WEIGHT FUNCTIONS FOR A CLASS OF LIAPUNOV FUNCTIONS IN THE PLANE*

By

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Abstract. In this paper we give a class of weight functions which generate Liapunov functions for a general second-order differential system. In the special case of a Lienard equation we give conditions under which these weight functions may be chosen so as to improve certain known estimates of regions of asymptotic stability. The procedure is applied to a well-known equation and new estimates are obtained.

1. Introduction. Two recent papers [1, 2] consider the problem of finding weight functions $\theta(x, y)$ such that functions

$$V_\theta(x, y) = \int_{h(x)}^{y} \theta(x, s)f(x, s)\, ds - \int_0^x \theta[s, h(s)]g[s, h(s)]\, ds \tag{1.1}$$

are Liapunov functions for a differential system $\dot{x} = f(x, y)$, $\dot{y} = g(x, y)$, where $f[x, h(x)] \equiv 0$. These papers show that in many cases there are many admissible values of θ. In [1] it is shown that, in the special case of the van der Pol equation, one can choose θ so as substantially to improve the estimate of the region of asymptotic stability of an isolated critical point over a well-known estimate.

In this paper we generalize the work done in [1]. In particular, we consider weight functions of the form

$$\theta(x, y) = \lambda_1 \int_0^y \phi(s)\, ds + \lambda_2 \int_0^x r(s, 0)\, ds \tag{1.2}$$

for the system

$$\dot{x} = \phi(y), \qquad \dot{y} = r(x, y), \tag{1.3}$$

and show that in a large class of cases, functions V_θ provide Liapunov functions for systems (1.3). In the special case of a Lienard equation we give general conditions under which one can choose constants λ_1, λ_2 in (1.2) so as to improve the estimate of the region of asymptotic stability over well-known estimates that may be obtained by using θ with $\lambda_1 = \lambda_2$. Finally, we apply the method to an equation that appears extensively in the literature [2, 3, 4, 5] and obtain improved estimates.

As was pointed out in [1], an unsolved problem is the general problem of finding θ so as to maximize the estimate of the region of asymptotic stability among admissible values

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of θ. This problem remains unsolved. However, the procedure found in Secs. 3 and 4 of this paper does provide a method for maximization over a subclass of admissible values of θ.

2. A general class of Liapunov functions. We consider the system

$$\dot{x} = \phi(y), \qquad \dot{y} = r(x, y), \tag{2.1}$$

and assume $\phi(y)$ and $r(x, y)$ are of class C' in an open rectangle R containing the origin and that the origin is the only equilibrium point of (1.1) in R. We further assume that $r(x, 0)$ is of class C' for all x. We state the following lemma, which is well known.

LEMMA. Suppose $r(x, 0) \in C'(-\infty, \infty)$ and $r(0, 0) = 0$. Then there exist functions $w(x)$ and $A(x)$, continuous on R, such that $r(x, 0) = xw(x)$ and $-\int_0^x r(s, 0)\,ds = x^2 A(x)$.

Proof. Let $\gamma(t) = r(tx, 0)$. Then $\gamma(1) - \gamma(0) = r(x, 0) = x\int_0^1 r_x(xt, 0)\,dt$. If we define $w(x) = \int_0^1 r_x(xt, 0)\,dt$ and let $A(x) = x^{-2}\int_0^x sw(s)\,ds$ for $x \neq 0$, then $\lim_{x\to 0} A(x) = \lim_{x\to 0} (2x)^{-1}(xw(x)) = w(0)/2 = r_x(0, 0)/2$. This proves the lemma.

Now let

$$P(x) = -\int_0^x r(s, 0)\,ds = x^2 A(x), \qquad I(y) = \int_0^y \phi(s)\,ds. \tag{2.2}$$

We define $\theta(x, y) = \lambda_1 I(y) + \lambda_2 P(x)$ where λ_1 and λ_2 are to be determined. From (1.1) and (1.2) we obtain

$$V_\theta(x, y) = \frac{\lambda_1 I^2(y)}{2} + \lambda_2 I(y)P(x) + \frac{\lambda_2 P^2(x)}{2}. \tag{2.3}$$

In order for V_θ to be positive definite, it is enough to assume that $\lambda_1 > 0$, $\lambda_2 > 0$.

