesq and K–de V) to three dimensions. These existing models were formulated in the late nineteenth century (1871 and 1895), the full equations for free surface waves being difficult to manipulate [1, 2].

We wish to investigate the free surface of an incompresssible fluid covering a flat bottom. The relevant equations are [3] (a suffix denotes partial differentiation):

$$\mathbf{v} = (u, v, w) = \nabla\phi, \qquad h = h_0 + \eta(x, y, t),$$

$$\nabla^2\phi = 0,$$

$$\left.\begin{aligned} \eta_t + \phi_x\eta_x + \phi_y\eta_y - \phi_z = 0 \\ gh + \phi_t + \tfrac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2) = 0 \end{aligned}\right\} \quad \text{at } z = h,$$

$$\phi_z = 0 \qquad\qquad \text{at } z = 0. \tag{1.1}$$

Here h is the local depth, h_0 the average depth, g the gravitational constant, $\mathbf{v}$ the fluid velocity and z the distance from the bottom. The equation of the surface is $z = h(x, y, t)$ and the last three equations are boundary conditions; the first two on the surface (constraints following from the fact that fluid elements move with the surface; and pressure balance on the surface); and the third on the bottom (no motion perpendicular to the bottom). For more details see [3].

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Boussinesq looked at the two-dimensional problem $(\partial/\partial y = 0)$ and found a much simpler set of equations in the small $h_0 k_0$ limit. They are derived in Appendix 1 and take the form:

$$h_t + (uh)_x = 0, \tag{1.2}$$

$$u_t + uu_x + gh_x + \tfrac{1}{3}c_0^2 h_0 h_{xxx} = 0, \qquad c_0^2 = gh_0 . \tag{1.3}$$

Now u is the average of ϕ_x over z between 0 and h.

We will also investigate an alternate form of the Boussinesq equation. In the long wavelength limit (1.3) yields approximately

$$\frac{Du}{dt} + gh_x \equiv u_t + uu_x + gh_x \approx 0. \tag{1.4}$$

We additionally assume

$$\frac{\partial}{\partial x}\frac{Du}{dt} \approx \frac{Du}{dt}$$

(in Appendix 1 a small parameter is introduced and $(Du/dx)_x - Du_x/dt$ is seen to be a higher order-term) and obtain

$$gh_{xx} \approx \frac{-Du_x}{dt}, \tag{1.5}$$

leading to our second form of (1.5):

$$u_t + uu_x + gh_x - (h_0^2/3)(u_{xt} + uu_{xx})_x = 0. \tag{1.6}$$

This equation, together with (1.2), will constitute our alternate form of the Boussinesq equations. Both forms are Galilean-invariant, as is (1.1) (not all forms of Boussinesq found in the literature are). In Appendix 1 (1.2), (1.3), and (1.6) are derived more rigorously and generalized to three dimensions.

An equation similar to (1.2) and (1.6), but accurate to a higher order in the expansion parameter of Appendix 1, was derived by Su and Gardner [4].

When wave motion is further restricted to the positive x direction we can finally obtain the Korteweg–de Vries equation

$$u_t + uu_x + (c_0 h_0^2/6)u_{xxx} = 0$$

(Appendix 2).

Collecting the three models, we have:

BOUSSINESQ I (BI):

$$h_t + hu_x = 0, \tag{1.7}$$

$$u_t + uu_x + gh_x + \tfrac{1}{3}c_0^2 h_0 h_{xxx} = 0. \tag{1.8}$$

BOUSSINESQ II (BII):

$$h_t + (hu)_x = 0, \tag{1.9}$$

$$u_t + uu_x + gh_x - \tfrac{1}{3}h_0^2(u_{xt} + uu_{xx}) = 0. \tag{1.10}$$

KORTEWEG DE VRIES (K–de V):

$$u_t + uu_x + (c_0 h_0/6)u_{xxx} = 0. \tag{1.11}$$

2. Three-dimensional equations. When $\partial/\partial y$ is reinstated in (1.1) a three-dimensional situation is described. However, in all models under consideration the height h is a function of x, y, and t and so from now on we will speak of $(2 + 1)$-dimensional models. The $(2 + 1)$-dimensional extension of (1.7) and (1.9) is

$$h_t + (hu)_x + (hv)_y = 0, \tag{2.1}$$

whereas (1.8) and (1.10) are unaltered (Appendix 1). We are now one equation short, and to close the system we note from the existence of a velocity potential that

$$v_x = u_y. \tag{2.2}$$

Finally, in Appendix 2 the $(2 + 1)$-dimensional generalization of (1.11) in the water wave context is shown to be

$$u_t + uu_x + \tfrac{1}{6}u_{xxx} + \tfrac{1}{2}v_y = 0, \qquad v_x = u_y, \tag{2.3}$$

where x is measured in units of h_0 and velocities in c_0. This equation has already been derived in the plasma physics context [5, 6], and in solid state theory [7].

