

—NOTES—

THE SOLITARY WAVE WITH SURFACE TENSION*

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1. Introduction. In this note, I discuss the effect of surface tension on a solitary wave. In general, the effect is small, but it can be decisive in that the wave is completely obliterated when surface tension is present, if the depth is small enough.

Let h be the depth of the fluid at infinity, ρg the weight density of the fluid, and T the surface tension. Then the critical parameter is the inverse Bond number, defined by

$$\tau = T/\rho g h^2. \tag{1.1}$$

I show there can be no solitary wave when $\tau \geq 1/3$. For water at 20 °C (for which $T = 74$ dynes/cm), this means there can be no solitary wave for depths less than approximately 0.48 cm. Naturally, surface tension makes its presence felt in a less extreme fashion at greater depths. Generally, the effects can be described as follows, depending on how one wants to look at the wave. If the horizontal dimensions are kept fixed, the effect of surface tension is to reduce both the Froude number F and the amplitude of the wave. Both the amplitude and the quantity F-1 are reduced by a factor $(1 - 3\tau)$ from their values when $\tau = 0$. The qualitative result that a tension will reduce the amplitude of the wave is to be expected on general physical grounds, of course. The quantitative factor $1 - 3\tau$ seems to be new, however.

One can also look at the wave keeping the amplitude fixed, independent of τ. The result then is that the Froude number also is independent of τ, while the wave crest is sharpened, the horizontal dimensions of the wave being reduced by a factor $(1 - 3\tau)^{1/2}$. Again, the qualitative effect of sharpening the crest is to be expected, but the constancy of the Froude number and the quantitative factor $(1 - 3\tau)^{1/2}$ seem to be new.

Because so much recent work on solitary waves is based on the approximate Korteweg-de Vries equation, it is perhaps important to point out explicitly that our analysis here is based, not on Korteweg-de Vries, but on the full nonlinear equations of water waves, with surface tension.

Finally, we remark that our discussion of surface tension is based on a new method for deriving the solitary wave. All the usual formulas fall out by setting $\tau = 0$. This method will be used in a future paper to discuss three-dimensional interaction between solitary waves.

2. The problem. The solitary wave is a two-dimensional, progressive wave in an irrotational, incompressible fluid having a free surface with surface tension acting on it. The fluid moves over a flat bottom and the only external force acting is gravity. What makes the

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wave "solitary" is that it has only one extreme point, and the free surface approaches a constant at infinity.

Introduce a coordinate system with the Y-axis vertical and pointing up, the X-axis coinciding with the bottom, and the origin moving horizontally with the phase speed of the flow. Then, we seek a nonconstant function $Y = H(X)$ describing the free surface, and a vector function $S(X, Y) = (U)X, Y), V(X, Y))$ giving the velocity in the fluid, such that:

$$U_Y = V_X, \qquad U_X + V_Y = 0, \quad \text{for} \quad 0 < Y < H(X), \tag{2.1a,b}$$

$$V = 0, \quad \text{when} \quad Y = 0; \tag{2.2}$$

$$V = UH_X, \quad \text{when} \quad Y = H(X); \tag{2.3}$$

$$gH - \frac{T}{\rho} \frac{H_{XX}}{(1 + H_X^2)^{3/2}} + \frac{1}{2}(U^2 + V^2) = \text{constant}, \qquad \text{when} \quad Y = H(X); \tag{2.4}$$

$$H(X) \to \text{constant}, \qquad U(X, Y) \to \text{constant}, \qquad V(X, Y) \to 0 \quad \text{as} \quad |X| \to \infty. \tag{2.5}$$

(2.1) expresses the irrotationality and incompressibility of the fluid, (2.2) and (2.3) make the bottom and the top streamlines, (2.4) is equivalent—via Bernoulli's equation—to the condition that the pressure at the free surface is proportional to the curvature [3], and (2.5) is the defining condition for a solitary wave. Uniform flow is excluded by the condition that H be nonconstant.

One way or another in problems of this sort, it seems necessary to stretch one of the independent variables relative to the other. In addition—and this is what makes our derivation different from the standard ones—we map the domain of the fluid onto the strip $\{(x, y)\colon -\infty < x < \infty, 0 < y < 1\}$ by setting

$$x = \varepsilon X/h, \qquad y = Y/H(X), \tag{2.6a,b}$$

where $h = \lim_{|X| \to \infty} H(X)$ (cf. (2.5)) and ε is the stretching parameter, measuring the ratio of the amplitude of the wave to the depth at infinity. Indeed, we suppose H to have the form

$$H(X) = h[1 + \varepsilon^2 \eta(x)]. \tag{2.7}$$

Here, η may also depend on ε, but this has been suppressed in the notation, since we assume η continuous as ε goes to zero. Of course, we are free to impose the normalization condition

$$\sup_x |\eta(x)| = 1. \tag{2.8}$$

We also write

$$U(X, Y) = U_0[1 + \varepsilon^2 u(x, y)], \qquad V(X, Y) = U_0 \varepsilon^3 v(x, y), \tag{2.9}$$

where $U_0 = \lim_{|X| \to \infty} U(X, Y)$. The form of the stretching introduced in (2.6a) as well as the peculiar way ε occurs in (2.8) are based on our prior experience of the behavior of solitary waves when there is no surface tension [1, 2].