Computing $\dot{V}_\theta$ along the trajectories of (2.1), we obtain

$$\begin{aligned}
\dot{V}_\theta(x, y) &= \phi(y)\lambda_2 I(y)[-r(x, 0)] + \phi(y)\lambda_2[-P(x)]r(x, 0) \\
&\quad + \lambda_1 I(y)\phi(y)r(x, y) + \lambda_2\phi(y)P(x)r(x, y), \\
&= \lambda_1\phi(y)I(y)[r(x, y) - r(x, 0)] + \phi(y)I(y)(\lambda_1 - \lambda_2)r(x, 0) \\
&\quad + \lambda_2\phi(y)[r(x, y) - r(x, 0)]P(x), \\
&= yD(x, y)\left[\frac{\phi(y)I(y)}{y^3}\lambda_1 y^2 + \frac{\phi(y)I(y)w(x)(\lambda_1 - \lambda_2)}{y^2 D(x, y)}xy + \frac{\phi(y)A(x)}{y}\lambda_2 x^2\right],
\end{aligned}$$

where $w(x)$ and $A(x)$ are given in the lemma and

$$D(x, y) = r(x, y) - r(x, 0) = \int_0^y r_y(x, s)\,ds.$$

Note that $yD(x, y) < 0$ for $y \neq 0$. Therefore, if $\dot{V}_\theta$ is to be negative semidefinite, the quantity above in square brackets must be positive semidefinite. Since this quantity is a quadratic form with variable coefficients, it will be positive semidefinite if its discriminant is nonpositive. In this case, for $y \neq 0$, the discriminant equals

$$\frac{\phi^2(y)I(y)}{y^6}\left[\frac{I(y)w^2(x)(\lambda_1 - \lambda_2)^2}{[y^{-1}D(x, y)]^2} - 4y^2\lambda_1\lambda_2 A(x)\right].$$

Thus the discriminant is nonpositive if

$$\frac{(\lambda_1 - \lambda_2)^2}{\lambda_1 \lambda_2} \le \frac{4A(x)[D(x, y)]^2}{w^2(x)I(y)},$$

which is equivalent to

$$\frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_1} \le 2 + \frac{4A(x)[D(x, y)]^2}{w^2(x)I(y)}. \tag{2.4}$$

THEOREM 1. Assume that $r(x, y)$ satisfies

(i) $r(x, 0) \in C'(-\infty, \infty)$,
(ii) $xr(x, 0) < 0$ for all $x \neq 0$,
(iii) $r_y(x, y) < 0$ neighboring (0, 0),
(iv) $r(0, 0) = 0$.

Further assume $\phi(y)$ is continuous neighboring the origin and locally satisfies $y\phi(y) > 0$ *for* $y \neq 0$. Then there exist positive constants λ_1 and λ_2 such that the function

$$V_\theta(x, y) = \frac{\lambda_1 I^2(y)}{2} + \lambda_2 I(y)P(x) + \frac{\lambda_2 P^2(x)}{2}$$

is a Liapunov function for (2.1). That is, $V_\theta(x, y)$ is positive definite and $\dot{V}_\theta(x, y) \le 0$ neighboring the origin.

Proof. Choose $\lambda_1 = \lambda_2$. The result follows from the preceding analysis.

With $\lambda_1 = \lambda_2 > 0$, the function (2.3) becomes $\lambda_1 V^2/2$, where $V = I(y) + P(x)$ is the function given in [2]. In this case the estimate of the stability region given by (2.3) is the same as the estimate given by V. There are, however, many general cases in which constants λ_1 and λ_2 may be chosen so as to improve the estimate given by V. One such general case will be explored thoroughly in the following section.

3. Estimates for Lienard equations. We consider in this section the equation

$$\ddot{y} + f(y)\dot{y} + g(y) = 0 \tag{3.1}$$

and the associated equivalent system

$$\dot{x} = g(y), \qquad \dot{y} = -x - F(y), \tag{3.2}$$

where $F(y) = \int_0^y f(s)\, ds$. If the conditions of Theorem 1 hold, then the origin is asymptotically stable. Further, the global behavior of (3.2) and the estimates by (2.3) depend heavily upon the behavior of g and F. We consider the case where (3.2) has an asymptotically stable equilibrium point and a single neighboring unstable equilibrium point. A separate analysis may be given under different conditions; for example, the case where the origin is a unique equilibrium point surrounded by a unique limit cycle (see [1]).

