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ASYMPTOTIC EXPANSION OF THE HANKEL TRANSFORM WITH EXPLICIT REMAINDER TERMS*

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1. Introduction. Let

$$F(x) = \int_0^\infty K(xt)\, f(t)\, dt \tag{1.1}$$

be the K-transform of f. In recent years various techniques have been developed to obtain explicit expressions for the remainder in the asymptotic expansions of functions defined by (1.1) as $x \to \infty$. A survey of such techniques is given by Wong [17]. It is well known that under some reasonable assumptions on f and K, the Parseval relation for the Mellin transform provides a powerful tool for obtaining the asymptotic expansion of $F(x)$. However, until recently the potential of this technique for obtaining an explicit expression for the remainder had been largely overlooked. If $M[K, s]$ is the Mellin transform of K evaluated at s, $M[f, 1-s]$ is the Mellin transform of f evaluated at $1-s$, and the integrals defining these transforms converge in a strip containing the line $\operatorname{Re} s = c$ then, formally, by the Parseval relation,

$$\int_0^\infty K(xt) f(t)\, dt = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-s}\, M[K, s]\, M[f, 1-s]\, ds. \tag{1.2}$$

Handelsman and Lew [3] had shown that for a large class of kernels K and functions f, $M[K, s]$ and $M[f, 1-s]$ can be analytically continued to meromorphic functions in the right half plane. If we can shift the line of integration from $\operatorname{Re} s = c$ to $\operatorname{Re} s = d > c$, then by the residue theorem,

$$F(x) = -\sum_{c<\operatorname{Re} s<d} \operatorname{Res}\,\{x^{-s}\, M[K, s]\, M[f, 1-s]\} + E \tag{1.3}$$

where

$$E = \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} x^{-s}\, M[K, s]\, M[f, 1-s]\, ds. \tag{1.4}$$

In [11] we proved that for a certain class of kernels K, the remainder E can be given as

$$E = x^{-p} \int_0^\infty K(xt, p)\, R_n^{(p)}(t)\, dt, \tag{1.5}$$

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where $K(t, p)$ is the inverse Mellin transform of $M[K, s+p]\,\Gamma(s)/\Gamma(s+p)$ and $R_n(t)$ is the nth remainder in the asymptotic expansion of $f(t)$ near the origin. In the present paper we apply the Mellin transform technique to obtain explicit expression for the remainder in the asymptotic expansion of the finite Hankel transform. Although we consider a specific kernel, $K(t) = t^{-\nu} J_\nu(t)$, the technique developed here is quite general and can be applied to other kernels such as the Bessel function $Y_\nu(t)$ and the Airy function $A_i(-t)$. For finite integral transforms with such kernels, the Handelsman-Lew technique [1, Chapter 4] cannot be applied without modification because, in general, $M[K, s]\, M[f, 1-s]$ in (1.2) does not approach zero as $|\mathrm{Im}\, s| \to \infty$ in a strip wide enough to the right of the line $\mathrm{Re}\, s = c$. In this paper we show that by first separating the terms which tend to zero slowly near the line $\mathrm{Re}\, s = c$, we can shift the line of integration in (1.2) to the right. The terms that we separate provide the contribution to the asymptotic approximation from the right endpoint of the interval of integration. The remainder is given by (1.4). For the computation of realistic error bounds, we need to express the remainder E in as simple a form as we possibly can. In [11], this was achieved by using simple integration by parts to express $M[K, s]\, M[f, 1-s]$ in an equivalent but different form. For the Hankel transform, we use the differential operator $t^{-1}\, d/dt$ to integrate by parts. In general, the differential operator is suggested by the form of the Mellin transform of the kernel. Although the asymptotic expansion of the Hankel transform is usually discussed with $J_\nu(t)$ as the transform kernel (see for example [12] and [16]), we use the kernel $J_\nu(t)/t^\nu$ because its Mellin transform has a very simple structure,

$$M[J_\nu(t)/t^\nu, s] = \frac{2^s\, \Gamma(\frac{1}{2}s)}{2^{\nu+1}\, \Gamma(\nu + 1 - \frac{1}{2}s)}. \tag{1.6}$$

Since the parameter ν appears only in the demoninator, it is easy to see how we can modify the basic technique in [11] to change $M[f, 1-s]$ so that the kernel in the remainder can be identified easily. By this technique we obtain the same form of the remainder as given by Wong by a different technique when the interval of integration is $(0, \infty)$. A comparison with his results is given in Sec. 3.

The error terms in the asymptotic approximation of the finite Hankel transform can also be obtained by other techniques such as that used by Olver [7] and Wong [16]. A completely different approach is used by Soni and Soni [12]. However, as mentioned earlier, our objective is to show that the basic Mellin transform technique can be modified to obtain explicit expressions for the remainder when the Mellin transform of the kernel is known and

$$f(t) \sim \sum_{m,n=0}^{\infty} a_{mn}\, t^{\alpha_m} (\log t)^n, \quad t \to 0+, \tag{1.7}$$

where $\mathrm{Re}\, \alpha_m \uparrow \infty$ as $m \to \infty$ and $\{n : a_{mn} \neq 0\}$ is finite for each m. As in [11], we assume for the sake of simplicity that the logarithmic singularities of f are only of the first degree. Logarithmic singularities of higher integral order can be treated similarly.