In all these models x and y have uneven status. It is tacitly assumed that there will be a basic nonlinear structure moving in the x direction, whereas the modulations will be in the x, y plane.

All our model equations were derived in the long wavelength, shallow water limit $h_0 k_0 \to 0$. However, it would be useful to know for what values of the dimensionless parameter $h_0 k_0$ these equations can be used. Sometimes model equations happen to be valid in regions that exceed what one has a right to expect from the derivation. This will in fact be seen to be the case here, especially for BI. To investigate our equations we will perform stability analyses for all three models and compare the results with those obtained from the full system (1.1) for arbitrary $h_0 k_0$ [8].

3. Stability according to generalized Boussinesq II (BII). The original version of the Boussinesq equation entails some additional complications (Sec. 4), so we will rather perversely begin with a stability analysis for BII. It can be written as a system of first- and second-order differential equations:

BII:

$$h_t + (hu)_x + (hv)_y = 0, \qquad u_t + uu_x + h_x - P_x = 0,$$

$$P = \tfrac{1}{3}(u_{xt} + uu_{xx}), \qquad v_x = u_y, \tag{3.1}$$

where x has been normalized to h_0 and velocities to $(gh_0)^{1/2}$ for convenience. We assume the existence of a stationary nonlinear wave and investigate it for stability with respect to small-amplitude, three-dimensional perturbations. This stationary wave propagates in the x direction ($\partial/\partial y = 0$) and, due to Galilean invariance, we can use (3.1) with $\partial/\partial t = 0$ to describe it. Thus in its own rest coordinate system the wave is given by

$$\frac{\partial hu}{\partial x} = 0, \qquad \frac{\partial}{\partial x}(h + \tfrac{1}{2}u^2 - P) = 0,$$

$$P = \tfrac{1}{3}uu_{xx}. \tag{3.2}$$

The first two equations can be integrated:

$$hu = u_0, \qquad h_0 = 1, \qquad h + \tfrac{1}{2}u^2 - P = 1 + \tfrac{1}{2}u_0^2, \tag{3.3}$$

and these, together with the third, give one equation for u:

$$u_{xx} = 3(u_0 u^{-2} + \tfrac{1}{2}u - u^{-1}(1 + \tfrac{1}{2}u_0^2)) \tag{3.4}$$

(in general there will be two constants of integration, but they can be reduced to one constant u_0 by rescaling; see Appendix 3). We will look for nonlinear solutions to (3.4) in the form

$$u = u_0 + a \cos \sigma + O(a^2)$$

where a is small but finite. The first-order equations in a yield

$$\begin{aligned} &\sigma = k_0 x + \chi, \\ &k_0^2 = 3(u_0^{-2} - 1), \qquad u_0 < 1, \\ &h_1 = -u_0^{-2} u_1. \end{aligned} \tag{3.5}$$

In second order

$$\begin{aligned} u_2 &= (3a^2/4k_0^2)(4u_0^{-3} - u_0^{-1})(1 - \tfrac{1}{3}\cos 2\sigma), \\ h_2 &= (a^2/4u_0^2 k_0^2)([10u_0 - 7]\cos 2\sigma - 3 - 6u_0^{-2}). \end{aligned} \tag{3.6}$$

The third-order expansion of (3.4) is of the form

$$\left(\frac{d^2}{dx^2} + k_0^2\right) u_3 = \text{terms involving } \cos \sigma + \text{nonres.}$$

Thus secular terms will appear in u_3 unless we introduce many space variables and allow χ to be a function of x_2 [9]. This leads to the condition

$$2\frac{d\chi}{dx_2} = (3a^2/8k_0)[3(6u_0^{-4} - u_0^{-2}) + 5(4u_0^{-3} - u_0^{-1})^2/k_0^2]. \tag{3.7}$$

This concludes the calculation of the nonlinear wave structure.