For this case we assume the following:

(i) $g(y) > 0$ for $y > 0$ and $g(y) \to \infty$ as $y \to \infty$,
(ii) $g(y_0) = 0$ for some $y_0 < 0$,
(iii) $g(y) < 0$ for $y \in (y_0, 0)$, (3.3)
(iv) $g(y) > 0$ for $y < y_0$ and $g(y) \to \infty$ as $y \to -\infty$,
(v) $f(y) > 0$ for all y, and
(vi) $f, g \in C'(-\infty, \infty)$.

To consider the regions of asymptotic stability we state the following theorem which has been proved elsewhere [4].

THEOREM 2. Suppose the system

$$\dot{X} = f(X)$$

has an isolated equilibrium point at the origin and suppose further that there exists a function $V(X)$, with $V(0) = 0$, of class C'' in E_n, which is positive definite neighboring the origin and has a finite number of critical points, finite or infinite. Then, there is a positive critical value k (possibly infinite) of V such that for each number a on the interval $0 < a < k$, the equations $V = a$ define closed manifolds with $(0) \in \{V < a\}$. If, throughout $\{V < a\}$, $\dot{V} \leq 0$, the set $\{V < k\}$ bounds a region of stability of the origin.

We now compare the estimates given by V_θ in (2.3) and

$$V(x, y) = \int_0^y g(s)\, ds + x^2/2.$$

Assuming (3.3), the only critical values are $\lambda_1(\int_0^{y_0} g(s)\, ds)^2/2$ and $\int_0^{y_0} g(s)\, ds$, respectively. Therefore, letting $k = \int_0^{y_0} g(s)\, ds$,

$$\frac{\lambda_1(\int_0^y g(s)\, ds)^2}{2} + \frac{\lambda_2 x^2 \int_0^y g(s)\, ds}{2} + \frac{\lambda_2 x^4}{8} = \frac{\lambda_1 k^2}{2} \qquad (y \geq y_0) \tag{3.4}$$

encloses a region of stability if it lies in a domain B containing the origin such that in B, for $y \neq 0$,

$$\lambda_1 \lambda_2^{-1} + \lambda_2 \lambda_1^{-1} \leq 2 + 2F^2(y)\left[\int_0^y g(s)\, ds\right]^{-1}. \tag{3.5}$$

Indeed, if (3.4) lies in such a region B, then the interior of (3.4) is a region of asymptotic stability. To see this, note first that $f(y) > 0$ for all y, so that by Green's theorem (3.2) has no nontrivial periodic solutions. It is then an immediate consequence of the Poincaré-Bendixson theorem (see [6, p. 184]) that the positive limit set of any solution starting inside (3.4) is either the origin or contains the origin and $(-F(y_0), y_0)$. But any trajectory starting inside (3.4) will not leave the interior of some $V_\theta = a < \lambda_1 k^2/2$ (see [4]) and hence cannot approach $(-F(y_0), y_0)$ along some sequence $\{t_n\} \to \infty$.

Since the determination of the constants λ_1 and λ_2 is dependent upon their ratio, we let $\lambda = \lambda_2 \cdot \lambda_1^{-1}$. Then (3.4) becomes

$$\frac{(\int_0^y g(s)\, ds)^2}{2} + \frac{\lambda x^2 \int_0^y g(s)\, ds}{2} + \frac{\lambda x^4}{8} = \frac{k^2}{2} \qquad (y \geq y_0), \tag{3.6}$$

and condition (3.5) becomes

$$\lambda + \lambda^{-1} \leq 2 + 2F^2(y)\left[\int_0^y g(s)\, ds\right]^{-1}. \tag{3.7}$$

To compare the estimates of the regions of asymptotic stability, we assume that for $y \neq 0$

$$2F^2(y)\left[\int_0^y g(s)\, ds\right]^{-1} \geq \alpha > 0 \tag{3.8}$$

inside a strip $-\delta_1 < y < \delta_2$ and that the curve (3.6) lies wholly within this strip. Then we choose λ such that

$$\lambda + \lambda^{-1} \leq 2 + \alpha. \tag{3.9}$$

Next, solve $V(x, y) = k$ for x^2. Also, solve for x^2 in (3.6). We obtain numbers x_1^2 and x_2^2, respectively. Using the notation of (2.2), $x_1^2 = 2(k - I(y))$ and

$$x_2^2 = 2(-I(y) + [k^2\lambda^{-1} + I^2(y)(1 - \lambda^{-1})]^{1/2}). \tag{3.10}$$

Then, if x_1^2 and x_2^2 are both defined at y, we have

$$\omega(\lambda, y) = x_2^2 - x_1^2 = \frac{2(1 - \lambda^{-1})(I(y) - k)(I(y) + k)}{k + [k^2\lambda^{-1} + I^2(y)(1 - \lambda^{-1})]^{1/2}}. \tag{3.11}$$

If $0 < \lambda < 1$, we have $\omega(\lambda, y) > 0$ and $\partial\omega/\partial\lambda < 0$ whenever $I(y) < k$.