2. Notation and Basic Assumptions. Let $a > 0$. We assume that $f(t)$ is a real or complex-valued function, absolutely integrable in $(0, a)$ and $f(t) = 0$ for $t > a$. For $K(t) = J_\nu(t)/t^\nu$, (1.1) reduces to

$$F(x) = \int_0^a (xt)^{-\nu} J_\nu(xt) f(t)\, dt. \tag{2.1}$$

In the rest of the paper we assume that $F(x)$ is defined by (2.1). Since $J_\nu(t)/t^\nu$ is continuously differentiable in $[0, \infty)$, $F(x)$ also is continuously differentiable. Following the notation used in [11] we write

$$f(t) = f_{n-1}(t) + R_n(t) \tag{2.2}$$

where

$$\begin{gathered} f_{n-1}(t) = \sum_{k=0}^{n-1} (a_k + b_k \log t)\, t^{\alpha_k} \\ -1 < \operatorname{Re} \alpha_0 \leq \operatorname{Re} \alpha_1 \leq \cdots \leq \operatorname{Re} \alpha_{n-1}, \\ |a_0| + |b_0| \neq 0, \end{gathered} \tag{2.3}$$

and

$$R_n(t) = O(t^{\alpha_n}),\ t \to 0+,\ \operatorname{Re} \alpha_{n-1} < \operatorname{Re} \alpha_n. \tag{2.4}$$

The variable s is complex; the real and imaginary parts of s are denoted by σ and τ respectively. The Mellin transform of a function $\phi(t)$ evaluated at s is denoted by $M[\phi, s]$ or $M[\phi(t), s]$. Thus

$$M[\phi, s] = M[\phi(t), s] = \int_0^\infty t^{s-1}\, \phi(t)\, dt,\ s = \sigma + i\tau, \tag{2.5}$$

whenever the above integral converges. As is usual, $M[\phi, s]$ also denotes the function which is an analytic continuation of the function element defined by (2.5) in the complex s-plane.

As in [12], $\mathscr{D}$ is the differential operator defined by

$$\mathscr{D} = \mathscr{D}^1 = t^{-1} \frac{d}{dt},\ \mathscr{D}^n = \mathscr{D}\, \mathscr{D}^{n-1}. \tag{2.6}$$

The function ψ is the logarithmic derivative of the gamma function and $(\alpha)_n$ is Pochhammer's symbol, $(\alpha)_n = \Gamma(\alpha + n)/\Gamma(\alpha)$.

3. Statement of results.

THEOREM 1. If $f(t)$ satisfies (2.2)–(2.4) and

(i) $f(t)$ is p times continuously differentiable in $(0, a]$ where

$$\operatorname{Re} \alpha_{n-1} < \operatorname{Re} \nu + p + \tfrac{1}{2}, \tag{3.1}$$

(ii) $f^{(k)}(t) = f_{n-1}^{(k)}(t) + O(t^{\alpha_n - k}),\ t \to 0+,\ k = 0, 1, \ldots, p$;

then

$$\begin{aligned} \int_0^a (xt)^{-\nu} J_\nu(xt) f(t)\, dt = {} & \sum_{k=0}^{n-1} 2^{\alpha_k - \nu} x^{-1-\alpha_k} \mathscr{R}(\nu, \alpha_k) \\ & + \sum_{k=0}^{p-1} (-1)^k a^{\nu+k+1} x^{-\nu-k-1} J_{\nu+k+1}(ax) (\mathscr{D}^k\, t^{-2\nu-1} f(t))_{t=a} \\ & + E(x), \end{aligned} \tag{3.2}$$

where

$$\mathscr{R}(\nu, \alpha_k) = \frac{\Gamma(\frac{1}{2} + \frac{1}{2}\alpha_k)}{\Gamma(\nu + \frac{1}{2} - \frac{1}{2}\alpha_k)} \times [a_k + b_k\{\log(2/x) + \tfrac{1}{2}\psi(\tfrac{1}{2} + \tfrac{1}{2}\alpha_k) + \tfrac{1}{2}\psi(\nu + \tfrac{1}{2} - \tfrac{1}{2}\alpha_k)\}] \tag{3.3}$$

and

$$E(x) = (-1)^p \int_0^\infty (xt)^{-\nu-p} J_{\nu+p}(xt)\, \mathscr{D}^p(t^{-2\nu-1} R_n(t))\, t^{2\nu+1+2p}\, dt. \tag{3.4}$$

If for some k, $\nu + \frac{1}{2} - \frac{1}{2}\alpha_k = -m$, $m = 0, 1, \ldots,$ then

$$\mathscr{R}(\nu, \alpha_k) = \tfrac{1}{2}(-1)^{m+1}\, \Gamma(m+1)\, \Gamma(\tfrac{1}{2} + \tfrac{1}{2}\alpha_k) b_k. \tag{3.5}$$