We now linearize (3.1) around our nonlinear wave $u(x)$, $h(x)$:

$$\begin{aligned} &\frac{\partial}{\partial t}\delta h + \frac{\partial}{\partial x}(h\,\delta u + u\,\delta h) + \frac{\partial}{\partial y}(h\,\delta v) = 0, \\ &\frac{\partial}{\partial t}u + \frac{\partial}{\partial x}(u\,\delta u + \delta h - \partial P) = 0, \\ &\delta P = \frac{1}{3}\left[\frac{\partial^2}{\partial x\,\partial t}\delta u + u_{xx}\,\delta u + u\,\delta u_{xx}\right], \\ &\frac{\partial\,\delta u}{\partial y} = \frac{\partial\,\delta v}{\partial x}. \end{aligned} \tag{3.8}$$

This is a system of equations in which the coefficients are periodic functions of x. We know from Floquet's theorem that solutions of the form

$$P(x)e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)},$$

where P is periodic with the same period as u and h, must exist. We will therefore take all perturbed quantities to be of this form and further assume k and ω to be small quantities of order a (this is equivalent to assuming slow variations of the nonlinear wave). Next we will look for resonances that arise in higher orders when the solutions up to a given order are multiplied by the coefficients of the equations. Physically, these resonances can be viewed as a wave-wave coupling effect involving two perturbed waves and the basic nonlinear wave. The procedure is well known [10, 11], and only an outline will be given here. Assume

$$\begin{aligned} \omega &= \omega_1 + \omega_2 + \cdots \qquad \omega_n \sim a^{n-p}k^p, \\ \mathbf{k} &= (k_x, k_y) = k(\cos\theta, \sin\theta), \\ \delta u(x,t) &= e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}[\delta u_0(x) + \delta u_1(x) + \cdots], \\ \delta h(x,t) &= e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}[\delta h_0(x) + \delta h_1(x) + \cdots]. \end{aligned} \tag{3.9}$$

Eq. (3.8) is solved in lowest order by

$$\delta u_0 = Ae^{i\sigma} + Be^{-i\sigma} \equiv \hat{A} + \hat{B}, \qquad \delta v_0 = 0, \qquad \delta h_0 = -u_0^{-1}\,\delta u_0.$$

The first-order form of (3.8) is

$$\frac{\partial}{\partial x}(\delta u_1 + u_0\,\delta h_1) = i\omega_1\,\delta h_0 - \frac{\partial}{\partial x}(h_1\,\delta u_0 + u_1\,\delta h_0), \tag{3.10}$$

$$\frac{\partial}{\partial x}(u_0\,\delta u_1 + \delta h_1 - \delta P_1) = -\frac{\partial}{\partial x}(u_1\,\delta u_0) + i\omega_1\,\delta u_0, \tag{3.11}$$

$$\delta P_1 - \tfrac{1}{3}u_0\,\delta u_{1xx} = \tfrac{1}{3}(i\omega_1\,\delta u_{0x} + u_{1xx}\,\delta u_0 + u_1\,\delta u_{0xx} + 2ik_x u_0\,\delta u_{0x}), \tag{3.12}$$

$$\frac{d}{dx}\,\delta v_1 = ik_y\,\delta u_0. \tag{3.13}$$

Secular terms will be avoided if

$$\omega_1 = k_x k_0^2 u_0^3/3$$

and the first-order solutions are easily found to be

$$\begin{aligned} \delta u_1 &= (3/u_0^2k_0^2)(C_1 - u_0C_2 + [4u_0^{-1} - u_0]X + u_0^{-2}k_0^{-2})(u_0 - 4u_0^{-1})(\hat{A}\hat{\alpha} + B\hat{\beta}), \\ \delta h_1 &= C_1/u_0 - (\omega_1/u_0^2k_0)(\hat{A} - \hat{B}) + 2u_0^{-2}(X + \hat{\alpha}\hat{A} + \hat{\beta}\hat{B}) - u_0^{-1}\,\delta u_1, \\ \delta v_1 &= (k_y/k_0)(\hat{A} - \hat{B}) + C_3, \end{aligned}$$

where $X = \alpha A + \beta B$ and $u_1 = a\cos\sigma \equiv \hat{\alpha} + \hat{\beta}$. The constants C_1, C_2 and C_3 are obtained upon integrating (3.10), (3.11), and (3.13) respectively. To obtain the values of these constants we must write out the second-order versions of (3.10), (3.11), and (3.13) (Appendix 4) and again demand that secular terms vanish. The calculation is simplest for the second-order version of (3.13):

$$\frac{\partial}{\partial x}\,\delta v_2 = -ik_x\,\delta v_1 + ik_y\,\delta u_1. \tag{3.14}$$

