A METHOD OF SOLUTION FOR AN ORDINARY DIFFERENTIAL EQUATION CONTAINING SYMBOLIC FUNCTIONS*

By H. H. PAN (*Polytechnic Institute of New York*)
AND R. M. HOHENSTEIN (*The Aerospace Corporation*)

1. Introduction. The following differential equation arises in some physical problems:

$$Ly = f(x) + \sum_{i=0}^{N} \beta_i \delta^{(i)}(x-\alpha) \tag{1}$$

in which L is an ordinary linear differential operator and δ is the Dirac delta function. δ', δ'', ..., $\delta^{(n)}$ are its first, second, ..., and nth derivatives with respect to the independent variable x.

If the coefficients in the differential operator L are constant, the solution to Eq. (1) can be obtained by the Laplace transform method. Otherwise, the transform method is not convenient to use. The purpose of this paper is to offer a method of solution for such cases. Of course the method also applies to the case of constant coefficients as well.

2. An identity. Before proceeding with the method, we would first like to prove the following identity which will be needed later:

$$f(x)\delta^{(n)}(x) = (-1)^n f^{(n)}(0)\delta(x) + (-1)^{n-1} n f^{(n-1)}(0)\delta'(x) + (-1)^{n-2}\frac{n(n-1)}{2!} f^{(n-2)}(0)\delta''(x) + \cdots + f(0)\delta^{(n)}(x). \tag{2}$$

Proof. Introduce a testing function $\phi(x)$ [1], multiply the left-hand side of (2) by $\phi(x)$ and integrate.

$$\int_{-\infty}^{+\infty} \{f(x)\delta^{(n)}(x)\}\phi(x)\,dx = \int_{-\infty}^{+\infty} \{f(x)\phi(x)\}\delta^{(n)}(x)\,dx \tag{3}$$

After a succession of integrations-by-parts, this leads to

$$\int_{-\infty}^{+\infty} \{f(x)\phi(x)\}\delta^{(n)}(x)\,dx = (-1)^n \int_{-\infty}^{+\infty} \{f(x)\phi(x)\}^{(n)}\delta(x)\,dx. \tag{4}$$

Since

$$\{f(x)\phi(x)\}^{(n)} = f^{(n)}(x)\phi(x) + nf^{(n-1)}(x)\phi'(x) + \frac{n(n-1)}{2!} f^{(n-2)}(x)\phi''(x) + \cdots f(x)\phi^{(n)}(x), \tag{5}$$

* Received February 22, 1980; revised version received July 7, 1980.

substitution from (5) into (4) results in the relation

$$\int_{-\infty}^{+\infty} \{f(x)\delta^{(n)}(x)\}\phi(x)\, dx = (-1)^n \Big| f^n(0)\phi(0) + nf^{(n-1)}(0)\phi'(0) + \frac{n(n-1)}{2!} f^{(n-2)}(0)\phi''(0) + \cdots + f(0)\phi^{(n)}(0) \Big|. \quad (6)$$

On the other hand, if the right-hand side of (2) is multiplied by $\phi(x)$ and integrated and the standard fact

$$\int_{-\infty}^{+\infty} \phi(x)\delta^{(k)}(x)\, dx = (-1)^k \int_{-\infty}^{+\infty} \phi^{(k)}(x)\delta(x)\, dx = (-1)^k \phi^{(k)}(0) \quad (7)$$

is used, one easily gets the same result as in (6). Therefore, the identity Eq. (2) is proved under the concept of symbolic functions [1].

3. Method of solution. Let us now consider the given differential equation (1). To avoid undue complexity, let us take

$$L = a(x)\frac{d^4}{dx^4} + b(x)\frac{d^3}{dx^3} + c(x)\frac{d^2}{dx^2} + d(x)\frac{d}{dx} + e(x), \quad (8)$$

$f(x) = 0$ and $N = 3$. This will make it easier to demonstrate the feature of the proposed method without actually sacrificing its generality.