Note that by condition (i) above, $f^{k)}(t)$ is continuous in $(0, a]$ for $k = 1, \ldots\, p$. Also, as stated in Sec. 2, $f(t) = 0$ for $t > a$. Therefore, for $k = 0, 1, \ldots, p$, $R_n^{(k)}(t) = f^{(k)}(t) - f_{n-1}^{(k)}(t)$ is continuous in $(0, \infty)$ except perhaps at $t = a$ where it may have a jump discontinuity. In Sec. 5 we shall show that the remainder $E(x)$ in the asymptotic approximation of $F(x)$ as given by (3.2) is of a lower order as $x \to \infty$ than the order of the larger of the two terms, $x^{-\nu-p-1/2}$ and $x^{-\alpha_{n-1}-1} \log x$. For the numerical computation of error bounds, it is more convenient to write

$$E(x) = E_1(x, \nu, p) - E_2(x, \nu, p) \tag{3.6}$$

where

$$E_1(x, \nu, p) = (-1)^p \int_0^a (xt)^{-\nu-p} J_{\nu+p}(xt)\, \mathscr{D}^p(t^{-2\nu-1} R_n(t))\, t^{2\nu+1+2p}\, dt, \tag{3.7}$$

$$E_2(x, \nu, p) = (-1)^p \int_a^\infty (xt)^{-\nu-p} J_{\nu+p}(xt)\, \mathscr{D}^p(t^{-2\nu-1} f_{n-1}(t)) t^{2\nu+1+2p}\, dt. \tag{3.8}$$

For fixed ν, p, and a, E_2 depends only on the behavior of f near the origin and can be given precisely. As $a \to \infty$, $E_2(x, \nu, p) \to 0$ and we have the following:

THEOREM 2. If $f(t)$ satisfies (2.2)–(2.4) and

(i) $f(t)$ is p times continuously differentiable in $(0, \infty)$ where p satisfies (3.1);

(ii) $f^{(k)}(t) = f_{n-1}^{(k)}(t) + O(t^{\alpha_n - k})$, $t \to 0+$, $k = 0, 1, \ldots,$ p;

(iii) The integrals

$$\int_0^N J_\nu(xt)\, t^{-\nu} f(t)\, dt, \int_0^N J_{\nu+p}(xt)\, t^{-\nu-k} f^{(p-k)}(t)\, dt,\ k = 0, 1 \tag{3.9}$$

converge as $N \to \infty$;

(iv) $f^{(k)}(t) = o(t^{\nu+1/2})$, $t \to \infty$, $k = 0, 1, \ldots, p-1$;

then

$$\int_0^\infty (xt)^{-\nu} J_\nu(xt) f(t)\, dt = \sum_{k=0}^{n-1} 2^{\alpha_k - \nu} x^{-1-\alpha_k}\, \mathscr{R}(\nu, \alpha_k) + E(x) \tag{3.10}$$

where $\mathscr{R}(\nu, \alpha_k)$ is defined by (3.3) and $E(x)$ by (3.4).

For $\alpha_k = \nu + k + \lambda - 1$, $k = 0, 1, \ldots, n$ and $b_k = 0$, $k = 0, 1, \ldots, n-1$ (see (2.3) and (2.4)), the above theorem is due to Wong [16]. The condition Q_3 in [16] needs strengthening. (Compare (2.2) in [16] with (3.9) above.) For a discussion of this, see [17, p. 409] and [12]. Also, (3.1) above is weaker than the corresponding condition (3.5) in [16]. We must mention here that Theorem 1 holds without the conditions (3.1). This condition is required only for the convergence of the integral (3.4) defining $E(x)$. We can write E in terms of E_1 and E_2. For fixed x and p, E_1 is an entire function of ν and E_2 is analytic in the half ν-plane $\operatorname{Re} \nu > \operatorname{Re} \alpha_{n-1} - p - \frac{1}{2}$. Furthermore, using integration by parts in (3.8) we can show that E_2 can be continued analytically into the whole complex ν-plane and has no singularities there. Thus, by analytic continuation, the expansion (3.2) is valid for every complex ν. In Theorem 2, we can remove the condition (3.1) provided that we impose some additional restriction on f. If the integrals in the condition (iii) converge uniformly with respect to ν on compact sets in the ν-plane, in particular if $t^{-\nu-1/2} f(t)$, $t^{-\nu-1/2} f^{(p)}(t)$ and $t^{-\nu-3/2} f^{(p-1)}(t)$ are absolutely integrable in $(1, \infty)$, then we can show that by analytic continuation, (3.10) holds for all complex ν.

