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SMALL-AMPLITUDE WAVES ON THE SURFACE OF A LAYER OF A VISCOUS LIQUID*

BY

L. CORTELEZZI AND A. PROSPERETTI

Università di Milano

Abstract. We study the initial-value problem posed by the small-amplitude waves on the surface of a layer of a viscous fluid of infinite lateral extent. The problem of the motion of the interface is reduced to an integro-differential equation which is solved by means of the Laplace transform. Explicit numerical results for illustrative cases are presented.

1. Introduction. A classical approach to several fluid-mechanics problems involving the small-amplitude motion of free surfaces consists in the separation of the time variable and in the analysis of the "normal modes" of the system [1, 2]. For several reasons which are discussed, for instance, in [3], this method is not suitable for the study of the related initial-value problem. For this purpose alternative techniques have been developed [3, 4] (or redeveloped) and applied in recent years to several interesting cases [3, 5–8].

In the present study we propose to treat by such methods the initial-value problem posed by the small-amplitude waves on the free surface of a layer of a viscous, incompressible fluid of unlimited lateral extent.

For standing waves, the results reported here are an extension to the case of finite depth of those of [3]. The distinctive feature of this problem with respect to the infinite-depth case is the added energy dissipation that comes from the viscous boundary layer adjacent to the lower boundary.

A secondary purpose of the present study is of a more technical nature and concerns a particular representation of the velocity field which is essentially an extension of Helmholtz's decomposition of a divergenceless field [9]. While this representation has been used more or less explicitly in several previous studies [2, 5–7], it has never been discussed in the detail desirable. Our reasons for discussing this matter are threefold. In the first place, this representation allows considerable generality in the type of waves considered (progressive, standing, two- or three-dimensional, etc.) while keeping the mathematical relations comparatively simple. Secondly, this technique can be extended to geometries other than the plane one considered here (see e.g. [2, 5]). Thirdly, it appears that the method can be useful in other contexts such as computational fluid mechanics in which different representations of divergenceless fields are currently being explored [10].

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2. Mathematical formulation. We consider a layer of a viscous, incompressible fluid of depth h and infinite lateral extent. The undisturbed equilibrium position of the free surface will be taken as the (x, y) plane. The z-axis points upwards against the direction of gravity. We shall denote by $\mathbf{x} = (x, y)$ position vectors in the horizontal planes and by $\mathbf{r} = (x, y, z)$ position vectors in space. If $\mathbf{K}$ denotes the unit vector in the z-direction we then have

$$\mathbf{r} = \mathbf{x} + \mathbf{K}z. \tag{1}$$

We take the free surface to be represented by the equation

$$F(\mathbf{r}, t) \equiv z - \eta(\mathbf{x}, t) = 0, \tag{2}$$

where η is small so as to allow the governing equations and boundary conditions to be linearized. This condition may be formulated more precisely requiring that

$$k|\eta| \ll 1, \qquad |\eta| \ll h,$$

where k is of the order of the dominant wave-number of the surface disturbance.

The motion of the fluid is governed by the (linearized) Navier-Stokes equations,

$$\partial\mathbf{u}/\partial t = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u}, \tag{3}$$

and by the condition of incompressibility,

$$\nabla \cdot \mathbf{u} = 0. \tag{4}$$

In these equations $\mathbf{u} = (u, v, w)$ is the velocity, ρ is the density, $\nu = \mu/\rho$ is the kinematic viscosity, and p is the perturbation pressure, i.e. the total pressure minus the gravitational potential gz.

Taking the curl of (3) gives the following equation for the vorticity $\boldsymbol{\omega}$:

$$\partial\boldsymbol{\omega}/\partial t = \nu\nabla^2\boldsymbol{\omega}, \tag{5}$$

where

$$\boldsymbol{\omega} = \nabla \times \mathbf{u}. \tag{6}$$

The boundary conditions require a vanishing velocity at the bottom of the fluid layer,

$$\mathbf{u} = 0 \qquad \text{on} \quad z = -h, \tag{7}$$

the vanishing of the tangential stresses at the free surface,

$$\tau_{xz} = 0, \qquad \tau_{yz} = 0 \qquad \text{on} \quad z = 0, \tag{8}$$

where

$$\tau_{ij} = \tfrac{1}{2}\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),$$

and the balancing of the normal stresses and surface tension forces on the free surface,

$$p - \rho g\eta + 2\mu\frac{\partial w}{\partial z} = -\sigma\nabla \cdot \mathbf{n} \qquad \text{on} \quad z = 0, \tag{9}$$

σ denoting the surface tension coefficient and $\mathbf{n}$ the unit normal to the free surface directed upward. The kinematic boundary condition must be added at the free surface, namely

$$w = \partial\eta/\partial t \qquad \text{on} \quad z = 0. \tag{10}$$

Notice that the conditions (8, 9, 10) have already been linearized. To the same degree of approximation the unit normal $\mathbf{n}$ is given by

$$\mathbf{n} = \left(-\frac{\partial\eta}{\partial x}, -\frac{\partial\eta}{\partial y}, 1\right). \tag{11}$$

The initial conditions are arbitrary as long as the incompressibility condition (4) is satisfied. For simplicity we shall confine ourselves to the case where the initial vorticity vanishes,

$$\boldsymbol{\omega}(\mathbf{r}, 0) = 0, \tag{12}$$

and the free surface is released with zero velocity at the initial instant $t = 0$,

$$\eta(\mathbf{x}, 0) = \eta_0(\mathbf{x}), \qquad \frac{\partial\eta}{\partial t}(\mathbf{x}, 0) = 0. \tag{13}$$

It will be clear from the following that conditions (12) and (13) determine uniquely the initial velocity and pressure fields. The effect of non-vanishing initial vorticity and free-surface velocity for the infinite-depth case is treated in [11].