Upon averaging over a period we obtain, from the periodicity of δv_2,

$$\langle \delta v_1 \rangle = C_3 = \tan\theta \langle \delta u_1 \rangle = 3\tan\theta u_0^{-2} k_0^{-2} [C_1 - u_0 C_2 + (4u_0^{-1} - u_0)X]. \quad (3.15)$$

The consistency conditions obtained from the second-order version of (3.10) and (3.11) are coupled and involve δv_1. With the aid of (3.15) they yield

$$C_1 = [u_0^{-2} - 2u_0^{-3} - 2u_0 + 3(\tan^2\theta) u_0^{-3} k_0^{-2}(1 - 4u_0^{-2})]X/\Delta, \quad (3.16)$$

$$C_2 = (2 + 2u_0^{-2} - u_0^2)X/\Delta, \qquad \Delta = 1 + u_0^2 + u_0^{-2} + 3(\tan^2\theta) k_0^{-2} u_0^{-4}, \quad (3.17)$$

and we are now in a position to write out the second-order equations with known right-hand sides (Appendix 4). If we add the first equation, multiplied by $-u_0^{-1}$ and integrated, to the integrated form of the second equation, and then the result to the third, we will obtain an equation for u_2 only, of the form

$$-\frac{1}{3} u_0 \left(\frac{d^2}{dx^2} + k_0^2 \right) \delta u_2 = (a_{11}A + a_{12}B)e^{i\sigma} + (a_{21}A + a_{22}B)e^{-i\sigma}$$

$$+ \text{ terms proportional to } e^{in}, \qquad n \neq \pm 1. \quad (3.18)$$

This time round secular terms will only be avoided if $\det(a_{ij})$ is zero. After a considerable amount of algebra this condition leads to a value for ω_2. Combining the result with the known form of ω_1, we obtain

$$\omega = \omega_1 + \omega_2 = \tfrac{1}{3} k_0^2 u_0^3 k \cos\theta \pm a u_0 k \sqrt{\cos^2\theta - u_0^{-4} k_0^2 \sin^2\theta} \sqrt{-2a_{12}}, \quad (3.19)$$

$$a_{12}(\theta) = \frac{k_0^2 u_0^4 (4u_0^5 - 25u_0^3 + 9u_0 - 13u_0^{-1} - 2u_0^{-3}) + \tan^2\theta (22u_0^{-1} - 11u_0 - 2u_0^3)}{u_0^2 (1 - u_0^2)(1 + u_0^2 + u_0^{-2} + 3\tan^2\theta k_0^{-2} u_0^{-4})}. \quad (3.20)$$