4. Proof of Theorem 1. Without loss of generality we may assume that $\nu > -\frac{1}{2}$. Since $t^{-\nu} J_\nu(t)$ is an entire function of ν, the expansion (3.2) can be extended to other values to ν by analytic continuation in the complex ν-plane. To prove that $F(x)$ can be expressed as the contour integral in (1.2), consider its Mellin transform. For $\operatorname{Re} s > 0$,

$$\int_0^N x^{s-1} F(x)\, dx = \int_0^a f(t)\, t^{-s} \int_0^{Nt} u^{-\nu+s-1} J_\nu(u)\, du\, dt. \tag{4.1}$$

The repeated integral on the right is obtained by using (2.1), then reversing the order of integration which is justified by the absolute convergence of the double integral and finally by applying a change of variable. As $N \to \infty$, the last integral converges to $M[f, 1-s] \cdot M[u^{-\nu} J_\nu(u), s]$ uniformly in $\delta_1 \le \operatorname{Re} s \le \delta_2$ where δ_1, δ_2 are arbitrary numbers that satisfy $0 < \delta_1 < \delta_2 < \min(1, 1 + \operatorname{Re} \alpha_0)$. By [15, p. 391],

$$M[u^{-\nu} J_\nu(u), s] = 2^{s-\nu-1}\, \Gamma(s/2)/\Gamma(\nu + 1 - s/2), \tag{4.2}$$

and by [13, p. 151],

$$\Gamma(s/2)/\Gamma(\nu + 1 - s/2) = O(|\tau|^{\sigma-\nu-1}),\ |\tau| \to \infty,\ s = \sigma + i\tau. \tag{4.3}$$

$M[f, 1-s]$ is analytic in $\operatorname{Re} s < 1 + \operatorname{Re} \alpha_0$. Furthermore, integrating once by parts,

$$\int_0^a t^{-s} f(t)\, dt = (1-s)^{-1} \left\{ a^{1-s} f(a) - \int_0^a t^{1-s} f'(t)\, dt \right\}. \tag{4.4}$$

From (4.3) and (4.4) it follows that if $0 < c < \min(1/2, 1 + \operatorname{Re} \alpha_0)$ and $\nu > -\frac{1}{2}$, then $M[f, 1-s]\, M[u^{-\nu} J_\nu(u), s]$ is absolutely integrable along the line $\operatorname{Re} s = c$. Since $F(x)$ is continuous, by [14, p. 47] we have

$$F(x) = (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} x^{-s}\, 2^{s-\nu-1}\, (\Gamma(s/2)/\Gamma(\nu + 1 - s/2)) M[f, 1-s]\, ds. \tag{4.5}$$

To obtain an asymptotic expansion for $F(x)$ as $x \to \infty$, we need to shift the line of integration in the above integral to the right. Since f satisfies (2.2)–(2.4), $M[f, 1-s]$ can be continued analytically into $\operatorname{Re} s < \operatorname{Re} \alpha_n + 1$ and its only singularities in the region $0 < \operatorname{Re} s < \operatorname{Re} \alpha_n + 1$ are poles of order two at most at $s = \alpha_k + 1$, $k = 0, 1, \ldots, (n-1)$ (see, for example, [3] and [11]). To obtain the principal part of $M[f, 1-s]$ at $s = \alpha_k + 1$, we write

$$M[f, 1-s] = \int_0^1 t^{-s}(f(t) - f_{n-1}(t))\, dt + \int_1^a t^{-s} f(t)\, dt$$
$$+ \sum_{k=0}^{n-1} (a_k(\alpha_k + 1 - s)^{-1} - b_k(\alpha_k + 1 - s)^{-2}). \tag{4.6}$$

The functions represented by the two integrals above have no sinularities in $\operatorname{Re} s < \operatorname{Re} \alpha_n + 1$. Thus, the poles of $M[f, 1-s]$ are the only singularities of the integrand in (4.5) in the region $0 < \operatorname{Re} s < \operatorname{Re} \alpha_n + 1$. Let $\operatorname{Res}(\alpha_k + 1)$ denote the residue of $x^{-s} \cdot 2^{s-\nu-1}(\Gamma(s/2)/\Gamma(\nu + 1 - s/2))M[f, 1-s]$ at $\alpha_k + 1$. Then, by (4.6), for $k = 0, 1, \ldots, n-1$,

$$\operatorname{Res}(\alpha_k + 1) = -2^{\alpha_k - \nu} x^{-1-\alpha_k} \mathscr{R}(\nu, \alpha_k) \tag{4.7}$$

where $\mathscr{R}(\nu, \alpha_k)$ is defined by (3.3).

By the condition (3.1) on p and the fact that $\operatorname{Re} \alpha_{n-1} < \operatorname{Re} \alpha_n$, we can choose a positive number d which satisfies

$$\operatorname{Re} \alpha_{n-1} + 1 < d < \min(\alpha_n + 1, \nu + p + 3/2). \tag{4.8}$$