3. The vorticity, velocity and pressure fields. It is convenient to introduce the special representation of the velocity field mentioned in Sec. 1 by considering the vorticity $\boldsymbol{\omega}$. Since by definition $\nabla \cdot \boldsymbol{\omega} = 0$, a vector potential $\mathbf{V}$ exists such that

$$\boldsymbol{\omega} = \nabla \times \mathbf{V}. \tag{14}$$

Clearly the velocity $\mathbf{u}$ itself is such a vector potential for $\boldsymbol{\omega}$. However, it is advantageous to use another vector potential which, as will be shown, can be taken to be of the form

$$\mathbf{V} = A\mathbf{K} + \nabla \times (B\mathbf{K}) \tag{15}$$

where $A = A(\mathbf{r}, t)$ and $B = B(\mathbf{r}, t)$ are suitable functions of space and time. The rationale for the decomposition (15) of the vector potential is that, in the linearized approximation, only the component of $\boldsymbol{\omega}$ parallel to the free surface, $\boldsymbol{\omega}_{\parallel}$, can be associated with surface waves. Obviously, if the representation (15) is valid, then

$$\boldsymbol{\omega}_{\parallel} = \nabla \times (A\mathbf{K}), \tag{16}$$

whereas $\boldsymbol{\omega}_{\perp}$, the component of $\boldsymbol{\omega}$ normal to the free surface, is given by

$$\boldsymbol{\omega}_{\perp} = \nabla \times \nabla(B\mathbf{K}). \tag{17}$$

Incidentally, we may observe that in the complete nonlinear formulation the free-surface motion is influenced by $\boldsymbol{\omega}_{\perp}$ only insofar as this vorticity component influences $\boldsymbol{\omega}_{\parallel}$ through the coupling introduced by the nonlinear terms.

To prove (15) we apply Biot and Savart's law to (14) to obtain [12]

$$\mathbf{V}(\mathbf{r}, t) = \frac{1}{4\pi} \int \frac{\boldsymbol{\omega}(\mathbf{r}', t) \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} d^3\mathbf{r}' + \nabla\psi, \tag{18}$$

where ψ is an arbitrary function of space and time and the integral is extended to the volume occupied by the fluid. Writing now in (18) $\boldsymbol{\omega}$ as $\boldsymbol{\omega}_{\parallel} + \boldsymbol{\omega}_{\perp}$ and applying some straightforward manipulations, we find

$$\mathbf{V} = \frac{1}{4\pi} \int \frac{\boldsymbol{\omega}_{\parallel}(\mathbf{r}', t) \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} d^3\mathbf{r}' + \nabla\psi$$
$$- \frac{1}{4\pi} \nabla \times \int |\mathbf{r} - \mathbf{r}'| \boldsymbol{\omega}_{\perp}(\mathbf{r}', t) d^3\mathbf{r}'. \tag{19}$$

Since $\boldsymbol{\omega}_{\perp} = (\boldsymbol{\omega} \cdot \mathbf{K})\mathbf{K}$, it is obvious that the second integral represents a vector directed along the z-axis. Upon comparison with (15) we thus have

$$B = -\frac{1}{4\pi} \int |\mathbf{r} - \mathbf{r}'|^{-1} \mathbf{K} \cdot \boldsymbol{\omega}(\mathbf{r}', t) d^3 r' + \beta(z, t), \tag{20}$$

where the function β (which must depend only on z and t to insure that $\mathbf{K} \times \nabla\beta = 0$) generates a family of gauge transformations. Thus the B-part of the vector potential describes the field $\boldsymbol{\omega}_{\perp}$, but is not uniquely determined by it.

Denote now the first integral in (19) by $\mathbf{C}$ and set

$$\mathbf{A} = \mathbf{C} + \nabla\psi.$$

We now show that it is possible to choose ψ in such a way that $\mathbf{A}$ is directed along the z-axis. To this end note that, by the definition of $\boldsymbol{\omega}_{\parallel}$,

$$(\nabla \times \mathbf{C})_z = \mathbf{K} \cdot \boldsymbol{\omega}_{\parallel} = 0.$$

Because of this property the line integral of $\mathbf{C}$ along a path lying on a z = constant plane depends only on the endpoints of the path. Therefore the relation

$$\psi(P) = \alpha(z, t) - \int_{P_0}^{P} \mathbf{C} \cdot d\mathbf{l},$$

where α is an arbitrary function of its arguments and the point P_0 is chosen in such a way that $\psi(P_0) = \alpha$, leads to a well-defined function which, by construction, is such that $\mathbf{A}$ is indeed directed along the z-axis. This completes our proof of (15).

The equations satisfied by the functions A and B can be obtained by substituting (14) and (15) into the linearized vorticity equation (5) with the result

$$\nabla(\mathscr{D}A) \times \mathbf{K} + \left(\nabla \frac{\partial}{\partial z} - \mathbf{K}\nabla^2\right)\mathscr{D}B = 0, \tag{21}$$

where $\mathscr{D}$ is the diffusion operator given by

$$\mathscr{D} = \frac{\partial}{\partial t} - \nu\nabla^2. \tag{22}$$

The component of the vector equation (21) parallel to the z-axis gives

$$\nabla_{\parallel}^2 \mathscr{D}B = 0, \tag{23}$$

where $\nabla_{\parallel}^2$ denotes the Laplacian operator in the (x, y) planes parallel to the undisturbed free surface,

$$\nabla_{\parallel}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Taking the curl of (21) and projecting on the z-axis gives

$$\nabla_{\parallel}^2 \mathscr{D} A = 0. \tag{24}$$

Eqs. (23), (24) and the requirement that A and B be bounded for $x, y \to \pm\infty$ lead to

$$\mathscr{D} A = f_A(z, t), \qquad \mathscr{D} B = f_B(z, t). \tag{25}$$

Mathematically, the relation between $\mathbf{V}$ and $\mathbf{u}$ is a gauge transformation of the type

$$\mathbf{V} = \mathbf{u} + \nabla\varphi. \tag{26}$$

Thus, from (14) and (26), we obtain the following representation of the velocity field:

$$\mathbf{u} = A\mathbf{K} + \nabla \times (B\mathbf{K}) - \nabla\varphi. \tag{27}$$

The scalar function φ must be determined so that the continuity equation (4) is satisfied. This condition leads to the equation

$$\nabla^2\varphi = \partial A/\partial z. \tag{28}$$

The expression (27) for the velocity field is an extension of Helmholtz's decomposition of a vector field in terms of a vector and a scalar potential [9].