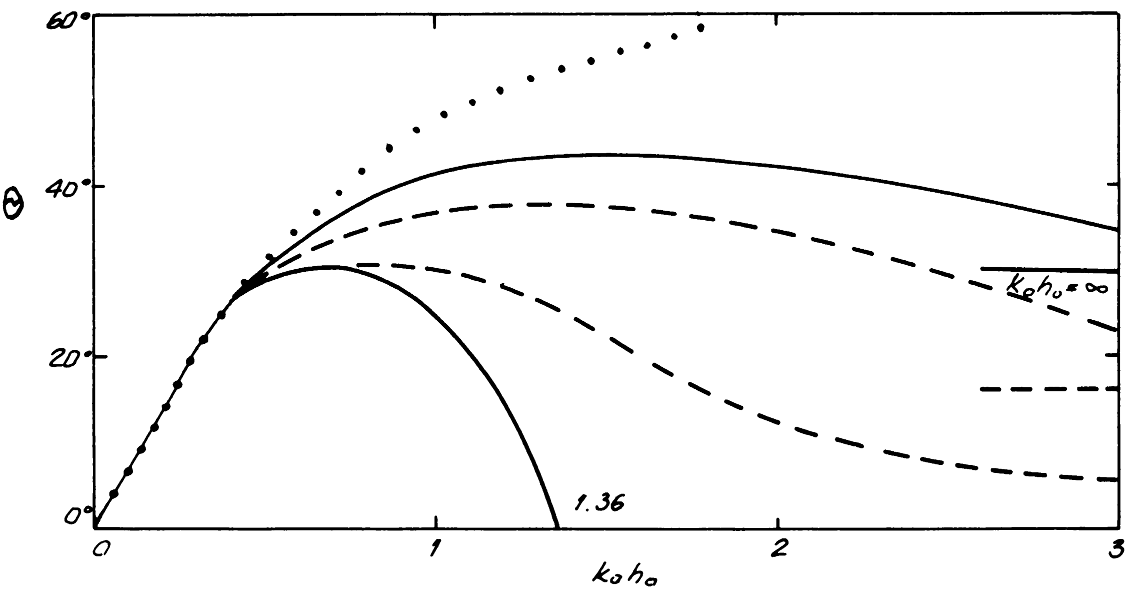

FIG. 1. Stability plot for water waves according to Hayes (solid lines) and BII (broken lines). Differences only become visible for $h_0 k_0 > \frac{1}{2}$, and serious for $h_0 k_0 > 1$. Dots correspond to K–de V.

Thus the value of $\omega(\mathbf{k})$ splits in two, as usually happens in the presence of nonlinear waves [12, 13]. Instability will set in between two critical angles θ given by

$$\tan^2\theta_1 = k_0^2 u_0^4,$$
$$\tan^2\theta_2 = k_0^2 u_0^4 \frac{2u_0^{-3} + 13u_0^{-1} - 9u_0 + 25u_0^3 - 4u_0^5}{3(22u_0^{-1} - 11u_0 - 2u_0^3)}. \tag{3.21}$$

For $k_0 \to 0$, $u_0^2 \to 1$ and these angles merge. They can be drawn as functions of k_0 only, if (3.5) is used in (3.21). They are shown in Fig. 1 (where h_0 has been reinstated). The critical angles θ_1 and θ_2 are compared there with those obtained by Hayes from the full equations (1.1) [8]. Agreement is surprisingly good for $h_0 k_0 < 1$ (after all, this was the expansion parameter!). The main difference between the two lower curves is that BII never gives instability for $\theta = 0$ (one-dimensional perturbations), whereas the full set does when $h_0 k_0 > 1.36$.

4. Stability according to generalized Boussinesq I (BI). Although BI looks simple enough, it furnishes two additional complications when an analysis similar to that of Sec. 3 is performed on it. These complications are most painlessly illustrated by taking the linearized forms of Eqs. (1.8), (2.1) and (2.2):

$$\frac{\partial\,\delta h}{\partial t} + \frac{\partial}{\partial x}(u\,\delta h + h\,\delta u) + \frac{\partial}{\partial y}(h\,\delta v) = 0,$$
$$\frac{\partial\,\delta u}{\partial t} + \frac{\partial}{\partial x}(u\,\delta u) + \frac{\partial\,\delta h}{\partial x} + \frac{1}{3}\frac{\partial^3 h}{\partial x^3} = 0, \qquad \frac{\partial\,\delta v}{\partial x} = \frac{\partial\,\delta u}{\partial y}, \tag{4.1}$$

and solving in the *linear* limit $a = 0$. Now the perturbed quantities are simply proportional to $e^{i(\mathbf{k}\mathbf{x}-\omega t)}$ and a linear dispersion relation is obtained. For $\theta = 0$ this dispersion relation reduces to

$$(\omega - ku_0)^2 = k^2 - \tfrac{1}{3}k^4. \tag{4.2}$$

Thus $\omega \approx 0$ for the following k:

$$k \approx 0, \tag{4.3}$$
$$k \approx \pm k_0, \qquad k_0 = \sqrt{3(1-u_0^2)}. \tag{4.4}$$

Stationary waves (and thus weakly nonlinear waves) can only exist for $k_0 < \sqrt{3}$. This implies that a drawing like Fig. 1 would become meaningless for $h_0 k_0 > \sqrt{3}$.

The second complication is a bit less trivial. By differentiating (4.2) we find that ω_k has three values for which $\omega = 0$:

$$V_1 = u_0 + 1, \qquad V_2 = -u_0 + u_0^{-1}, \qquad V_3 = u_0 - 1 \tag{4.5}$$

(the second value V_2 is degenerate and corresponds to both $k = k_0$ and $k = -k_0$). For one particular value of u_0 $(=\tfrac{1}{2})$, $V_1 = V_2$ and this presents a new possibility of wave-wave coupling between the stationary modes

$$k = \pm k_0, \qquad \omega = 0, \tag{4.6}$$

and the modes

$$\omega \approx V_1 k, \qquad \omega \approx V_2(k \pm k_0). \tag{4.7}$$

This new mode-mode coupling mechanism will in general lead to a third critical angle θ_3 (from the above we expect this angle to be zero for $u_0 = \frac{1}{2}$).