If $\operatorname{Re} s \geq \nu + 2$, the integrand in (4.5) does not approach zero as $|\operatorname{Im} s| \to \infty$ unless $f(a) = 0$ (see (4.3) and (4.4)). Therefore, for $d \geq \nu + 2$, we cannot shift the line of integration in (4.5) from $\operatorname{Re} s = c$ to $\operatorname{Re} s = d$. However, by the residue theorem and (4.7) it follows that

$$F(x) = \sum_{k=0}^{n-1} 2^{\alpha_k - \nu} x^{-1-\alpha_k} \mathscr{R}(\nu, \alpha_k)$$
$$+ (2\pi i)^{-1} \int_C x^{-s} 2^{s-\nu-1} (\Gamma(s/2)/\Gamma(\nu + 1 - s/2))\, M[f, 1-s]\, ds, \tag{4.9}$$

where C is the contour shown in Fig. 1. $M[f, 1-s]$ can be expressed in a form which is more desirable than (4.4), by applying integration by parts as follows. For $\operatorname{Re} s < 1 + \operatorname{Re} \alpha_0$,

$$\int_0^a f(t)\, t^{-s}\, dt = \frac{a^{1-s}}{2(\nu + 1 - s/2)} f(a)$$
$$- \frac{1}{2(\nu + 1 - s/2)} \int_0^a \mathscr{D}(t^{-2\nu-1} f(t))\, t^{2\nu+3-s}\, dt, \tag{4.10}$$

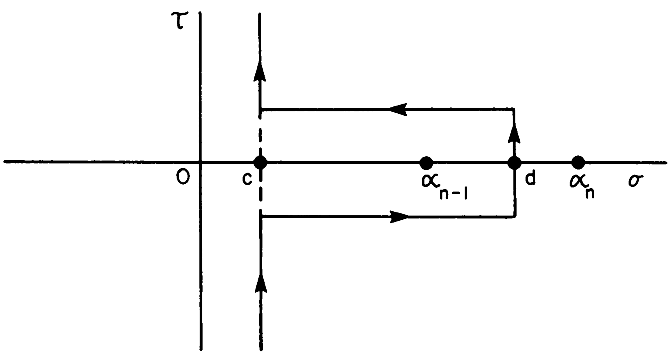

FIG. 1.

where $\mathscr{D}$ is defined by (2.6). We integrate by parts in this manner p times and in the final integral replace $f(t)$ by $R_n(t) + f_{n-1}(t)$. Thus,

$$M[f, 1-s] = \sum_{k=0}^{p-1} \frac{(-1)^k 2^{-k-1} a^{2\nu+2k+2-s}}{(\nu+1-s/2)_{k+1}} (\mathscr{D}^k t^{-2\nu-1} f(t))_{t=a}$$
$$+ \frac{(-2)^{-p}}{(\nu+1-s/2)_p} \int_0^a \mathscr{D}^p(t^{-2\nu-1} R_n(t))\, t^{2\nu+2p+1-s}\, dt$$
$$+ \frac{(-2)^{-p}}{(\nu+1-s/2)_p} M[h(t), 1-s] \tag{4.11}$$

where

$$h(t) = \mathscr{D}^p(t^{-2\nu-1} f_{n-1}(t))\, t^{2\nu+2p+1}, \quad 0 < t \le a,$$
$$= 0, \quad t > a. \tag{4.12}$$

But

$$\mathscr{D}^p(t^{-2\nu-1} f_{n-1}(t))\, t^{2\nu+2p+1} = \sum_{k=0}^{n-1} (A_{kp} + B_{kp} \log t)\, t^{\alpha_k} \tag{4.13}$$

where A_{kp}, B_{kp} are constants. Therefore,

$$M[h, 1-s] = \sum_{k=0}^{n-1} A_{kp} a^{\alpha_k - s+1}(\alpha_k - s + 1)^{-1}$$
$$+ \sum_{k=0}^{n-1} B_{kp} a^{\alpha_k - s+1}(\alpha_k - s + 1)^{-1}\{\log a - (\alpha_k - s + 1)^{-1}\}. \tag{4.14}$$

Furthermore, the integral in (4.11) converges absolutely in $\operatorname{Re} s < \operatorname{Re} \alpha_n + 1$. Therefore, $M[f, 1-s]$ can be continued analytically into the half plane $\operatorname{Re} s < \operatorname{Re} \alpha_n + 1$. Now denote the integral in (4.9) by I. By substituting the expression obtained for $M[f, 1-s]$ in (4.11), we can decompose I into the desired form. By means of the relation $\Gamma(\nu+1-s/2)(\nu+1-s/2)_{k+1} = \Gamma(\nu+k+2-s/2)$, we write

$$I = \sum_{k=0}^{p-1} (-1)^k (\mathscr{D}^k t^{-2\nu-1} f(t))_{t=a} I_{1,k} + (-1)^p I_2 + (-1)^p I_3 \tag{4.15}$$

where

$$I_{1,k} = (2\pi i)^{-1} \int_C \frac{x^{-s} 2^{s-\nu-k-2} a^{2\nu+2k+2-s} \Gamma(s/2)}{\Gamma(\nu+k+2-s/2)}\, ds, \tag{4.16}$$

$$I_2 = (2\pi i)^{-1} \int_C \frac{x^{-s} 2^{s-\nu-p-1} \Gamma(s/2)}{\Gamma(\nu+p+1-s/2)}$$
$$\cdot \int_0^a \mathscr{D}^p(t^{-2\nu-1} R_n(t)) t^{2\nu+2p+1-s}\, dt\, ds, \tag{4.17}$$

$$I_3 = (2\pi i)^{-1} \int_C \frac{x^{-s} 2^{s-\nu-p-1} \Gamma(s/2)}{\Gamma(\nu+p+1-s/2)} M[h, 1-s]\, ds. \tag{4.18}$$