Substitution of (27) into the Navier-Stokes equation (3) gives, by (25),

$$\nabla\left(\frac{p}{\rho} - \mathscr{D}\varphi + f_A z\right) = 0,$$

from which

$$p = \rho\left[\mathscr{D}\varphi - \int_{-h}^{z} f_A(\zeta, t)\, d\zeta + g_0(t)\right]. \tag{29}$$

It will be shown in the next section that the functions f_A, f_B and g_0 can be made to vanish by a suitable gauge transformation.

Turning now to the boundary conditions, we find that the requirement (8) of vanishing tangential stresses at the interface becomes

$$\mu\left[\frac{\partial}{\partial y}\left(\frac{\partial B}{\partial z}\right) + \frac{\partial}{\partial x}\left(A - 2\frac{\partial\varphi}{\partial z}\right)\right] = 0, \qquad \mu\left[-\frac{\partial}{\partial x}\left(\frac{\partial B}{\partial z}\right) + \frac{\partial}{\partial y}\left(A - 2\frac{\partial\varphi}{\partial z}\right)\right] = 0.$$

These relations can be viewed as Cauchy-Riemann relations between the functions $\partial B/\partial z$ and $A - 2\mathscr{D}\varphi/\partial z$. Boundedness for $|\mathbf{x}| \to \infty$ can only be achieved if

$$A - 2\frac{\partial\varphi}{\partial z} = g_1(t) \qquad \text{on} \quad z = 0, \tag{30}$$

$$\frac{\partial B}{\partial z} = g_2(t) \qquad \text{on} \quad z = 0, \tag{31}$$

where, as will be presently shown, g_1 and g_2 can be made to vanish. The kinematic boundary condition at the free surface (10) is satisfied provided that

$$A - \frac{\partial \varphi}{\partial z} = \frac{\partial \eta}{\partial t} \qquad \text{on} \quad z = 0. \tag{32}$$

On the bottom of the fluid layer we must impose the no-slip condition (7). The vanishing of the normal velocity w requires

$$A - \partial\varphi/\partial z = 0 \qquad \text{on} \quad z = -h. \tag{33}$$

The vanishing of the tangential velocity gives the pair of Cauchy-Riemann equations

$$\frac{\partial \mathbf{B}}{\partial y} - \frac{\partial \varphi}{\partial x} = 0, \qquad \frac{\partial \mathbf{B}}{\partial x} + \frac{\partial \varphi}{\partial y} = 0 \qquad \text{on} \quad z = -h,$$

from which, as before,

$$\varphi = h_0(t), \qquad \mathbf{B} = h_1(t) \quad \text{on} \quad z = -h. \tag{34}$$

It will be shown that h_0 and h_1 can also be taken as zero.

4. Gauge transformation and the final mathematical formulation. It was shown in the previous section that the vorticity field is left unaltered by the gauge transformation

$$A(\mathbf{r}, t) = A'(\mathbf{r}, t) + \alpha(z, t), \tag{35}$$

$$\mathbf{B}(\mathbf{r}, t) = \mathbf{B}'(\mathbf{r}, t) + \beta(z, t). \tag{36}$$

In order that the velocity field (27) be left unchanged by this transformation as well, it is necessary to set

$$\varphi(\mathbf{r}, t) = \varphi'(\mathbf{r}, t) + \chi(z, t), \tag{37}$$

where the function χ must have the form

$$\chi(z, t) = \int_{-h}^{z} \alpha(y, t)\, d\zeta + F_0(t). \tag{38}$$

By setting $F_0 = h_0$ and comparing with (34) it is seen that φ' can be taken to vanish at the bottom of the fluid layer. Substitution of (35), (36) into Eqs. (25) shows that A' and $\mathbf{B}'$ can be taken to satisfy homogeneous equations if α and β are chosen to be solutions of

$$\frac{\partial \alpha}{\partial t} - \nu \frac{\partial^2 \alpha}{\partial z^2} = f_A(z, t), \qquad \frac{\partial \beta}{\partial t} - \nu \frac{\partial^2 \beta}{\partial z^2} = f_B(z, t). \tag{39}$$

We are still at liberty to impose arbitrary initial and boundary conditions on these functions. Since we shall take $\boldsymbol{\omega}(\mathbf{r}, 0) = 0$ it is natural to assume also $\alpha(z, 0) = \beta(z, 0) = 0$. Taking $\partial\beta/\partial z = g_2(t)$ on $z = 0$ and $\beta = h_1(t)$ on $z = -h$ leaves homogeneous boundary conditions for $\partial \mathbf{B}'/\partial z$ and $\mathbf{B}'$ on the boundary planes. This choice specifies completely the problem for β leading to a unique solution for this function. For α Eq. (30) suggests setting

$$\alpha(0, t) = -g_1(t),$$

so as to make the corresponding condition homogeneous. A last boundary condition can be imposed on α. This residual freedom can be used to simplify the expression for the perturbation pressure. Indeed, from (38) and (25), after an integration by parts, we find

$$\mathscr{D}\chi = \nu \frac{\partial \alpha}{\partial z}\bigg|_{-h} + \int_{-h}^{z} f_A(y, t)\, d\zeta + \frac{dh_0}{dt}.$$