A calculation similar to that of Sec. 3 yields

$$\tan^2 \theta_1 = (2u_0^2 + 1)(1 - u_0^2)u_0^{-4}, \tag{4.8}$$

$$\tan^2 \theta_2 = \frac{(23u_0^4 - 4u_0^2 - 10u_0^6)(1 - u_0^2)}{(2u_0^8 + 5u_0^6 - 4u_0^4)}, \tag{4.9}$$

$$\tan^2 \theta_3 = (u_0^{-2} - 4)(1 - u_0^2), \qquad u_0^2 \le 1. \tag{4.10}$$

Again, these angles can be obtained in terms of $k_0 h_0$ only. For $h_0 k_0 < 1$ they look much like those for the other two models (Hayes and BII; again θ_1 and θ_2 merge and give the right slope at zero). However, the lower stable region will now be bounded on the right by θ_3 and will be topologically like Hayes' case. The critical value of $h_0 k_0$ is 1.5, as can be seen from (4.10) and (4.4). This is not very different from Hayes' value 1.36 (actually known before Hayes [10, 14]). For larger values of $h_0 k_0$, however, the model breaks down and finally becomes meaningless for $h_0 k_0 > \sqrt{3}$.

5. Three-dimensional K–de V. In Appendix 2 the three-dimensional K–de V equation is derived for water waves. An analysis similar to that of Sec. 3 for this equation has already been performed in the plasma physics context [13]. The dispersion relation is, in our notation,

$$\omega = \tfrac{1}{3}h_0^2 k_0^2 \pm \frac{a h_0 k_0}{2\sqrt{3}} \sqrt{(h_0^2 k_0^2 - \tan^2 \theta)^2/(h_0^2 k_0^2 + \tan^2 \theta)}k \cos \theta. \tag{5.1}$$

(This equation was obtained from Eq. (7.4) of the reference with the substitution $\omega/k_x \to \alpha\omega/k_x$, $\tan^2 \theta \to \alpha \tan^2 \theta$, $\alpha = 2\sqrt{3}/(h_0 k_0)^2$. This substitution is sanctioned by the fact that if $u(x, y, t)$ and $v(x, y, t)$ solve (2.3), so will $\alpha u(\alpha^{-1/2}x, \alpha^{-1}y, \alpha^{-3/2}t)$ and $\alpha^{3/2}v(\alpha^{-1/2}x, \alpha^{-1}y, \alpha^{-3/2}t)$.)

Thus $\theta_1 = \theta_2 = \tan^{-1}(h_0 k_0)$ and a single, degenerate curve with the right slope at zero is obtained (dotted line in Fig. 1). However, the instability region for small $h_0 k_0$ is in any case confined to an extremely narrow solid angle. Thus the fact that the width has shrunk to zero for the K–de V equation is perhaps not surprising.

A comparison of all three models—BI, BII, and K–de V—with the full equations shows that whereas the Boussinesq models offer an excellent picture of the stability problem for weakly nonlinear waves when $h_0 k_0 < 1$, K–de V is somewhat simplified and certainly limited to $h_0 k_0 < \frac{1}{2}$.

6. Summary. Generalized $(2 + 1)$-dimensional forms of the Boussinesq equations are seen to furnish very good models for the modulations of weakly nonlinear water waves for $h_0 k_0 < 1$. A $(2 + 1)$-dimensional extension of the Korteweg–de Vries equation, on the other hand, gives a somewhat simplified theory, less exact than Boussinesq, but still qualitatively good for $h_0 k_0 < \frac{1}{2}$.

Note added in proof: A one-dimensional analysis of the Su–Gardner equation, mentioned in Sec. 1 and derived in [4], yields stability for all $h_0 k_0$. In this respect it is thus seen to be no better than our BII.