The integrand in $I_{1,k}$ has no singularity in $\operatorname{Re} s > 0$. Therefore the coutour C can be replaced by the straight line $\operatorname{Re} s = c$, $0 < c < 1/2$. Along this line, the integral converges absolutely for $k = 0, 1, \ldots, p-1$. By using [15, p. 192, Eq. (7)], we obtain

$$I_{1,k} = x^{-\nu-k-1}a^{\nu+k+1}J_{\nu+k+1}(ax). \tag{4.19}$$

In consequence of the condition (ii), the inner integral in I_2 converges absolutely in $\operatorname{Re} s < \operatorname{Re} \alpha_n + 1$. Therefore, the integrand has no singularity in the strip $0 < \operatorname{Re} s < \operatorname{Re} \alpha_n + 1$ and we can replace the contour C by the straight line $\operatorname{Re} s = c$ in this case also. By [15, p. 391], for each fixed x, $0 < x < \infty$,

$$M[(xt)^{-\nu-p}J_{\nu+p}(xt), s] = x^{-s}2^{s-\nu-p-1}\Gamma(s/2)/\Gamma(\nu+p+1-s/2). \tag{4.20}$$

By using the estimate (4.3), we conclude that $M[(xt)^{-\nu-p}J_{\nu+p}(xt), c+i\tau]$ as a function of τ belongs to $L(-\infty, \infty)$. Furthermore, the Mellin transform of the function, which equals $\mathscr{D}^p(t^{-2\nu-1}R_n(t))t^{2\nu+2p+1}$ for $0 < t \le a$ and is zero for $t > a$, converges absolutely for $s = -c$. Therefore, by the Parseval relation for the Mellin transform [14, Theorem 42],

$$I_2 = \int_0^a (xt)^{-\nu-p}J_{\nu+p}(xt)\mathscr{D}^p(t^{-2\nu-1}R_n(t))t^{2\nu+2p+1}\,dt. \tag{4.21}$$

In the case of I_3, we cannot replace the contour C by the straight line $\operatorname{Re} s = c$ as we did for the integrals considered above. However, in consequence of (4.3) and (4.14), we can replace C by the straight line $\operatorname{Re} s = d$. The integral along this line may not converge absolutely. It is possible to use (4.13) and evaluate the integral term by term but it is more convenient to prove that

$$I_3 = -\int_a^\infty (xt)^{-\nu-p}J_{\nu+p}(xt)\mathscr{D}^p(t^{-2\nu-1}f_{n-1}(t))t^{2\nu+2p+1}\,dt, \tag{4.22}$$

by verifying the conditions for the validity of the Parseval relation. Referring back to the inequality (4.8), we see that for each fixed x, the Mellin transform of $(xt)^{-\nu-p}J_{\nu+p}(xt)$ which is given in (4.20), converges uniformly in a strip containing $\operatorname{Re} s = d$. If we define

$$\begin{aligned} g(t) &= 0, & 0 \le t \le a,\\ &= \mathscr{D}^p(t^{-2\nu-1}f_{n-1}(t))t^{2\nu+2p+1}, & a < t, \end{aligned} \tag{4.23}$$

then, as a consequence of (4.13), the Mellin transform of $g(t)$ converges absolutely in a strip containing $\operatorname{Re} s = 1-d$ and $M[g, 1-s] = -M[h, 1-s]$. By means of (4.13) again, we can verify that for each fixed x, $0 < x < \infty$ and for $r = d$, the function

$$\xi^{1-d}(\xi^r + \xi^{-r})^{-1}\int_1^N (xt)^{-\nu-p}J_{\nu+p}(xt)g(\xi t)\,dt \tag{4.24}$$

is bounded uniformly in $N \ge 1$, $0 < \xi < \infty$; and when $N \to \infty$, it represents a function continuous at $\xi = 1$. Therefore, by [14, Theorem 43], (4.22) holds.

By (4.19), (4.21), and (4.22), we obtain I which represents the contour integral in (4.9). Since $I_2 = E_1$ and $I_3 = E_2$, we obtain the expansion (3.2).

Proof of Theorem 2. Let $a \to \infty$ in (3.2). We observe that for $k = 0, 1, \ldots, p$,

$$\mathscr{D}^k(t^{-2\nu-1}f(t))t^{2\nu+2k+1} = \sum_{l=0}^{k} C_{lk}t^{k-l}f^{(k-l)}(t). \tag{4.25}$$

Therefore, by the condition (iv), for $k = 0, 1, \ldots, p-1$,

$$a^{\nu+k+1} J_{\nu+k+1}(ax)(\mathscr{D}^k t^{-2\nu-1} f(t))_{t=a} \to 0. \tag{4.26}$$

By the condition (3.1), $E_2 \to 0$ and finally, by (4.25) and the conditions (iii), (iv) and (3.1), $E_1 \to E$. Thus we obtain (3.10).