Substitution of (37) into (29) and use of this relationship shows that the function g_0 can be made to vanish if, on $z = -h$, α satisfies

$$\nu \frac{\partial \alpha}{\partial z}\bigg|_{-h} = \frac{dh_0}{dt} - g_0(t).$$

Thus, the pressure field is simply given by

$$p = \rho\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)\varphi'. \tag{40}$$

The series of gauge transformations just introduced considerably simplifies the mathematical problem formulated in Secs. 2 and 3. Dropping the prime on A, B and φ we can summarize the final formulation as follows

$$\mathscr{D}A = 0, \tag{41}$$

$$\nabla^2\varphi = \frac{\partial A}{\partial z}, \qquad \mathscr{D}B = 0. \tag{42}$$

The boundary conditions on $z = 0$ are

$$\frac{\partial \varphi}{\partial z} = \frac{\partial \eta}{\partial t}, \qquad A = 2\frac{\partial \eta}{\partial t}, \qquad \frac{\partial B}{\partial z} = 0, \tag{43}$$

where use has been made of (30), (32) in the first two. On $z = -h$ we have instead

$$\varphi = 0, \qquad A - \frac{\partial \varphi}{\partial z} = 0, \qquad B = 0. \tag{44}$$

5. Formal solution. We now wish to show that it is possible to express the equation of motion of the free surface and all the other relevant quantities in terms of suitable Green's functions for the problems posed by (41), (42). As is clear from (9), (27), (40) (and as had been anticipated), the B-component of the vorticity does not affect the motion of the free surface and therefore will not be considered further.

In view of the linearity of the problem it is convenient to make use of Fourier transforms in the (x, y) plane. For any quantity $Q(\mathbf{x}, z, t)$ we define

$$\hat{Q}(\mathbf{k}, z, t) = (2\pi)^{-1} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, e^{i\mathbf{k}\cdot\mathbf{x}}\, Q(\mathbf{x}, z, t), \tag{45}$$

where $\mathbf{k} = (k_x, k_y)$. From now on the dependence on $\mathbf{k}$ will be understood and only that on z and t will be indicated explicitly where necessary.

The first step is to solve the problem for φ. Taking the Fourier transform of (42), (43), (44) we find

$$\left(\frac{\partial^2}{\partial z^2} - k^2\right)\hat{\varphi} = \frac{\partial \hat{A}}{\partial z}, \qquad \hat{\varphi}(0, t) = \frac{\partial \hat{\eta}}{\partial t}, \qquad \frac{\partial \hat{\varphi}}{\partial z}(-h, t) = \hat{A}_h, \tag{46}$$

where $\hat{A}_h = \hat{A}(-h, t)$ is the value of $\hat{A}$ at the bottom of the layer and $k = (k_x^2 + k_y^2)^{1/2}$. Consider the Green's function $\hat{G}(z; z')$ satisfying

$$\left(\frac{\partial^2}{\partial z'^2} - k^2\right)\hat{G} = \delta(z - z'), \qquad \frac{\partial \hat{G}}{\partial z'}(z; 0) = \frac{\partial \hat{G}}{\partial z'}(z; -h) = 0. \tag{47}$$

It is a simple matter to show that the solution to (46) can be expressed as

$$\hat{\varphi}(z, t) = \int_{-h}^{0} \hat{G}(z; z')\frac{\partial \hat{A}}{\partial z'}\, dz' - \hat{G}(z; 0)\frac{\partial \hat{\eta}}{\partial t} + \hat{G}(z; -h)\hat{A}_h. \tag{48}$$

The value $\hat{A}_h$ of $\hat{A}$ at the lower rigid boundary can be determined by imposing the vanishing of $\hat{\varphi}$ for $z = -h$.

The Fourier transform $\hat{p}$ of the dynamical pressure p is given, according to (40), by

$$\hat{p} = \left[\frac{\partial}{\partial t} - \nu\left(\frac{\partial^2}{\partial z^2} - k^2\right)\right]\hat{\varphi}. \tag{49}$$

By recalling that, since $\hat{G}(z; z') = \hat{G}(z'; z)$, the Green's function satisfies the same equation in z' as it does in z and using (47), (41), (43), (44), a straightforward computation gives

$$\hat{p}(z, t) = \hat{G}(z; 0)\left[\frac{\partial^2 \hat{\eta}}{\partial t^2} + 2\nu k^2 \frac{\partial \hat{\eta}}{\partial t}\right] - \nu k^2 \hat{G}(z; -h)\hat{A}_h. \tag{50}$$

Observing now that

$$\frac{\partial \hat{w}}{\partial z} = \frac{\partial}{\partial z}\left(\hat{A} - \frac{\partial \hat{\varphi}}{\partial z}\right) = -k^2 \hat{\varphi}, \tag{51}$$

we can substitute (48), (50) into the equation of motion of the free surface to find

$$\hat{G}_0\left(\frac{\partial^2 \hat{\eta}}{\partial t^2} + 4\nu k^2 \frac{\partial \hat{\eta}}{\partial t}\right) - \left(g + \frac{\sigma}{\rho}k^2\right)\hat{\eta} - 3\nu k^2 \hat{G}(0; -h)\hat{A}_h - 2\nu k^2 \int_{-h}^{0} \hat{G}(0; z')\frac{\partial \hat{A}}{\partial z'}\, dz' = 0 \tag{52}$$

where

$$\hat{G}_0 = \lim_{z \to 0} G(z; 0).$$

In writing this equation use has been made of the fact that, in the linearized approximation, $\nabla \cdot \mathbf{n} \simeq -\nabla_{\|}^2 \eta$.