With the form of the operator as given in (8), Eq. (1) becomes

$$a(x)y^{iv} + b(x)y''' + c(x)y'' + d(x)y' + e(x)y = \beta_0\delta(x-\alpha) + \beta_1\delta'(x-\alpha) + \beta_2\delta''(x-\alpha) + \beta_3\delta'''(x-\alpha) \quad (9)$$

Since this is a nonhomogeneous equation, the solution will be expressed in the usual manner as if the right-hand terms were ordinary functions

$$y(x) = y_h(x) + y_p(x), \quad (10)$$

where $y_h(x)$ is the solution to the reduced homogeneous equation $Ly = 0$ and $y_p(x)$ is a particular solution. For the particular solution, we assume the following form:

$$\begin{aligned} y_p(x) &\equiv 0 && \text{for} \quad x < \alpha \\ &= G(x)H(x-\alpha) && \text{for} \quad x \geq \alpha \end{aligned} \quad (11)$$

where $H(x)$ is the unit step function and $G(x)$ is an unknown function. Since $H(x-\alpha) = 0$ for $x < \alpha$ one might incline to write $y_p(x) = G(x)H(x-\alpha)$ for all x. However, we prefer the above form for a reason which will become clear later on. It is noted that in taking $y_p(x)$ in this form, Eq. (9) is identically satisfied for $x < \alpha$ and for $x \geq \alpha$ the substitution of $G(x)H(x-\alpha)$ into the left-hand side of Eq. (9) will yield the correct singularities to match those on the right-hand side. It will be shown later that the unknown function $G(x)$ can be so determined as also to satisfy Eq. (9) for $x \geq \alpha$.

It follows from Eq. (11) for $x \geq \alpha$ that

$$
\begin{aligned}
y_p'(x) &= G'(x)H(x-\alpha) + G(x)\delta(x-\alpha) = G'(x)H(x-\alpha) + G(\alpha)\delta(x-\alpha) \\
y_p''(x) &= G''(x)H(x-\alpha) + G'(\alpha)\delta(x-\alpha) + G(\alpha)\delta'(x-\alpha) \\
&\ \vdots \\
y_p^{(k)}(x) &= G^{(k)}(x)H(x-\alpha) + \sum_{j=0}^{k-1} G^{(k-1-j)}(\alpha)\delta^{(j)}(x-\alpha)
\end{aligned}
\tag{12}
$$

and

$$
\begin{aligned}
d(x)y_p'(x) &= d(x)G'(x)H(x-\alpha) + d(\alpha)G(\alpha)\delta(x-\alpha) \\
c(x)y_p''(x) &= c(x)G''(x)H(x-\alpha) + [c(\alpha)G'(\alpha) - c'(\alpha)G(\alpha)]\delta(x-\alpha) \\
&\quad + c(\alpha)G(\alpha)\delta'(x-\alpha), \\
b(x)y_p'''(x) &= b(x)G'''(x)H(x-\alpha) + [b(\alpha)G''(\alpha) - b'(\alpha)G'(\alpha) + b''(\alpha)G(\alpha)] \\
&\quad \cdot \delta(x-\alpha) + [b(\alpha)G'(\alpha) - 2b'(\alpha)G(\alpha)]\delta'(x-\alpha) \\
&\quad + b(\alpha)G(\alpha)\delta''(x-\alpha), \\
a(x)y_p^{iv}(x) &= a(x)G^{iv}(x)H(x-\alpha) + [a(\alpha)G'''(\alpha) - a'(\alpha)G''(\alpha) \\
&\quad + a''(\alpha)G'(\alpha) - a'''(\alpha)G(\alpha)]\delta(x-\alpha) + [a(\alpha)G''(\alpha) - 2a'(\alpha)G'(\alpha) \\
&\quad + 3a''(\alpha)G(\alpha)]\delta'(x-\alpha) + [a(\alpha)G'(\alpha) - 3a'(\alpha)G(\alpha)]\delta''(x-\alpha) \\
&\quad + a(\alpha)G(\alpha)\delta'''(x-\alpha).
\end{aligned}
\tag{13}
$$

To obtain the results in Eq. (13), the identity (2) has been used.