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Appendix 1. For convenience we normalize all lengths to h_0 and velocities to $(gh_0)^{1/2}$. Then (1.1) takes the form

$$\mathbf{v} = (u, v, w) = \nabla\phi, \qquad \nabla^2\phi = 0, \tag{A1.1, A1.2}$$

$$\left.\begin{aligned} \eta_t + \phi_x\eta_x + \phi_y\eta_y - \phi_z &= 0 \\ \eta + \phi_t + \tfrac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2) &= 0 \end{aligned}\right\} \quad \text{at } z = 1 + \eta, \qquad \begin{aligned}&(A1.3)\\&(A1.4)\end{aligned}$$

$$\phi_z = 0 \qquad\qquad \text{at } z = 0. \tag{A1.5}$$

We next stretch the coordinates x, y, t according to the scheme

$$\xi = \varepsilon^{1/2}x, \qquad \rho = \varepsilon y, \qquad \tau = \varepsilon^{1/2}t, \tag{A1.6}$$

and assume amplitudes of waves and/or solitons to be small and of the form

$$\eta = \varepsilon\eta^{(1)} + \varepsilon^2\eta^{(2)} + \cdots$$

$$\phi = \varepsilon^{1/2}\phi^{(1)} + \varepsilon^{3/2}\phi^{(2)} + \cdots$$

Eqs. (A1.1), (A1.6), etc. imply

$$u = \varepsilon u^{(1)} + \varepsilon^2 u^{(2)} + \cdots,$$

$$v = \varepsilon^{3/2}v^{(1)} + \varepsilon^{5/2}v^{(2)} + \cdots. \tag{A1.7}$$

The general solution to (A1.2) satisfying (A1.5) is, in the new system,

$$\phi = f + \sum_{m=1}^{\infty} (-)^m \frac{z^{2m}}{(2m)!}\left[\varepsilon\frac{\partial^2}{\partial\xi^2} + \varepsilon^2\frac{\partial^2}{\partial\rho^2}\right]^m f. \tag{A1.8}$$

So far $f(x, y, t)$ is arbitrary. Upon differentiation of (A1.8) by x we obtain

$$\phi_x = u - \frac{\varepsilon}{2}z^2 u_{xx} + O(\varepsilon^2). \tag{A1.9}$$

We must also satisfy (A1.3) and (A1.4). Retaining terms up to and including $\varepsilon^{5/2}$, these two conditions imply

$$\eta_\tau^{(1)} + w_\xi^{(1)} + \varepsilon(\eta_\tau^{(2)} + w_\xi^{(2)} + [\eta^{(1)}w^{(1)}]_\xi - \tfrac{1}{6}w_{\xi\xi\xi}^{(1)} + v_\rho^{(1)}) + O(\varepsilon^2) = 0,$$
$$w_\tau^{(1)} + \eta_\xi^{(1)} + \varepsilon(w_\tau^{(2)} + w^{(1)}w_\xi^{(1)} - \tfrac{1}{2}w_{\xi\tau\tau}^{(1)} + \eta_\xi^{(2)}) + O(\varepsilon^2) = 0,$$
$$w = f_x. \tag{A1.10}$$

Averaging (A1.9) over the depth yields

$$w = \bar{u} + \frac{\varepsilon}{6}\bar{u}_{xx} + O(\varepsilon^2) \tag{A1.11}$$

whereas it will be sufficient to take

$$v = \bar{v} + O(\varepsilon). \tag{A1.12}$$

When (A1.11) and (A1.12) are used in (A1.10) and we go back to the original variables, we obtain after some manipulations

$$h_t + (h\bar{u})_x + (h\bar{v})_y = 0, \qquad \bar{u}_t + \bar{u}\bar{u}_x + h_x + \tfrac{1}{3}h_{xxx} = 0. \tag{A1.13}$$

Finally, to lowest order from (A1.1),

$$\bar{v}_x = \bar{u}_y.$$

This completes the derivation of BI.

The expression

$$\frac{\partial}{\partial x}\frac{Du}{dt} - \frac{Du_x}{dt} = u_x^2 = \varepsilon^3 u_\xi^{(1)} + o(\varepsilon^3) \tag{A1.14}$$

is of third order and so can be neglected. Eq. (A1.13) is thus equivalent to (1.9) and (1.10) under this expansion scheme. (Actually, the unwanted term (A1.14) would enter (1.10) differentiated by x and thus would be of order $\frac{7}{2}$, a full ε above all significant terms.)