By known properties of the parabolic equation (41), we can express the field $\hat{A}$ in the form

$$\hat{A}(z, t) = \int_0^t \left[2\hat{M}_0(z, t - \tau)\frac{\partial \hat{\eta}(\tau)}{\partial \tau} + \hat{M}_h(z, t - \tau)\hat{A}_h(\tau)\right] d\tau, \tag{53}$$

where

$$\left[\frac{\partial}{\partial t} - \nu\left(\frac{\partial^2}{\partial z^2} - k^2\right)\right]\hat{M}_{0,h} = 0,$$

$$\hat{M}_0(0, t) = \hat{M}_h(-h, t) = \delta(t), \qquad \hat{M}_0(-h, t) = \hat{M}_h(0, t) = 0.$$

The meaning of (53) is that the vorticity at a point z at time t comes from the diffusion of the vorticity produced at the free surface and at the bottom of the fluid layer during the previous motion. Substituting (53) into (52) we obtain the following form for the equation of motion of the free surface:

$$\begin{aligned} &\hat{G}_0\left(\frac{\partial^2\hat{\eta}}{\partial t^2} + 4\nu k^2\frac{\partial\hat{\eta}}{\partial t}\right) - \left(g + \frac{\sigma}{\rho}k^2\right)\hat{\eta} \\ &\quad - \nu k^2\left[3\hat{G}(0, -h)\hat{A}_h + 2\int_0^t \hat{A}_h(\tau)\int_{-h}^0 \hat{G}(0; z)\frac{\partial\hat{M}_h}{\partial z}(z, t-\tau)\right] \\ &\quad\quad - 2\nu k^2\int_0^t\frac{\partial\hat{\eta}}{\partial\tau}\int_{-h}^0 2G(0; z)\frac{\partial\hat{M}_0}{\partial z}(z, t-\tau)\,dz\,d\tau = 0. \end{aligned} \tag{54}$$

In interpreting the physical meaning of the different terms of this equation, it should be recalled that the rate of energy dissipation in a viscous fluid depends on the velocity gradients at the free surface and on the instantaneous vorticity distribution in the fluid [3]. For an initially irrotational motion, as the one of present concern, the vorticity is generated at the boundaries and diffuses into the body of the fluid. Roughly speaking, the last two terms in Eq. (54) describe the diffusion of vorticity from the bottom of the layer and from the free surface, while the second term, proportional to $\partial\hat{\eta}/\partial t$, accounts for the surface dissipation.

In conclusion, let us observe that it will be shown that the quantity $\hat{A}_h$ depends on the surface velocity through a relationship of the form

$$\hat{A}_h(t) = \int_0^t \hat{N}(t-\tau)\frac{\partial\hat{\eta}}{\partial\tau}\,d\tau. \tag{55}$$

Thus, as had been found in the study of related problems [3–8], the equation of motion of the free surface exhibits a "memory" effect of the free surface velocity.

6. Explicit solution. The solution of Eq. (41) for $\hat{A}$ is readily found by means of Laplace transforms. Setting

$$\tilde{A}(\mathbf{k}, z, s) = \mathscr{L}(\hat{A}) = \int_0^\infty e^{-st}\hat{A}(\mathbf{k}, z, t)\,dt,$$

we find

$$\tilde{A} = 2\mathscr{L}\left(\frac{\partial\hat{\eta}}{\partial t}\right)\frac{Sh\lambda(z+h)}{Sh\lambda h} - \tilde{A}_h\frac{Sh\lambda z}{Sh\lambda h},$$

where

$$\lambda = (k^2 + s/\nu)^{1/2}. \tag{56}$$

Recalling the convolution theorem for the Laplace transform and comparing with (53), we have

$$\hat{M}_0 = \frac{1}{h} e^{-k^2 \nu t} \frac{\partial}{\partial z} \theta_4\left(\frac{z+h}{2h} \middle| \frac{i\pi \nu t}{h^2}\right),$$

$$\hat{M}_h = -\frac{1}{h} e^{-k^2 \nu t} \frac{\partial}{\partial z} \theta_4\left(\frac{z}{2h} \middle| \frac{i\pi \nu t}{h^2}\right),$$

where $\theta_4(u \mid v)$ is Jacobi's theta function of the fourth kind.

The Green's function $\hat{G}$ is also readily found to be

$$\hat{G} = -\frac{Chk(z+h)Chkz'}{kShkh}, \qquad z' > z;$$

$$\hat{G} = -\frac{ChkzChk(z'+h)}{kShkh}, \qquad z > z'.$$

The requirement that $\hat{\varphi}(-h, t) = 0$, after substitution of these expression into (48), gives the following result for $\tilde{A}_h$, the Laplace transform of the bottom value of $\hat{A}$:

$$\tilde{A}_h(s) = \frac{1}{k} \frac{2k\lambda\ Shkh - (\lambda^2 + k^2) Sh\lambda h}{\lambda\ Shkh\ Ch\lambda h - k\ Sh\lambda h\ Chkh} \mathscr{L}\left(\frac{\partial \hat{\eta}}{\partial t}\right),$$

which proves (55).