Upon substitution of each term from Eqs. (11) and (13), Eq. (9) becomes

$$
\begin{aligned}
&(LG)H(x-\alpha) + \{a(\alpha)G'''(\alpha) - [a'(\alpha) - b(\alpha)]G''(\alpha) + [a''(\alpha) - b'(\alpha) + c(\alpha)]G'(\alpha) \\
&\quad - [a'''(\alpha) - b''(\alpha) + c'(\alpha) - d(\alpha)]G(\alpha)\}\delta(x-\alpha) \\
&\quad + \{a(\alpha)G''(\alpha) - [2a'(\alpha) - b(\alpha)]G'(\alpha) + [3a''(\alpha) - 2b'(\alpha) + c(\alpha)]G(\alpha)\}\delta'(x-\alpha) \\
&\quad + \{a(\alpha)G'(\alpha) - [3a'(\alpha) - b(\alpha)]G(\alpha)\}\delta''(x-\alpha) + a(\alpha)G(\alpha)\delta'''(x-\alpha) \\
&\qquad = \beta_0\delta(x-\alpha) + \beta_1\delta'(x-\alpha) + \beta_2\delta''(x-\alpha) + \beta_3\delta'''(x-\alpha).
\end{aligned}
\tag{14}
$$

Equating like coefficients in $\delta^{(i)}(x-\alpha)$ on both sides of Eq. (14) including $i = -1$, we obtain the following sets of conditions:

$$
LG = a(x)G^{iv} + b(x)G''' + c(x)G'' + d(x)G' + e(x)G = 0 \tag{15}
$$

and at $x = \alpha$

$$
\begin{aligned}
&a(\alpha)G'''(\alpha) - [a'(\alpha) - b(\alpha)]G''(\alpha) + [a''(\alpha) - b'(\alpha) + c(\alpha)]G'(\alpha) \\
&\qquad - [a'''(\alpha) - b''(\alpha) + c'(\alpha) - d(\alpha)]G(\alpha) = \beta_0, \\
&a(\alpha)G''(\alpha) - [2a'(\alpha) - b(\alpha)]G'(\alpha) + [3a''(\alpha) - 2b'(\alpha) + c(\alpha)]G(\alpha) = \beta_1, \\
&a(\alpha)G'(\alpha) - [3a'(\alpha) - b(\alpha)]G(\alpha) = \beta_2, \\
&a(\alpha)G(\alpha) = \beta_3.
\end{aligned}
\tag{16}
$$

Solution of Eq. (16) gives the values of $G(x)$, $G(x)$, $G''(x)$ and $G'''(x)$ at $x = \alpha$. Since we pick the function $G(x)$ to start at $x = \alpha$ these values are the initial values of $G(x)$. The differential equation (15) in conjunction with the initial conditions (16) uniquely determines the unknown function G. It is to be expected that G satisfies the homogeneous equation because the right-hand side of Eq. (9) is zero except at $x = \alpha$ where the singularities occur.

For the special case in which $\beta_0 = 1$ and $\beta_1 = \beta_2 = \beta_3 = 0$, Eq. (9) becomes

$$Ly = \delta(x - \alpha). \tag{17}$$

The solution to Eq. (17) is the Green's function of the operator L. Therefore, the proposed method also presents an alternate way for obtaining the Green's function which to the authors' knowledge is not to be found in the literature.

4. Physical problems giving rise to the differential equation of type (1). In the following, two examples are cited which give rise to the type of differential equation as in (1).

Example 1. Flexural vibrations of a tapered beam carrying a large number of heavy bodies. In principle this problem can be solved by the classical method. In that method the governing equation is written for each section between the heavy bodies and these equations are spliced together by imposing the proper junction conditions. However, if the number of the heavy bodies is large, this method becomes unwieldy, especially in the treatment of forced vibrations. In contrast, the problem can be handled in a much more efficient way by use of symbolic functions. The equation can be written in a single expression valid for the entire length of the beam as follows:

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 y}{\partial x^2}\right] + \frac{\rho A(x)}{g}\frac{\partial^2 y}{\partial t^2} - \frac{\partial}{\partial x}\left[\sum_{i=1}^{R} J_i\delta(x - a_i)\frac{\partial^3 y}{\partial x\,\partial t^2}\right] + \sum_{i=1}^{R} M_i\delta(x - a_i)$$
$$\times \frac{\partial^2 y}{\partial t^2} = \sum_{j=1}^{S} F_j(t)\delta(x - b_j) + \sum_{k=1}^{T} C_k(t)\delta'(x - c_k) \tag{18}$$

where R is the number of heavy bodies, S the number of driving forces and T the number of driving couples. J_i and M_i are the rotatory inertia and the mass of the ith heavy body, respectively.