We summarize the above analysis.

THEOREM 3. Let conditions (3.3) hold. Assume that the curve (3.6) lies in a strip $-\delta_1 < y < \delta_2$ and that (3.8) holds in this strip. If $0 < \lambda < 1$ and $\lambda + \lambda^{-1} \leq 2 + \alpha$, then the estimate of the region of asymptotic stability given by (3.6) wholly contains the estimate given by $V(x, y) = k$. With the above procedure, maximal improvement over the estimate $V(x, y) = k$ is obtained by choosing λ such that $0 < \lambda < 1$ and $\lambda + \lambda^{-1} = 2 + \alpha$.

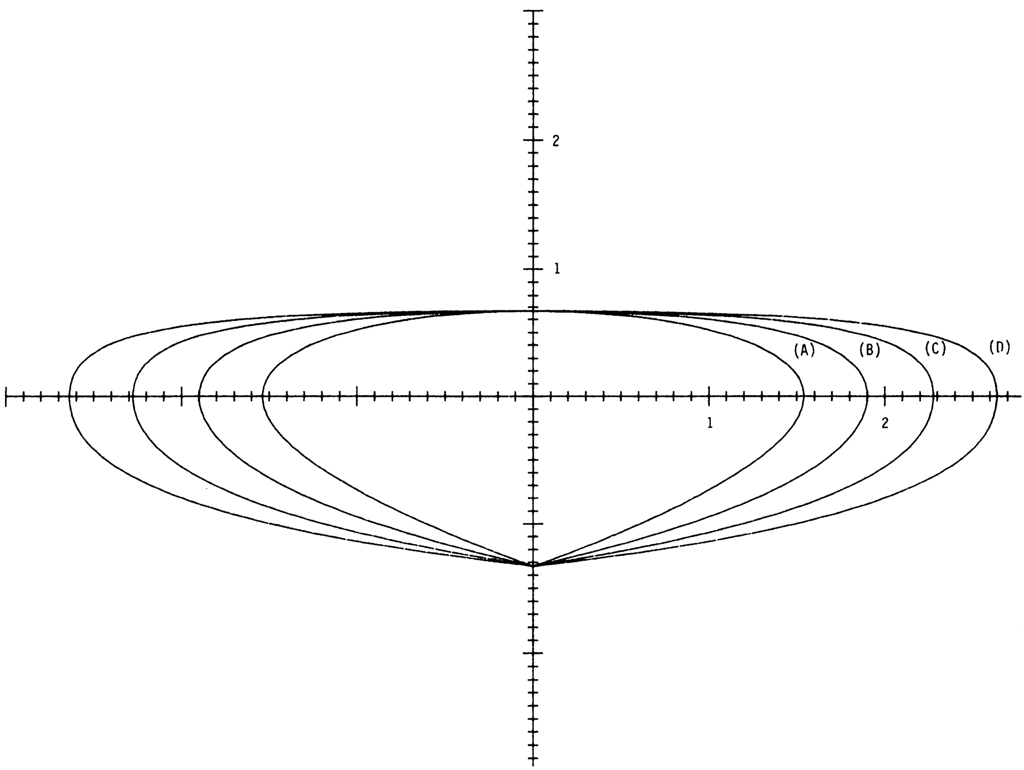

FIG. 1. (A) curve (4.6), $b = 2$; (B) curve (4.5), $a = 1$, $b = 2$; (C) curve (4.5), $a = 2$, $b = 2$; (D) curve (4.5), $a = 3$, $b = 2$.