Substituting these results into the equation of motion (54), we finally find

$$\frac{\partial^2 \hat{\eta}}{\partial t^2} + 4\nu k^2 \frac{\partial \hat{\eta}}{\partial t} + \omega_0^2 \hat{\eta} + 2\nu k^2 \int_0^t \hat{K}(t-\tau) \frac{\partial \hat{\eta}}{\partial \tau}\, d\tau = 0, \tag{57}$$

where

$$\omega_0^2 = \left(g + \frac{\sigma}{\rho} k^2\right) kThkh \tag{58}$$

is the angular frequency of the inviscid problem [1] and the Laplace transform $\tilde{K}$ of the kernel $\hat{K}$ is given by

$$\tilde{K}(s) = 2\nu k(k - \lambda Thkh) + \nu \frac{\{4\lambda kShkh[ke^{-\lambda h}(k\ Chkh + \lambda\ Shkh) - (k^2 + \lambda^2)] + (k^2 + \lambda^2)^2 Sh\lambda h\}}{2k\ Chkh[k\ Chkh\ Sh\lambda h - \lambda\ Shkh\ Ch\lambda h]} \tag{59}$$

The integrodifferential equation (57) can be solved to obtain $\tilde{\eta}$, the Laplace transform of the surface elevation $\hat{\eta}$. Recalling that for $t = 0$ we have taken $\partial \hat{\eta}/\partial t = 0$, we find

$$\tilde{\eta}(s) = \frac{1}{s}\left[1 - \frac{\omega_0^2}{s^2 + 4\nu k^2 s + \omega_0^2 + 2\nu k^2 \tilde{K}(s)}\right] \hat{\eta}(0). \tag{60}$$

In general this result cannot be inverted analytically, and one has to make use of numerical methods. We shall present some illustrative numerical results in the next section. Here let us note that, in the manner described in [3, 8], it is possible to obtain from (60) information regarding the spectrum of the problem at hand. More precisely, if all the equations of Sec. 2 are Fourier-transformed and then a time dependence of the type $\exp(-\sigma t)$ is postulated for all time-dependent quantities, the problem becomes an eigenvalue problem. It has been shown in [3, 8] that the discrete eigenvalues σ are given by the roots of the following characteristic equation obtained from the denominator of (60):

$$\sigma^2 - 4\nu k^2\sigma + \omega_0^2 + 2\nu k^2\tilde{K}(-\sigma) = 0. \tag{61}$$

In addition to this point, or discrete, spectrum, the problem also possesses a continuous spectrum made up of all the real, positive values of σ such that

$$\sigma \geq \nu k^2. \tag{62}$$

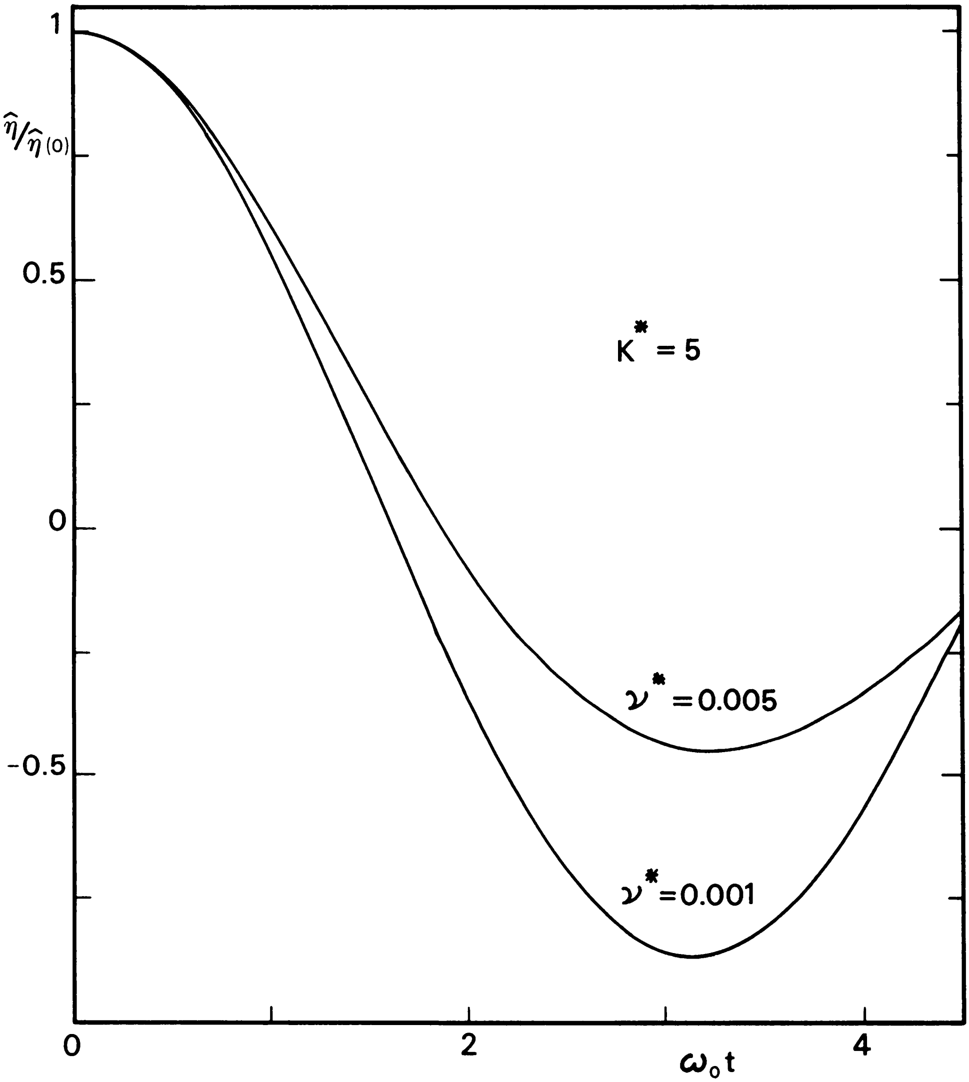

FIG. 1. The dimensionless surface elevation as a function of $\omega_0 t$ for $k^* = kh = 5$ and different values of the dimensionless viscosity $\nu^* = \nu/h^2\omega_0$.

