ON MONOTONICITY OF THE OPTIMAL STRAIN PATH IN LINEAR VISCOELASTICITY*

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Abstract. We consider the problem of finding the strain path $e(t)$, $0 \le t \le T$ which minimizes the work done by a one-dimensional linear viscoelastic material. The material is assumed to be initially unstrained; the time interval is fixed; and, the final strain is specified. It has previously been shown[1] that this problem has a unique solution which is referred to as the optimal strain path. We prove

(i) The optimal strain path is monotone.

(ii) An estimate of the work done on the optimal strain path.

The first proves a conjecture of Gurtin, MacCamy and Murphy, while the second is an alternative proof to a result of Day.

1. Introduction. We consider a one-dimensional linear viscoelastic body which has been unstrained for all time $t \le 0$. Thus the stress $\sigma(t)$, at time t, is given by

$$\sigma(t) = \int_0^t G(t-s)\dot{e}(s)\,ds, \tag{1}$$

with $e(t)$ the strain, at time t, and G the *relaxation function.*

Let $T > 0$ and $e_T \neq 0$ be prescribed and consider the problem of minimizing the work

$$W(e) = \int_0^T \sigma(t)\dot{e}(t)\,dt \tag{2}$$

over all strain paths in the class

$$S = \{e \in C^1[0, T],\ e(0) = 0,\ e(T) = e_T\}. \tag{3}$$

We shall assume $G \in C^3[0, T]$ and that for all $t \in [0, T]$

$$G(t) > 0, \qquad \dot{G}(t) < 0, \qquad \ddot{G}(t) \ge 0. \tag{4}$$

Gurtin, MacCamy and Murphy [1] have proven that there does not exist a minimizer for W within the class S. We shall therefore consider a weak formulation of the problem. Integrate (1) by parts, using (3); substitute the result into (2); and again integrate by parts to obtain

* Received February 2, 1980. The author wishes to thank Johns Hopkins University for supporting this research.

[1] Gurtin, MacCamy and Murphy [1] prove that the problem has no smooth solution, but that a weak formulation of the problem has a unique solution which is smooth on the open interval $(0, T)$.

PROBLEM W. Minimize

$$W(e) = \frac{1}{2} G(0) e_T^2 + e_T \int_0^T \dot{G}(T - t) e(t)\, dt - \dot{G}(0) \int_0^T e(t)^2\, dt$$
$$- \frac{1}{2} \int_0^T \int_0^T \ddot{G}(|t - s|) e(s) e(t)\, ds\, dt \tag{5}$$

over all $e \in L^2(0, T)$.

We call a solution of problem W an *optimal strain path.*

THEOREM GMM [1]. There exists a unique optimal strain path e. Moreover,

(i) $e \in C^1[0, T]$, but $e(0) \neq 0$, $e(T) \neq e_T$,
(ii) $e(t) + e(T - t) = e_T$ for all $t \in [0, T]$,
(iii) e satisfies the Euler equation

$$e_T \dot{G}(T - t) = 2\dot{G}(0) e(t) + \int_0^T \ddot{G}(|t - s|) e(s)\, ds.$$

See [1] or Theorem 1 for proof.

We now proceed to investigate certain fundamental properties of the optimal path. Specifically, we are interested in whether it can undergo a change in sign and whether it is monotone in time, as conjectured in [1]. We are also interested in obtaining the work estimate for the optimal path which Day [2] obtained for smooth paths.

2. Monotonicity of the optimal path. We first show that the optimal path e cannot undergo a change in sign.

PROPOSITION. If e_T is positive (negative) then the optimal strain path is strictly positive (negative).

Proof. We suppose $e_T > 0$, and we note that the proof is similar when e_T is negative. Let e be the optimal strain path. It is then clear from (4), (5), and GMM(i) that

$$W(|e|) \leq W(e).$$

Since e minimizes W, it follows that e is non-negative. To obtain the strict inequality assume $e(t_0) = 0$ for some $t_0 \in [0, T]$. By Theorem GMM(iii) we then have

$$e_T \dot{G}(T - t_0) = \int_0^T \ddot{G}(|t_0 - s|) e(s)\, ds. \tag{6}$$

The left-hand side of (6) is strictly negative by (4), but the first part of this proposition and (4) also tell us that the right-hand side of (6) is non-negative, a contradiction.

We now prove that the optimal strain path is indeed monotone.

THEOREM 1. If e_T is positive (negative) then the optimal strain path e is an increasing (a decreasing) function of time. Moreover, if $\ddot{G}(t)$ is strictly positive for $t \in [0, T]$ then e is strictly increasing (decreasing).