The question is whether (18) in its present form can be solved. The answer is yes! By setting the right-hand side of (18) to zero, with the assumption of harmonic motion, we find the spatial equation to be

$$\frac{d^2}{dx^2}\left[EI(x)\frac{d^2 Y}{dx^2}\right] - \frac{\rho A(x)\omega^2}{g} Y + \omega^2\sum_{i=1}^{R} J_i\frac{dY(a_i)}{dx}\delta'(x - a_i) - \omega^2\sum_{i=1}^{R} M_i Y(a_i)\delta(x - a_i) = 0. \tag{19}$$

This equation is of the type (1). Hence, its solution can be readily found by the present method. The solution to (18) can then be obtained by an eigenfunction expansion in the eigenfunctions of (19), again in a single expression. The detailed solution of this problem will not be presented here. Readers interested in the procedure for the complete solution following the solution to the spatial equation (19) are referred to [2] where a similar problem is treated.

Before leaving this problem, let us consider a specialized case as an illustration of this method. In Eq. (18) let all $F_j(t)$'s be set to zero, set $T = 1$, wherein both C_1 and y are independent of t and $c_1 = l/2$ with $I(x) = I_0(1 + mx)^3$. This has now been reduced to a static problem of finding the deflection of a linearly tapered beam subject to a couple C_1 at its midlength. To make the problem specific, the beam is assumed to be free at $x = 0$ and clamped at $x = l$. According to the present method the solution can be written in the form

$$y(x) = Y(x) + G(x)H\left(x - \frac{l}{2}\right). \tag{20}$$

Going through the procedure as prescribed before, we find

$$EI_0 G(x) = \frac{C_1}{2m^2}\frac{1}{(1 + mx)} + \frac{C_1 x}{2m\left(1 + m\dfrac{l}{2}\right)^2} - \frac{C_1}{2m^2\left(1 + m\dfrac{l}{2}\right)} - \frac{C_1 l}{4m\left(1 + m\dfrac{l}{2}\right)^2}. \tag{21}$$

This leads to the correct solution

$$\begin{aligned} EI_0 y = {} & \frac{C_1}{2m}\left[\frac{1}{(1 + ml)^2} - \frac{1}{\left(1 + m\dfrac{l}{2}\right)^2}\right]x \\ & - \frac{C_1}{2m}\left[\frac{l}{(1 + ml)^2} - \frac{l}{2\left(1 + m\dfrac{l}{2}\right)^2} + \frac{1}{m(1 + ml)} - \frac{1}{m\left(1 + m\dfrac{l}{2}\right)}\right] \\ & + EI_0 G(x)H\left(x - \frac{l}{2}\right), \end{aligned} \tag{22}$$

which can be verified by the conventional splicing approach.

Example 2. Response of a viscoelastic incomplete ring to a suddenly applied force at its free end. The pair of governing equations describing the motion of a viscoelastic ring is as follows [3, 4]:

$$\begin{aligned} Q(\beta^{iv} + 2\beta'' + \beta) &= \frac{2(1 + k)}{J}\rho R^3 P\ddot{v} - \frac{2(1 + k)}{J}Pw(\theta, t), \\ v'' &= \frac{R}{1 + k}(\beta - k\beta'') \end{aligned} \tag{23}$$

where β is the twist and v is the deflection, P, Q are the viscoelastic differential operators, and $w(\theta, t)$ is the forcing function. In obtaining (23) we have assumed the Poisson's ratio to be constant.

By means of a dual eigenfunction expansion of the form

$$\beta = \sum_{n=0}^{\infty} c_n(t)\beta_n \quad \text{and} \quad v = \sum_{n=0}^{\infty} c_n(t)v_n \tag{24}$$

where β_n and v_n are the pairs of eigenfunctions in the associated elastic problem, the differential equation governing the generalized Fourier coefficient c_n's is found from (23) to be

$$\rho R^3 P\left(\frac{d}{dt}\right)\ddot{c}_i + \frac{J\lambda_i}{2R}Q\left(\frac{d}{dt}\right)c_i = \frac{1}{N_i}P\left(\frac{d}{dt}\right)\int_0^{\alpha} wv_i \, d\theta \tag{25}$$

in which the ring extends from $\theta = 0$ at the clamped end to $\theta = \alpha$ at the free end.

