ON A SECOND SOLUTION OF VAN DER POL'S DIFFERENTIAL EQUATION*

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1. Some time ago, Shen published in this journal [1] a method to form a Weierstrass elliptic function $\wp(t\,|\,2\omega_1, 2\omega_3)$, ω_1 being real and ω_3 being purely imaginary, which is the solution of the Van der Pol's differential equation

$$\frac{d^2x}{dt^2} + \mu(x^2 - 1)\frac{dx}{dt} + \kappa^2 x = 0 \tag{1}$$

under the initial condition that $x = x_0$ when $t = t_0$, μ and κ being real constants. He formed the solution by taking

$$-\mu^2 g_3 = \tfrac{1}{4} g_2^2, \tag{2}$$

where g_2 and g_3 are two invariants in the following differential equation satisfied by $\wp(t\,|\,2\omega_1, 2\omega_3)$:

$$\left(\frac{d\wp}{dt}\right)^2 = 4\wp^3 - g_2\wp - g_3. \tag{3}$$

The solution requires that both g_2 and g_3 must be real and besides g_3 must be negative. Shen asserted that g_3 is negative if

$$q > (505)^{-1/2}, \tag{4}$$

approximately, where $q = \exp(i\pi\omega_3/\omega_1)$, and that g_2 is real if both μ and κ are comparatively smaller than x_0. Under these restrictions, Shen proceeded to determine ω_1 and ω_3 through the use of complete elliptic integrals under the further condition that

$$0 < g_2 < (16/27)\mu^4. \tag{5}$$

Finally, he showed that there exists always one and only one cycle on the phase plane for such a solution.

2. Shen is to be commended for his contribution. The purpose of this note is to indicate that there exists a second solution in which ω_3 is no longer purely imaginary whereas ω_1 remains real, if the condition in (5) is not satisfied.

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For convenience of discussion, we introduce two invariants σ_4 and σ_6 in place of g_2 and g_3 as follows:

$$\begin{aligned} \sigma_4 &= \tfrac{1}{60} g_2 = \sum_{m,n=-\infty}^{\infty}{}' \frac{1}{(2m\omega_1 + 2n\omega_3)^4}, \\ \sigma_6 &= \tfrac{1}{140} g_3 = \sum_{m,n=-\infty}^{\infty}{}' \frac{1}{(2m\omega_1 + 2n\omega_3)^6}, \end{aligned} \tag{6}$$

where the prime on each summation sign denotes the omission of simultaneous zeros of m and n. In a succession of papers [2, 3, 4], the present writer, partly with the collaboration of Tsai, computed the values of σ_4 and σ_6 in the following cases:

(i) in the rectangular case $2\omega_1 = 1$ and $2\omega_3 = ci$, for $1 \le c < \infty$;

(ii) in the rhombic case $2\omega_1 = 1$ and $2\omega_3 = \frac{1}{2} + ci$, for $0 < c < \infty$.

In the former case, the values for $0 < c < 1$ can easily be deduced from the tabulated values. These two cases are the only cases in which σ_4 and σ_6 are both real. It is found that the values vary in the following manner:

(i) In the rectangular case,

$$\begin{aligned} &\sigma_4 \quad \text{or} \quad g_2 > 0 \qquad (0 < c < \infty), \\ &\sigma_6 \quad \text{or} \quad g_3 \begin{cases} < 0 & (0 < c < 1) \\ = 0 & (c = 1) \\ > 0 & (1 < c < \infty). \end{cases} \end{aligned} \tag{7}$$

(ii) In the rhombic case,

$$\begin{aligned} &\sigma_4 \quad \text{or} \quad g_2 \begin{cases} > 0 & (0 < c < \sqrt{3}/6) \\ = 0 & (c = \sqrt{3}/6) \\ < 0 & (\sqrt{3}/6 < c < \sqrt{3}/2) \\ = 0 & (c = \sqrt{3}/2) \\ > 0 & (\sqrt{3}/2 < c < \infty), \end{cases} \\ &\sigma_6 \quad \text{or} \quad g_3 \begin{cases} < 0 & (0 < c < \frac{1}{2}) \\ = 0 & (c = \frac{1}{2}) \\ > 0 & (\frac{1}{2} < c < \infty). \end{cases} \end{aligned} \tag{8}$$