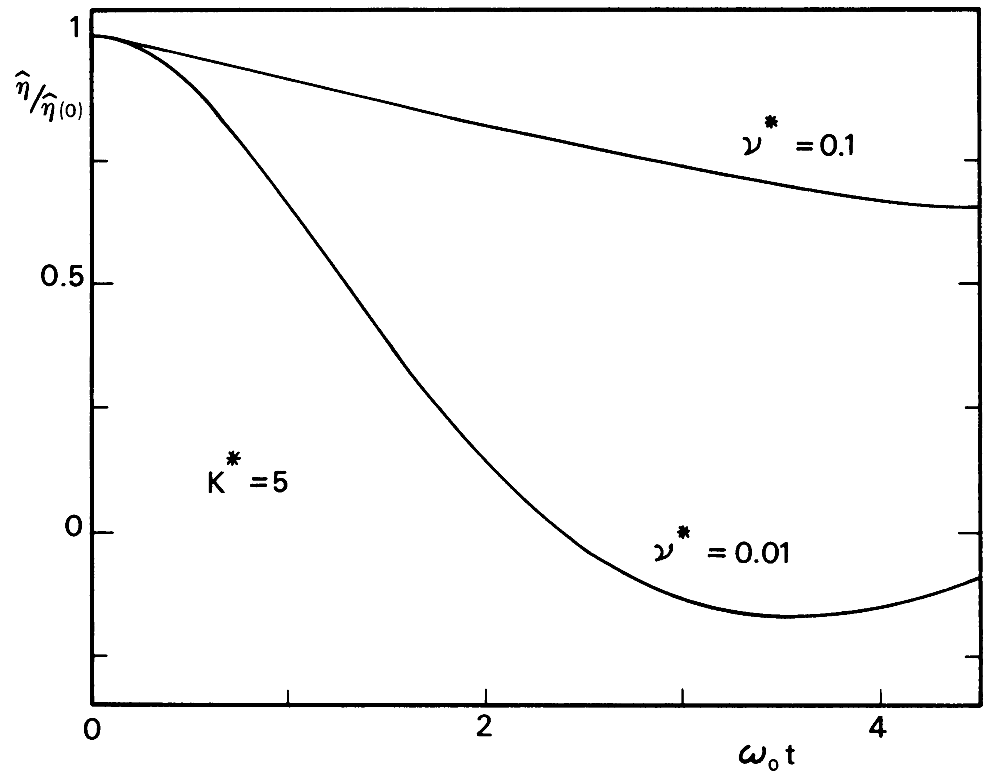

FIG. 2. The dimensionless surface elevation as a function of $\omega_0 t$ for $k^* = kh = 5$ and different values of the dimensionless viscosity $\nu^* = \nu/h^2\omega_0$.

It has been shown in [8] that this feature is connected with the branch cut in the complex s-plane made necessary by the multiple-valuedness of the quantity λ defined by (56).

The asymptotic behavior of $\hat{\eta}(t)$ for large times is proportional to $\exp(-\sigma t)$, where σ is the root of (61) with smallest real part [3, 8]. Thus the surface elevation will exhibit an oscillatory behavior when this root is complex, whereas the motion will be monotonic when this root is real.

7. Results. Before proceeding to show some numerical results we shall consider the limiting case in which the depth of the fluid layer is large with respect to the dominant wave-length so that $\varepsilon = \exp(-2kh) \ll 1$. The case of infinite depth has already been considered in detail in [3]. Here we find a first-order correction valid for large but finite depth. To first order in ε it is straightforward to obtain from (59) the result

$$\tilde{K}(s) = 4\nu^2k^3(k-\lambda) + \varepsilon\left[4\nu^2k^3\lambda - \omega_0^2 + \frac{4\nu^2k(k^2+\lambda^2)^2}{k-\lambda}\right]. \tag{63}$$

The first term in this equation coincides with that given in [3]. The inversion of (60) with $\tilde{K}(s)$ given by (63) is readily achieved [3], with the result

$$\hat{\eta}(t) = \hat{\eta}(0)\left\{1 + \omega_0^2(1-\varepsilon)\sum_{i=1}^{6}\frac{z_i}{\sigma_i}\exp[(z_i^2 - k^2\nu)t] \times \operatorname{erfc}(z_i t^{1/2})\right\}, \tag{64}$$

where $(-z_i)$ are the roots of

$$(z^2 - k^2\nu)(z^4 + 2\nu k^2 z^2 - 4\nu^{3/2}k^3 z + \omega_0^2 + k^4\nu^2) - \varepsilon[4\nu^{1/2}kz^5 + 4\nu k^2 z^4 + 4\nu^{3/2}k^3 z^3 + (8k^4\nu^2 + \omega_0^2)z^2 + 8\nu^{5/2}k^5 z + k^2\nu(\omega_0^2 - 4\nu^2 k^4)] = 0, \quad (65)$$