In terms of the new unknowns u, v, and η, (2.1)–(2.5) become

$$\left.\begin{aligned} u_y &= \varepsilon^2 v_x + \varepsilon^4(\eta v_x - y\eta_x v_y) \\ u_x + v_y &= \varepsilon^2(y\eta_x u_y - \eta u_x) \end{aligned}\right\}, \qquad 0 < y < 1; \tag{2.10}$$

$$v = 0, \qquad \text{when} \quad y = 0; \tag{2.11}$$

$$v = \eta_x + \varepsilon^2 \eta_x u, \qquad \text{when} \quad y = 1; \tag{2.12}$$

$$\eta - \frac{\varepsilon^2 \tau \eta_{xx}}{(1 + \varepsilon^6 \eta_x^2)^{3/2}} + F\left(u + \frac{\varepsilon^2}{2} u^2 + \frac{\varepsilon^4}{2} v^2\right) = 0, \qquad \text{when} \quad y = 1; \tag{2.13}$$

$$\eta(x), \qquad u(x, y), \qquad \text{and} \quad v(x, y) \to 0 \quad \text{as} \quad |x| \to \infty. \tag{2.14}$$

In (2.13), τ is the inverse Bond number (1.1) and

$$F = U_0^2/gh \tag{2.15}$$

is the Froude number, which is not known *a priori* since U_0 is not known. Also, in deriving (2.13), the constant in (2.4) has been chosen in a convenient way.

3. The formal solution. Eqs. (2.10)–(2.14) are obviously susceptible to formal solution by expanding everything in a series of powers of ε^2. Thus, we write $u = u^{(0)} + \varepsilon^2 u^{(1)} + \cdots$, $v = v^{(0)} + \varepsilon^2 v^{(1)} + \cdots$, $\eta = \eta^{(0)} + \varepsilon^2 \eta^{(1)} + \cdots$, and $F = F^{(0)} + \varepsilon^2 F^{(1)} + \cdots$, substitute into (2.10)–(2.15) and equate coefficients of like powers of ε.

The equations for the zero-order approximation are

$$u_y^{(0)} = 0, \qquad u_x^{(0)} + v_y^{(0)} = 0, \qquad \text{for} \quad 0 < y < 1; \tag{3.1}$$

$$v^{(0)} = 0, \qquad \text{when} \quad y = 0; \tag{3.2}$$

$$v^{(0)} = \eta_x^{(0)}, \qquad \text{when} \quad y = 1; \tag{3.3}$$

$$\eta^{(0)} + F^{(0)} u^{(0)} = 0, \qquad \text{when} \quad y = 1; \tag{3.4}$$

$$\eta^{(0)}, \qquad u^{(0)}, \qquad v^{(0)} \to 0 \quad \text{as} \quad |x| \to \infty. \tag{3.5}$$

Eqs. (3.1)–(3.4) are easily solved, and they give

$$F^{(0)} = 1, \qquad u^{(0)}(x, y) = -\eta^{(0)}(x), \qquad v^{(0)}(x, y) = y\eta_x^{(0)}(x), \tag{3.6}$$

where $\eta^{(0)}$ is free, subject only to (3.5).

$\eta^{(0)}$ is determined at the next step. Using (3.6), we find the equations for the next approximation to be

$$u_y^{(1)} = y\eta_{xx}^{(0)}, \qquad u_x^{(1)} + v_y^{(1)} = \eta^{(0)}\eta_x^{(0)}, \qquad \text{when} \quad 0 < y < 1; \tag{3.7}$$

$$v^{(1)} = 0, \qquad \text{when} \quad y = 0; \tag{3.8}$$

$$v^{(1)} = \eta_x^{(1)} - \eta^{(0)}\eta_x^{(0)}, \qquad \text{when} \quad y = 1; \tag{3.9}$$

$$\eta^{(1)} - \tau\eta_{xx}^{(1)} - F^{(1)}\eta^{(0)} + u^{(1)} + \tfrac{1}{2}(\eta^{(0)})^2 = 0, \qquad \text{when} \quad y = 1; \tag{3.10}$$

$$\eta^{(1)}, \qquad u^{(1)}, \qquad \text{and} \quad v^{(1)} \to 0 \quad \text{as} \quad |x| \to \infty. \tag{3.11}$$