Appendix 2. We now assume that the basic nonlinear wave or soliton propagates in the positive x direction with a constant velocity near one ($(gh_0)^{1/2}$ before renormalization). We stretch the coordinates according to the scheme

$$\xi = \varepsilon^{1/2}(x - t), \qquad \tau = \varepsilon^{3/2}t$$

and all other quantities as in Appendix 1. Once again (A1.8) is obtained. Conditions (A1.3) and (A1.4) both give the same equation to lowest order in ε:

$$\eta^{(1)} = u^{(1)} = f_\xi. \tag{A2.1}$$

To close the system we must go to next order:

$$\eta_\tau^{(1)} - \eta_\xi^{(2)} + u_\xi^{(2)} + 2u^{(1)}u_\xi^{(1)} - \tfrac{1}{6}u_{\xi\xi\xi}^{(1)} + v_\rho^{(1)} = 0,$$
$$u_\tau^{(1)} - u_\xi^{(2)} + \eta_\xi^{(2)} + u^{(1)}u_\xi^{(1)} + \tfrac{1}{2}u_{\xi\xi\xi}^{(1)} = 0. \tag{A2.2}$$

Adding these two equations we obtain, in view of (A2.1),

$$u_\tau^{(1)} + \tfrac{3}{2}u^{(1)}u_\xi^{(1)} + \tfrac{1}{6}u_{\xi\xi\xi}^{(1)} + \tfrac{1}{2}v_\rho^{(1)} = 0. \tag{A2.3}$$

Finally,

$$v_{\xi}^{(1)} = u_{\rho}^{(1)} \tag{A2.4}$$

is again obtained from the identity $\phi_{xy} = \phi_{yx}$.

Thus we have obtained the three-dimensional K–de V equation (Kadomtsev–Pitvyashvili equation) as a (2 + 1)-dimensional model for shallow water waves. Complete agreement with (2.3) could be obtained via a trivial renormalization.

Appendix 3. A more general form of the integrated equations in (3.3) would be

$$hu = u_0, \qquad h + \tfrac{1}{2}u^2 - P = T.$$

However, the transformation $h = \alpha h'$, $u = \alpha^{1/2}u'$, $P = \alpha P'$, $u_0 = \alpha^{3/2}u_0'$, where $\alpha^{1/2}$ is the positive root of

$$u_0 X^5 + X^2 - T$$

will lead to

$$h'u' = u_0', \qquad h' + \tfrac{1}{2}u'^2 - P' = 1 + \tfrac{1}{2}u_0'^2.$$

To preserve the form of the remaining equations (3.1) we would need the further transformations $x = x'$, $t = \alpha^{-1/2}t'$, $v = \alpha^{1/2}v'$.

Appendix 4. The second-order equations are:

$$\begin{aligned}\frac{\partial}{\partial x}(\delta u_2 + u_0\,\delta h_2) = \frac{-\partial}{\partial x}&(h_2\,\delta u_0 + u_2\,\delta h_0 + h_1\,\delta u_1 + u_1\,\delta h_1)\\ &- ik_x(\delta u_1 + h_1\,\delta u_0 + u_0\,\delta h_1 + u_1\,\delta h_0)\\ &- ik_y\,\delta v_1 + i\omega_1\,\delta h_1 + i\omega_2\,\delta h_2,\end{aligned} \tag{A4.1}$$

$$\begin{aligned}\frac{\partial}{\partial x}(u_0\,\delta u_2 + \delta h_2 - \delta P_2) = \frac{-\partial}{\partial x}&(u_2\,\delta u_0 + u_1\,\delta u_1)\\ &- ik_x(u_0\,\delta u_1 + u_1\,\delta u_0 + \delta h_1 - \delta P_1)\\ &+ i\omega_1\,\delta u_1 + i\omega_2\,\delta u_0,\end{aligned} \tag{A4.2}$$

$$\begin{aligned}\delta P_2 - \tfrac{1}{3}u_0\,\delta u_{2xx} = \tfrac{1}{3}[&-i\omega_1\,\delta u_{1x} - i\omega_2\,\delta u_{0x} + k_x\omega_1\,\delta u_0 + \delta u_0\,\delta u_{2xx}\\ &+ \delta u_1 u_{1xx} + u_1\,\delta u_{1xx} + u_2\,\delta u_{0xx} + 2u_0\,\delta u_{0xx_2} - u_0 k_x^2\,\delta u_0\\ &+ 2ik_x u_0\,\delta u_{1x} + 2ik_x u_1\,\delta u_{0x}].\end{aligned} \tag{A4.3}$$