A SHARPENING OF MASLOV'S METHOD FOR FINDING ASYMPTOTIC SOLUTIONS*

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1. Introduction. Using a certain Hamiltonian flow, Maslov's method of characteristics determines the asymptotic limit (i.e., first term in the asymptotic series) of the solution of linear partial differential equations near turning points. By applying the same technique to an associated non-Hamiltonian flow we determine the full asymptotic series of the solution.

2. We assume that the reduced Helmholtz equation

$$\Delta\psi(\bar{x}) + \tau^2 f(\bar{x})\psi(\bar{x}) = 0, \tag{1}$$

τ a large parameter, has an asymptotic solution—for brevity, near turning points of the highest order—of the form

$$\psi(\bar{x}) - \int A(\bar{x}, \bar{p}, \tau)\exp\{i\tau(\bar{x}\cdot\bar{p} - S(\bar{p}))\}\, d\bar{p} = O(\tau^{-\infty}), \tag{2}$$

where $S(\bar{p})$ is such that $\bar{x} - \nabla_p S(\bar{p}) = 0$ determines the Lagrangian manifold of Maslov near the turning point [1]. Then the eikonal, i.e.

$$\bar{p}\cdot\bar{p} - f(\bar{x}) = 0,$$

associated with (1) is Maslov's Hamiltonian [2],

$$H = \bar{p}\cdot\bar{p} - f(\bar{x}). \tag{3}$$

3. Carrying the differentiation (1) across the integral (2), we obtain

$$\int \exp\{i\tau(\bar{x}\cdot\bar{p} - S(\bar{p}))\}\{(i\tau)^2(\bar{p}\cdot\bar{p} - f(\bar{x}))A + 2i\tau(\bar{p}\cdot\nabla_x A) + \Delta_x A\}\, d\bar{p} = O(\tau^{-\infty}). \tag{4}$$

The first term in the integral is Maslov's Hamiltonian (3) on the manifold. Expanding $\bar{p}\cdot\bar{p} - f(\bar{x})$, we get

$$\bar{p}\cdot\bar{p} - f(\bar{x}) = \bar{p}\cdot\bar{p} - f(\nabla_p S(\bar{p})) + (\bar{x} - \nabla_p S(\bar{p}))\cdot\bar{D}(\bar{x}, \bar{p}) = (\bar{x} - \nabla_p S(\bar{p}))\cdot\bar{D}(\bar{x}, \bar{p}),$$

where the remainder term is

$$\bar{D}(\bar{x}, \bar{p}) = \bar{D} = -\int_0^1 \nabla_x f(t(\bar{x} - \nabla_p S(\bar{p})) + \nabla_p S(\bar{p}))\, dt.$$

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By substituting into (4) and noting

$$\int \nabla_p \cdot [\exp\{i\tau(\bar{x} \cdot \bar{p} - S(\bar{p}))\}A\bar{D}]\, d\bar{p}$$

$$= \int \exp\{i\tau(\bar{x} \cdot \bar{p} - S(\bar{p}))\}\{i\tau A(\bar{x} - \nabla_p S(\bar{p})) \cdot \bar{D} + \bar{D} \cdot \nabla_p A + A\nabla_p \cdot \bar{D}\}\, d\bar{p}$$

and taking the surface integral over a sufficiently large radius so that it vanishes, (4) becomes

$$\int \exp\{i\tau(\bar{x} \cdot \bar{p} - S(\bar{p}))\}i\tau\left(-\bar{D} \cdot \nabla_p A - A\nabla_p \cdot \bar{D} + 2\bar{p} \cdot \nabla_x A + \frac{1}{i\tau}\Delta_x A\right) d\bar{p} = O(\tau^{-\infty}). \quad (6)$$

4. Eq. (6) is a requirement that the asymptotic series in τ of the integral should be trivial. We may achieve this by requiring that

$$-\bar{D} \cdot \nabla_p A + 2\bar{p} \cdot \nabla_x A - A\nabla_p \cdot \bar{D} + \frac{1}{i\tau}\Delta_x A = 0 \quad (7)$$

in a neighborhood of the Lagrangian manifold [3]. Eq. (7) leads to a transport equation in such a neighborhood if we introduce the flow

$$\bar{x}' = 2\bar{p}, \qquad \bar{p}' = -\bar{D}(\bar{x}, \bar{p}), \quad (8)$$

where the primes indicate time derivatives. Now (7) will hold in such a neighborhood if we allow the asymptotic series

$$A(\bar{x}, \bar{p}, \tau) = \sum_{k=0}^{\infty} A_k(\bar{x}, \bar{p})(i\tau)^{-k}$$

to evolve according to the transport equation

$$A_k' - A_k\nabla_p \cdot \bar{D} + \Delta_x A_{k-1} = 0. \quad (9)$$

5. With A determined by (9), the integral in (2) has the same asymptotic series expansion as the solution $\psi(x)$. Instead of the flow (8), Maslov [2, 3] uses the Hamiltonian flow associated with the Hamiltonian $H = \bar{p} \cdot \bar{p} - f(\bar{x})$. In that case the integral in (2) will only have the same asymptotic limit (i.e., initial term in the asymptotic series) as the solution $\psi(\bar{x})$.

The two flows agree in the Lagrangian manifold. In fact, any flow which agrees with either on the Lagrangian manifold will lead to an integral whose asymptotic limit is the same as that of $\psi(\bar{x})$.

References

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[3] J. J. Duistermaat, *Fourier integral operators*, Courant Institute Lecture Notes, New York University, New York, 1973, p. 18