We next consider the case where the incomplete ring is initially at rest. If the suddenly applied force at the free end can be described by the expression

$$w(\theta, t) = W^*\delta(\theta - \alpha)e^{-\gamma t}H(t), \tag{26}$$

Eq. (25) becomes

$$\rho R^3 P\left(\frac{d}{dt}\right)\ddot{c}_i + \frac{J\lambda_i}{2R} Q\left(\frac{d}{dt}\right)c_i = \frac{W^* v_i(\alpha)}{N_i} P\left(\frac{d}{dt}\right)\{e^{-\gamma t}H(t)\}. \tag{27}$$

Since P and Q are linear differential operators, (27) is easily seen to be a differential equation of the type (1). For example, if the property of the viscoelastic material considered in the problem is such that the operators P and Q are in the following forms:

$$P\left(\frac{d}{dt}\right) = 1 + p_1\frac{d}{dt} + p_2\frac{d^2}{dt^2} + p_3\frac{d^3}{dt^3}, \tag{28}$$

$$Q\left(\frac{d}{dt}\right) = q_0 + q_1\frac{d}{dt} + q_2\frac{d^2}{dt^2} + q_3\frac{d^3}{dt^3}, \tag{29}$$

Eq. (27) becomes

$$\begin{aligned}\rho R^3 P\left(\frac{d}{dt}\right)\ddot{c}_i + \frac{J\lambda_i}{2R} Q\left(\frac{d}{dt}\right)c_i = \frac{W^* v_i(\alpha)}{N_i}\{&(1 - p_1\gamma + p_2\gamma^2 - p_3\gamma^3)e^{-\gamma t}H(t) \\ &+ (p_1 - p_2\gamma + p_3\gamma^2)\delta(t) + (p_2 - p_3\gamma)\dot{\delta}(t) + p_3\ddot{\delta}(t)\}.\end{aligned} \tag{30}$$

Since the incomplete ring is initially at rest, all the initial conditions for (25) or (30) are zero at $t = 0^-$. With the present method, recognizing that in this case the solution to the reduced homogeneous equation is zero, one can transform (30) with its homogeneous initial conditions into a differential equation completely free of any symbolic function:

$$\rho R^3 P\left(\frac{d}{dt}\right)\ddot{c}_i + \frac{J\lambda_i}{2R} Q\left(\frac{d}{dt}\right)c_i = \frac{W^* v_i(\alpha)}{N_i}(1 - p_1\gamma + p_2\gamma^2 - p_3\gamma^3)e^{-\gamma t} \quad \text{for} \quad t > 0 \tag{31}$$

but with a set of nonhomogeneous initial conditions as follows:

$$\begin{aligned} &c_i(0^+) = 0, & &c_i^{(1)}(0^+) = 0, \\ &c_i^{(2)}(0^+) = \frac{W^* v_i(\alpha)}{\rho R^3 N_i}, & &c_i^{(3)}(0^+) = -\frac{W^* v_i(\alpha)\gamma}{\rho R^3 N_i}, \\ &c_i^{(4)}(0^+) = \frac{W^* v_i(\alpha)}{\rho R^3 N_i}\left(\gamma^2 - \frac{J\lambda_i q_3}{2\rho R^4 p_3}\right). \end{aligned} \tag{32}$$

This can be solved in the usual way.

References

[1] B. Friedman, *Principles and techniques of applied mathematics*, John Wiley & Sons, New York, 1956

[2] H. H. Pan, *Transverse vibration of an Euler beam carrying a system of heavy bodies*, J. Appl. Mech. **32**, 434–437 (1965)

[3] H. H. Pan, *Orthogonality condition for the normal modes in the out-of-plane twist-bending vibrations of an elastic ring*, Int. J. of Mech. Sci. **8**, 601–603 (1966)

[4] H. H. Pan, *Out-of-plane vibrations of elastic and viscoelastic incomplete rings*, presented at the Eighth U.S. National Congress of Applied Mechanics, June (1978)