4. Example. We now consider the equation

$$\ddot{y} + a\dot{y} + 2by + 3y^2 = 0 \tag{4.1}$$

where $a > 0$, $b > 0$. This equation has been studied extensively (e.g. see [2, 3 or 4]). This equation is equivalent to the system

$$\dot{x} = 2by + 3y^2, \qquad \dot{y} = -x - ay. \tag{4.2}$$

For this system the equation (3.6) becomes

$$\frac{(by^2 + y^3)^2}{2} + \lambda \frac{x^2}{2}(by^2 + y^3) + \lambda \frac{x^4}{8} = \left[\frac{4b^3}{27}\right]^2 \cdot \frac{1}{2} \qquad \left(y \geq \frac{-2b}{3}\right). \tag{4.3}$$

This curve encloses a region of asymptotic stability for (4.2) if it lies in a strip $-\delta_1 < y < \delta_2$ such that in this strip

$$\lambda + \lambda^{-1} \leq 2 + 2a^2(b + y)^{-1}. \tag{4.4}$$

From (4.3) note that $-2b/3 \leq y \leq b/3$. Therefore, if $\lambda + \lambda^{-1} \leq 2 + 3a^2(2b)^{-1}$, then (4.4) holds. Solving this inequality for λ and choosing $0 < \lambda < 1$ and $\lambda + \lambda^{-1} = 2 + 3a^2(2b)^{-1}$, we obtain

$$\lambda_0 = 2\left[2 + \frac{3a^2}{2b} + \left(\left[2 + \frac{3a^2}{2b}\right]^2 - 4\right)^{1/2}\right]^{-1}.$$

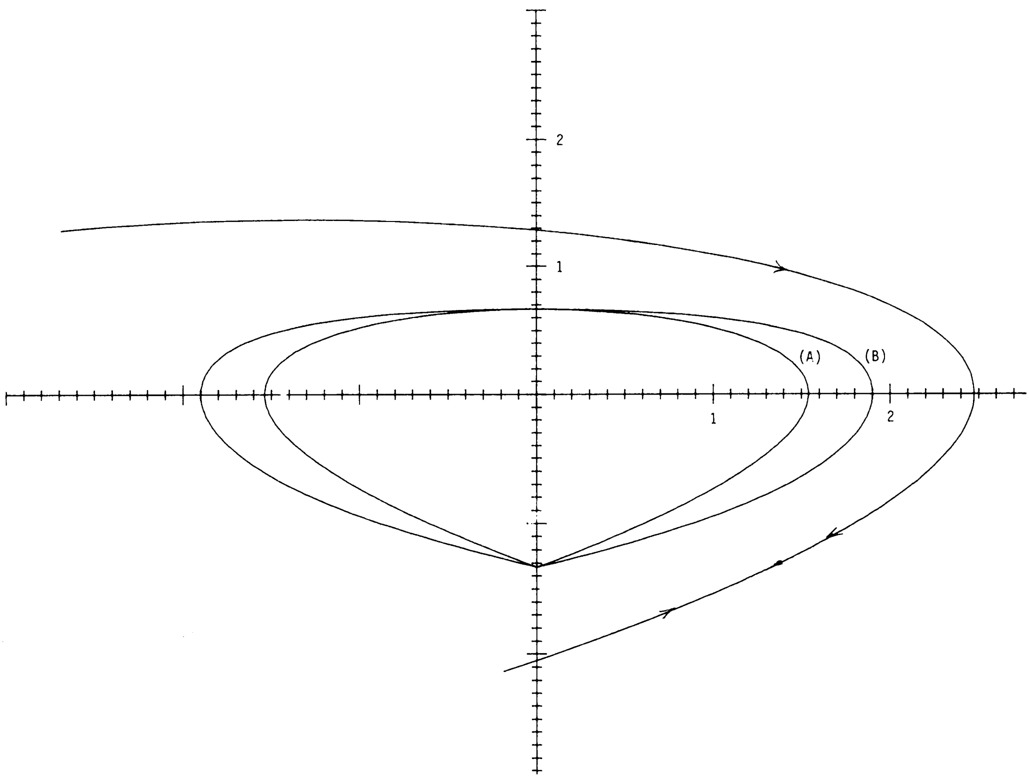

FIG. 2. (A) Curve (4.6), $b = 2$; (B) curve (4.5), $a = 1$, $b = 2$.