Proof. Again assume $e_T > 0$ and consider the problem of minimizing

$$H(f) = -e(0)\int_0^T [\ddot{G}(t) + \ddot{G}(T-t)]f(t)\,dt - \dot{G}(0)\int_0^T f(t)^2\,dt$$
$$-\frac{1}{2}\int_0^T\int_0^T \ddot{G}(|t-s|)f(s)f(t)\,ds\,dt$$

over all $f \in C[0, T]$. Following [1], we note that H is the sum of one linear and two quadratic terms. By the lemma in the appendix, the sum of the quadratic terms is non-negative. We therefore conclude that h is a minimizer of H *if and only if* its first variation is zero,

$$\int_0^T \beta(t)\left\{-e(0)[\ddot{G}(t) + \ddot{G}(T-t)] - 2\dot{G}(0)h(t) - \int_0^T \ddot{G}(|t-s|)h(s)\,ds\right\}dt = 0$$

for all $\beta \in C[0, T]$. This is in turn equivalent to the Euler equation

$$-e(0)[\ddot{G}(t) + \ddot{G}(T-t)] = 2\dot{G}(0)h(t) + \int_0^T \ddot{G}(|t-s|)h(s)\,ds. \tag{7}$$

Eq. (7) is a Fredholm integral equation of the second kind. Thus by the Fredholm alternative,[2] Eq. (7) has a unique solution $h \in L^2(0, T)$. Our smoothness assumptions on G and Eq. (7) imply that $h \in C[0, T]$ and therefore that h is the minimizer of H.

It is clear from the proof of the proposition that h is non-negative (since $H(|h|) \le H(h)$). It also follows that if $\ddot{G}(t)$ is strictly positive on $[0, T]$, then so is h. To conclude the proof we show $h = \dot{e}$.

Consider Eq. GMM(iii),

$$e_T\dot{G}(T-t) = 2\dot{G}(0)e(t) + \int_0^T \ddot{G}(|t-s|)e(s)\,ds.$$

If we integrate the integral by parts and take the derivative of the whole equation with respect to t, we arrive at

$$e(T)\ddot{G}(T-t) - e_T\ddot{G}(T-t) - e(0)\ddot{G}(t) = 2\dot{G}(0)\dot{e}(t) + \int_0^T \ddot{G}(|t-s|)\dot{e}(s)\,ds.$$

With the aid of GMM(ii), at $t = 0$, we find that the left-hand side of the above equation is identical to the left-hand side of (7). Thus $\dot{e}$ satisfies Eq. (7), and hence, $\dot{e} = h$ by uniqueness.

3. A work estimate. We now obtain upper and lower bounds for the work done on the optimal strain path.

THEOREM 2. Let e be the optimal strain path. Then

$$\frac{1}{2}G\left(\frac{T}{2}\right)e_T^2 \le W(e) \le \frac{1}{2}\frac{G(0) + G(T)}{2}e_T^2.$$

Remark. If G is linear, $\ddot{G} = 0$, then the inequalities in Theorem 2 are both equalities.

[2] The Fredholm alternative requires that the homogeneous equation, (7) with $e(0) = 0$, have only the zero solution. This follows from the lemma in the appendix.

Proof of Theorem 2. If we multiply GMM(iii) by $e(t)$, integrate over $[0, T]$ with respect to t, and substitute into (5), we arrive at

$$W(e) = \frac{1}{2}\left[G(0)e_T^2 + e_T \int_0^T \dot{G}(T-t)e(t)\,dt\right]. \tag{8}$$

Now, break up the integral in (8) into two intervals: $[0, T/2]$, $[T/2, T]$; and on the second interval use the identity GMM(ii). The change of variables $s = T - t$ then yields

$$W(e) = \frac{1}{2}\left[G\left(\frac{T}{2}\right)e_T^2 + e_T \int_0^{T/2} [\dot{G}(T-s) - \dot{G}(s)]e(s)\,ds\right]. \tag{9}$$

Since (4) tells us $\dot{G}$ is increasing, we know $\dot{G}(T-s) - \dot{G}(s)$ is positive on $[0, T/2]$. The proposition then gives us Day's work estimate

$$W(e) \geq \frac{1}{2} G\left(\frac{T}{2}\right)e_T^2.$$

To obtain the upper bound, use Theorem 1 and GMM(ii), at $t = T/2$, to conclude that

$$e(s) \leq e(T/2) = e_T/2, \qquad \text{for} \quad s \in [0, T/2].$$

If we substitute this inequality into Eq. (9), we arrive at

$$W(e) \leq \frac{1}{2}\left[G\left(\frac{T}{2}\right)e_T^2 + \frac{1}{2}e_T^2 \int_0^{T/2} [\dot{G}(T-s) - \dot{G}(s)]\,ds\right].$$

The result now follows upon integration.

4. Appendix.

LEMMA [1, 3]. Let $f \in C[0, T]$. Then

$$2\dot{G}(0)\int_0^T f(t)^2\,dt + \int_0^T \int_0^T \ddot{G}(|t-s|)f(s)f(t)\,ds\,dt$$
$$= -\frac{1}{2}\int_0^T \int_0^T \dddot{G}(|t-s|)[f(t) - f(s)]^2\,ds\,dt + \int_0^T [\ddot{G}(t) + \ddot{G}(T-t)]f(t)^2\,dt, \tag{10}$$

and if

$$2\dot{G}(0)f(t) + \int_0^T \ddot{G}(|t-s|)f(s)\,ds = 0 \tag{11}$$

for all $t \in [0, T]$, then $f \equiv 0$.

Proof. See [1] to obtain (10). Assume that (11) holds. Multiply (11) by $f(t)$ and integrate over $[0, T]$ to conclude that (10) equals zero. The desired result follows from (4).

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