(3.7), (3.8) and (3.9) give

$$u^{(1)}(x, y) = -f(x) + \frac{y^2}{2}\eta_{xx}^{(0)}(x),$$

$$v^{(1)}(x, y) = y[f_x(x) + \eta^{(0)}(x)\eta_x^{(0)}(x)] - \frac{y^3}{6}\eta_{xxx}^{(0)}(x),$$

where f is so far free. Substituting these formulas into (3.10) and the result into (3.9), we find that $\eta^{(0)}$ must satisfy the ordinary differential equation

$$(\tfrac{1}{3} - \tau)\eta^{(0)}_{xxx} = F^{(1)}\eta^{(0)}_x - 3\eta^{(0)}\eta^{(0)}_x. \tag{3.12}$$

4. The case $\tau \geq \frac{1}{3}$. Suppose first that $\tau = \frac{1}{3}$. Then (3.12) shows that $\eta^{(0)}$ is constant. Thus, by (2.7), H is constant (to $O(\varepsilon^4)$), and this violates our original hypothesis.

Next, if $\tau \neq \frac{1}{3}$, (3.12) can be integrated twice to give

$$(\tfrac{1}{3} - \tau)(\eta^{(0)}_x)^2 = (F^{(1)} - \eta^{(0)})(\eta^{(0)})^2. \tag{4.1}$$

This shows that $\frac{1}{3} - \tau$ and $F^{(1)} - \eta^{(0)}$ must have the same sign, so that, sending x to infinity, $\frac{1}{3} - \tau$ and $F^{(1)}$ must have the same sign. Thus, when $\tau > \frac{1}{3}$, $F^{(1)} < 0$. Integrating (4.1) when $(\frac{1}{3} - \tau)$ and $F^{(1)}$ are both negative, we find there is no solution which is finite for all x.

Thus, as said in the introduction, there can be no solution to our problem when $\tau \geq \frac{1}{3}$, at least when η, u and v are analytic in ε. This proof is completely valid only for the lowest-order terms in the expansions. However, similar arguments show that there can be no higher-order terms either when $\tau \geq \frac{1}{3}$.

5. The solitary wave. When $\tau < \frac{1}{3}$, the argument of Sec. 4 still applies, this time to show that $F^{(1)} > 0$. To solve (4.1) with a minimum of square roots and other unpleasantnesses, write $F^{(1)} = 4a^2(1 - 3\tau)/3$. Then, integrating (4.1), we find

$$\eta^{(0)} = \frac{4a^2}{3}(1 - 3\tau)\operatorname{sech}^2 ax, \tag{5.1}$$

where (3.5) has been used.

Returning to the original variables F and H, we see that

$$F = 1 + \tfrac{4}{3}\varepsilon^2 a^2(1 - 3\tau) + O(\varepsilon^4 a^4) \tag{5.2}$$

and

$$H = h[1 + \tfrac{4}{3}\varepsilon^2 a^2(1 - 3\tau)\operatorname{sech}^2 \frac{\varepsilon a X}{h} + O(\varepsilon^4 a^4)]. \tag{5.3}$$

Clearly everything here is a function of the product εa and not of ε or a alone, and it is εa that measures the amplitude of H. Replacing εa by a, say, and setting $\tau = 0$, we find that (5.2) and (5.3) reduce to the usual formulas [1, 2] for a solitary wave with no surface tension. As described in Sec. 1, the effect of surface tension is to reduce the quantity F-1 and the amplitude of the wave, both by a factor $1 - 3\tau$.

(5.3) shows clearly that $h/(\varepsilon a)$ is the characteristic length associated with the wave, and (5.1)–(5.2) is the appropriate form for the solution with fixed characteristic length. On the other hand, it may also be desirable to consider the wave for fixed amplitude. In that case, we impose the condition (2.8) on (5.1) to find $4a^2(1 - 3\tau) = 3$. With this value of a, (5.2) and (5.3) become

$$F = 1 + \varepsilon^2 + O(\varepsilon^4), \qquad H = h\left[1 + \varepsilon^2 \operatorname{sech}^2 \frac{\varepsilon X\sqrt{3}}{2h\sqrt{1 - 3\tau}} + O(\varepsilon^4)\right]. \tag{5.4}$$

For a fixed amplitude, then, we see from (5.4) that the effect of surface tension is to sharpen the crest of the wave by shortening its characteristic length by a factor $(1 - 3\tau)^{1/2}$.

A final remark before ending. Although we have carried out the analysis only to $O(\varepsilon^2)$, it is obvious that the process can be taken as far as desired by the method suggested in Sec. 2: expanding all quantities occurring in (2.10)–(2.14) in series of powers of ε^2, equating coefficients of like powers, and solving the resulting equations recursively. However, the calculation to $O(\varepsilon^2)$ is surely enough for most purposes.

References

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