5. Order of the remainder. We will prove that the asymptotic behavior of $F(x)$ as $x \to \infty$, can be obtained from (3.2) without any restriction on ν provided that $E(x)$ denotes the function obtained from (3.4) by analytic continuation in the complex ν-plane. To show that $E(x)$ is indeed defined in the whole complex ν-plane, consider the integrals

$$G(x, \nu) = \int_a^\infty t^\alpha \, t^{-\nu-p} \, J_{\nu+p}(xt) \, dt \tag{5.1}$$

and

$$H(x, \nu) = \int_a^\infty t^\alpha \log t \; t^{-\nu-p} \, J_{\nu+p}(xt) \, dt. \tag{5.2}$$

G and H are analytic functions of ν in $\operatorname{Re} \nu > \operatorname{Re} \alpha - p - 1/2$. By using

$$\int y^\mu J_{\mu-1}(y) \, dy = y^\mu J_\mu(y) \tag{5.3}$$

to integrate G by parts l times we obtain

$$\begin{aligned} G(x, \nu) = & \sum_{k=1}^{l} (-x)^{-k} \, 2^{k-1} \, a^{\alpha-\nu-p-k+1} \left(\frac{\alpha}{2} - \frac{1}{2} - \nu - p\right)_{k-1} J_{\nu+p+k}(ax) \\ & + (-x)^{-l} \, 2^l \left(\frac{\alpha}{2} - \frac{1}{2} - \nu - p\right)_l \int_a^\infty t^\alpha \, t^{-\nu-p-l} \, J_{\nu+p+l}(xt) dt. \end{aligned} \tag{5.4}$$

This provides the analytic continuation of G into the half plane $\operatorname{Re} \nu > \operatorname{Re} \alpha - p - l - \frac{1}{2}$. Since l is arbitrary, G can be continued analytically into the whole complex ν-plane. Furthermore, if we choose l so that the integral on the right in (5.4) converges absolutely, we conclude that for any fixed ν, α, and p,

$$G(x, \nu) = O(x^{-3/2}), \; x \to \infty. \tag{5.5}$$

We use the same technique to show that H can be continued analytically into the whole complex ν-plane and for any fixed ν, α, and p,

$$H(x, \nu) = O(x^{-3/2} \log x), \; x \to \infty. \tag{5.6}$$

By (4.13), (5.5) and (5.6) it follows that for $p \geq 1$ and any fixed ν,

$$E_2(x, \nu, p) = O(x^{-\nu-p-3/2} \log x), \; x \to \infty, \tag{5.7}$$

where E_2 is defined by (3.8). Now we consider E_1. Let

$$(-1)^p \, E_1(x, \nu, p) = I_4 + I_5 \tag{5.8}$$

where

$$I_4 = \int_0^{1/x} (xt)^{-\nu-p} \, J_{\nu+p}(xt) \, \mathscr{D}^p(t^{-2\nu-1} \, R_n(t)) \, t^{2\nu+2p+1} \, dt \tag{5.9}$$

and

$$I_5 = \int_{1/x}^{a} (xt)^{-\nu-p} J_{\nu+p}(xt)\, \mathscr{D}^p(t^{-2\nu-1} R_n(t))\, t^{2\nu+2p+1}\, dt. \tag{5.10}$$

By (2.2) and the condition (ii) of Theorem 1,

$$\mathscr{D}^p(t^{-2\nu-1} R_n(t))\, t^{2\nu+2p+1} = O(t^{\alpha_n}),\ t \to 0+. \tag{5.11}$$

Since for fixed ν and p, $y^{-\nu-p} J_{\nu+p}(y)$ is bounded in $0 \le y \le 1$,

$$I_4 = O\left(\int_0^{1/x} t^{\alpha_n}\, dt\right) = O(x^{-\alpha_n-1}),\ x \to \infty. \tag{5.12}$$

We can obtain sharper estimates for the asymptotic behavior of I_5 as $x \to \infty$ if we consider the cases $\operatorname{Re}\nu \ge \operatorname{Re}\alpha_n - p + 1/2$ and $\operatorname{Re}\nu < \operatorname{Re}\alpha_n - p + 1/2$ separately. For $\operatorname{Re}\nu \ge \operatorname{Re}\alpha_n - p + 1/2$, by using (5.11) and the fact that for fixed ν and p, $y^{1/2} J_{\nu+p}(y)$ is bounded when $y \ge 1$,

$$I_5 = O\left(\int_{1/x}^{a} (xt)^{-\nu-p} (xt)^{-1/2} t^{\alpha_n}\, dt\right) = O\left(x^{-\alpha_n-1} \int_1^{ax} u^{-\nu-p+\alpha_n-1/2}\, du\right),\ x \to \infty. \tag{5.13}$$