and

$$\sigma_k = \prod_{\substack{i=1 \\ i \neq k}}^{6} (z_i - z_k).$$

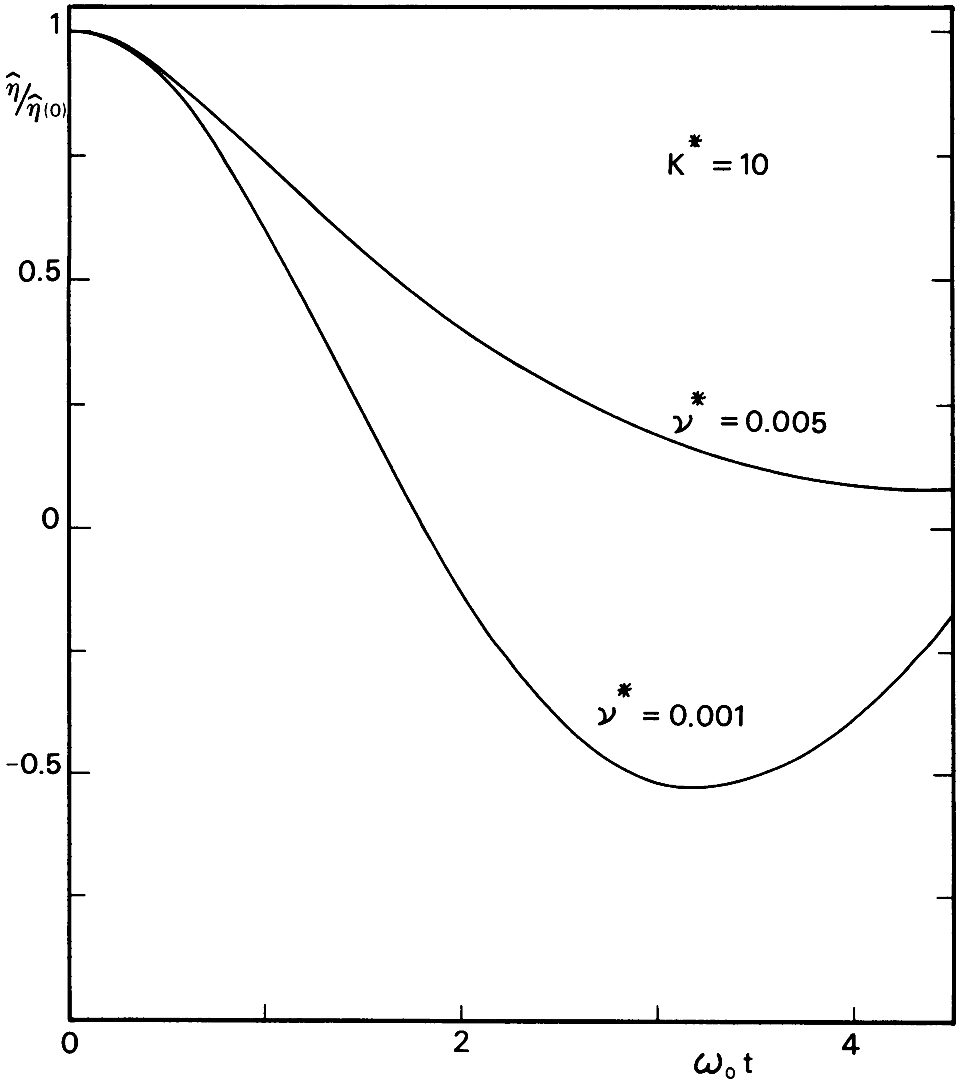

FIG. 3. The dimensionless surface elevation as a function of $\omega_0 t$ for $k^* = kh = 10$ and different values of the dimensionless viscosity $\nu^* = \nu/h^2\omega_0$.

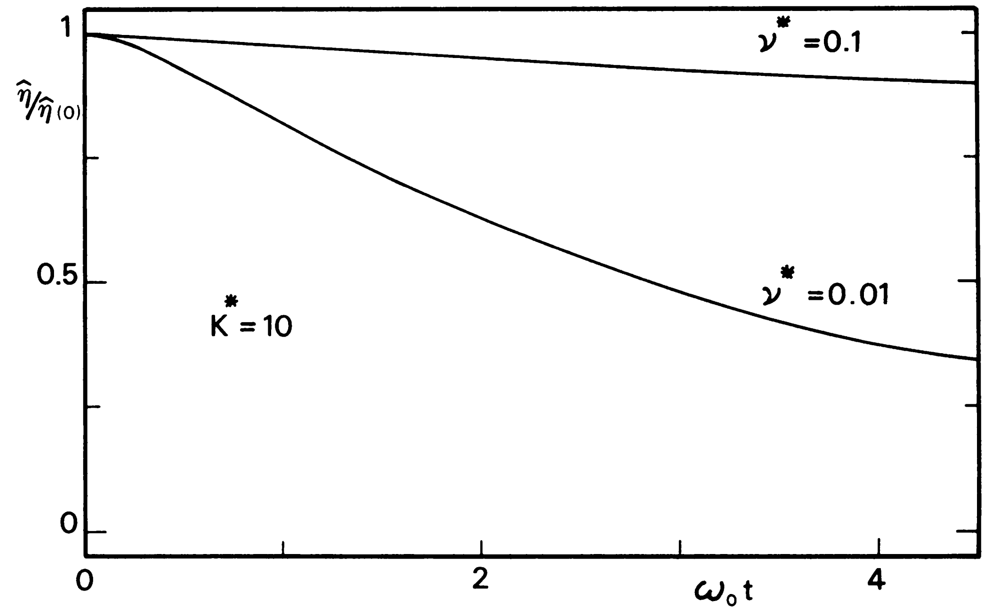

FIG. 4. The dimensionless surface elevation as a function of $\omega_0 t$ for $k^* = kh = 10$ and different values of the dimensionless viscosity $\nu^* = \nu/h^2\omega_0$.

For $\varepsilon = 0$ two roots of (65) are $\pm k\nu^{1/2}$. The remaining quartic coincides with that studied in [3].

The numerical inversion of the complete result (60) has been obtained by the method of [13]. Some examples of the results are shown in Figs. 1–4 for several values of the nondimensional viscosity ν^* and wave number k^*,

$$\nu^* = \nu/h^2\omega_0, \qquad k^* = kh,$$

where ω_0 is given by (58). This nondimensionalization is very convenient to minimize the number of parameters in the problem, but some care is necessary in interpreting the results obtained in this way. For instance, if a comparison for equal k and ν but different values of h, h_1 and h_2, is required, then the appropriate values of the parameters ν^* must be in the ratio $\nu_1^*/\nu_2^* = (h_2/h_1)^2$.

It is clear from the figures that the motion of the free surface is damped and depending on the value of ν can be either oscillatory or aperiodic. It can also be seen that the rate of energy dissipation increases with decreasing values of the depth of the fluid layer.

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