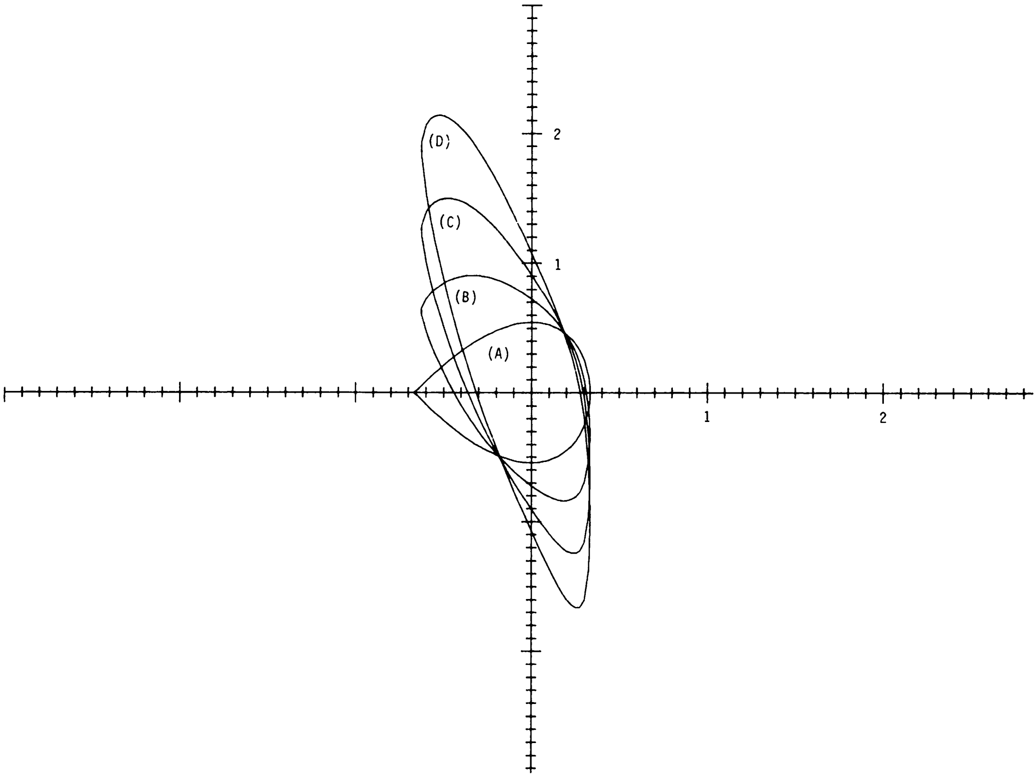

FIG. 3. (A) Curve (4.8), $b = 1$; (B) curve (4.7), $a = 1$, $b = 1$; (C) curve (4.7), $a = 2$, $b = 1$; (D) curve (4.7), $a = 3$, $b = 1$.

Thus,

$$\frac{(by^2 + y^3)^2}{2} + \lambda_0 \frac{x^2}{2}(by^2 + y^3) + \lambda_0 \frac{x^4}{8} = \left[\frac{4b^3}{27}\right]^2 \cdot \frac{1}{2} \qquad \left(y \geq -\frac{2b}{3}\right) \tag{4.5}$$

determines the maximal region of asymptotic stability over the weight functions $\theta(x, y) = \lambda_1(by^2 + y^3) + \lambda_2 x^2/2$ for (4.2).

From the preceding section it should be noted that (4.5) totally encloses

$$V(x, y) = by^2 + y^3 + x^2/2 = 4b^3/27. \tag{4.6}$$

Indeed, it can be shown that for b fixed, $\omega(\lambda, y) \simeq 2a[3(k^2 - I^2(y))(2b)^{-1}]^{1/2}$ as $a \to \infty$ for $-2b/3 \leq y \leq b/3$, where $\omega(\lambda, y)$ is defined in (3.11). This is shown in Fig. 1 where $b = 2$ and a varies. Fig. 2 shows V_θ and V where $a = 1$, $b = 2$ in comparison with the actual region of asymptotic stability.

It is also useful to transform (4.5) back to the phase plane and compare it with a previous choice for V. From (4.2), $x = -\dot{y} - ay$, so (4.5) becomes

$$\frac{(by^2 + y^3)^2}{2} + \lambda_0(by^2 + y^3)\frac{(\dot{y} + ay)^2}{2} + \lambda_0 \frac{(\dot{y} + ay)^4}{8} = \left[\frac{4b^3}{27}\right]^2 \cdot \frac{1}{2} \tag{4.7}$$

in the phase plane. For (4.1),

$$\dot{y}^2/2 + by^2 + y^3 = 4b^3/27 \tag{4.8}$$

is an estimate of the region of asymptotic stability that has been employed elsewhere (see [3, 5]). These curves are compared in Fig. 3 where $b = 1$ and a varies.

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