3. Suppose that in general (i) in the rectangular case, $2\omega_1 = a$ and $2\omega_3 = bi$, and (ii) in the rhombic case, $2\omega_1 = a$ and $2\omega_3 = \frac{1}{2}a + bi$. No generality is lost if the real constants a and b in either case are both taken as positive. Obviously, the same variation of σ_4 and σ_6 as described above prevails if $b/a = c$. It is thus seen that Shen's solution indeed belongs to the rectangular case in which $0 < c < 1$. In this case, the three roots e_1, e_2 and e_3 of the cubic equation

$$4\wp^3 - g_2 \wp - g_3 = 0 \tag{9}$$

are all real under the condition in (5).

On the other hand, if the condition in (5) is not satisfied, that is, if

$$g_2 < 0 \quad \text{or} \quad g_2 > (16/27)\mu^4, \tag{10}$$

then one of the three roots of the cubic equation remains real but the other two roots become complex conjugate. Shen's equation (5) is a quadratic equation of g_2. It gives two real roots under the restriction imposed on x_0. One root is positive and the other is negative. The positive root which satisfies the condition in (5) leads to the Shen's solution. The negative root which satisfies the first condition in (10) instead leads to a second solution. It may be anticipated that this second solution belongs to the rhombic case since g_2 is always positive in the rectangular case. Furthermore, it is seen from the variation of σ_4 and σ_6 in the rhombic case that when g_2 and g_3 are both negative, the value of c is $0 < c < 1/2$. Here, the value of q becomes

$$q = ie^{-\pi c}. \tag{11}$$

4. In this solution, the values of ω_1 and q may be found directly from the following equations:

$$g_2 = \frac{1}{12}\left(\frac{\pi}{\omega_1}\right)^4 (\theta_3^8 - \theta_3^4\theta_4^4 + \theta_4^8),$$

$$g_3 = \frac{1}{432}\left(\frac{\pi}{\omega_1}\right)^6 (2\theta_3^4 - \theta_4^4)(2\theta_4^4 - \theta_3^4), \tag{12}$$

where

$$\theta_3 = 1 + 2\sum_{n=1}^{\infty} q^{n^2}, \qquad \theta_4 = 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2}. \tag{13}$$

Elimination of ω_1 leads to an equation of p with real coefficients,

$$F(p, g_2, g_3) = 0, \tag{14}$$

where

$$p = -q^2 = e^{-2\pi c}. \tag{15}$$

The equation can be solved numerically for a real root of p, less than unity but greater than $e^{-\pi}$ or 0.0432 nearly, when g_2 and g_3 are given. Subsequently, ω_1 is found by substitution. Finally, we find

$$2\omega_3 = \omega_1 + \frac{i\omega_1}{\pi}\log\left(\frac{1}{p}\right). \tag{16}$$

Let e_1 be the real root of the cubic equation. The other two roots e_2 and e_3 are complex conjugate. Again, let

$$y \equiv \frac{dx}{dt} = \frac{d}{dt}\wp(t\,|\,2\omega_1, 2\omega_3) \tag{17}$$

and write

$$y^2 = 4(x - e_1)(x - e_2)(x - e_3). \tag{18}$$

Since e_1 is always negative when $0 < c < \frac{1}{2}$ [3], the value of y is purely imaginary for $x < e_1$. The path on the phase plane is shown in the accompanying Fig. 1.

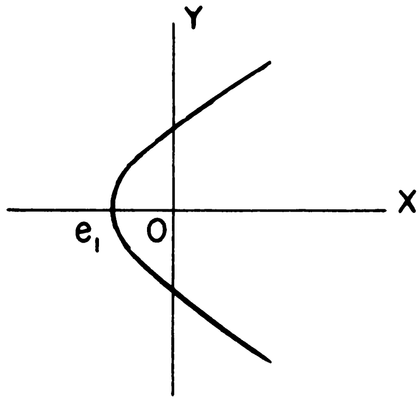

FIG. 1.

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