The last integral is bounded as $x \to \infty$ unless $\nu = \alpha_n - p + 1/2$, in which case it is of the same order as $\log x$. Therefore, for $\operatorname{Re}\nu \ge \operatorname{Re}\alpha_n - p + 1/2$,

$$I_5 = O(x^{-\alpha_n-1} \log x),\ x \to \infty. \tag{5.14}$$

For $\operatorname{Re}\nu < \operatorname{Re}\alpha_n - p + 1/2$, we use the asymptotic behavior of $J_{\nu+p}(xt)$. By [15, p. 199],

$$\begin{aligned} I_5 &= \int_{1/x}^{a} x^{-\nu-p-1/2} g(t) \cos(xt - (\nu+p)\pi/2 - \pi/4)\, dt \\ &\quad + O\left(\int_{1/x}^{a} x^{-\nu-p-3/2} t^{-1} |g(t)|\, dt\right),\ x \to \infty, \end{aligned} \tag{5.15}$$

where $g(t) = t^{\nu+p+1/2} \mathscr{D}^p(t^{-2\nu-1} R_n(t))$. By (5.11), $g(t) = O(t^{\alpha_n-\nu-p-1/2})$ as $t \to 0+$. Therefore, $g(t)$ is absolutely integrable in $(0, a)$ and we can apply the Riemann-Lebesque lemma to the first integral in (5.15). The order of the second integral as $x \to \infty$ is given by

$$\begin{aligned} \int_{1/x}^{a} x^{-\nu-p-3/2} t^{\alpha_n-\nu-p-3/2}\, dt &= O(x^{-\nu-p-3/2}) + O(x^{-\alpha_n-1}),\ \nu \ne \alpha_n - p - 1/2, \\ &= O(x^{-\alpha_n-1} \log x),\ \nu = \alpha_n - p - 1/2. \end{aligned} \tag{5.16}$$

Therefore, for $\operatorname{Re}\nu < \operatorname{Re}\alpha_n - p + 1/2$,

$$I_5 = o(x^{-\nu-p-1/2}) + O(x^{-\alpha_n-1} \log x),\ x \to \infty. \tag{5.17}$$

By (5.14) it follows that the above estimate for I_5 holds without any restriction on ν. Thus, referring back to (5.8) and (5.12),

$$E_1 = o(x^{-\nu-p-1/2}) + O(x^{-\alpha_n-1} \log x),\ x \to \infty. \tag{5.18}$$

By (5.7), E_2 is of a lower order than E_1 as $x \to \infty$. Therefore, the order of the remainder E

in (3.2) is determined by that of E_1 in (5.18). If $\operatorname{Re}\alpha_{n-1} < \operatorname{Re}\nu + p - 1/2 < \operatorname{Re}\alpha_n$, E is of a lower order than the order of the last term in each one of the finite sums in (3.2) but in general, some of the terms in the finite sums may be of a lower order than the order of E. For example, if for a fixed ν, positive integers n and p are chosen so that $\operatorname{Re}\nu + p - 1/2 \leq \operatorname{Re}\alpha_{n-1}$, some of the terms in the first sum will, in general, be of a lower order than the order of E. In such a case, the asymptotic approximation of $F(x)$ must include only those terms in (3.2) which are of a higher order than the order of E and the remainder should include E and the remaining terms.

In Theorem 2, we can write

$$E = E_1 - E_2 + (-1)^p I_6 \tag{5.19}$$

where

$$I_6 = \int_a^\infty (xt)^{-\nu-p} J_{\nu+p}(xt) \mathscr{D}^p(t^{-2\nu-1} f(t))\, t^{2\nu+2p+1}\, dt \tag{5.20}$$

and a is some fixed number which satisfies $0 < a < \infty$. If we assume that the integrals in the condition (iii) converge absolutely as $N \to \infty$, then by referring to (4.25) and the condition (iv), the integral I_6 converges absolutely and, by [15, p. 199] and the Riemann-Lebesque lemma, $I_6 = o(x^{-\nu-p-1/2})$, $x \to \infty$. Therefore, the behavior of E is again given by that of E_1 in (5.18).

If the integrals in the condition (iii) converge uniformly as $N \to \infty$ for all x sufficiently large, but not necessarily absolutely, then $I_6 = o(x^{-\nu-p})$ as $x \to \infty$. (For a discussion of this, see [16].)

6. Applications

Example 1. Let

$$g(x) = \int_0^x t^\lambda J_\nu(t)\, dt. \tag{6.1}$$

This integral has been studied and tabulated for various values of x, λ, and ν (for example, see [2, 5, 15]). Tables for large values of x when $\nu = \lambda = 0$ have also been given by Schmidt [9]. By a change of variable,

$$g(x) = x^{\lambda+\nu+1} \int_0^1 (xu)^{-\nu} J_\nu(xu) u^{\lambda+\nu}\, du.$$

We use (3.2) with $a = 1$, $f(u) = u^{\lambda+\nu}$, λ and ν real, $\alpha_0 = \lambda + \nu$, $a_0 = 1$, and $b_0 = 0$. α_1 can be taken as large as we want. Referring to (2.6),

$$\mathscr{D}^k(u^{-2\nu-1}\, u^{\lambda+\nu}) = (-1)^k 2^k \left(\frac{1}{2} + \frac{1}{2}\nu - \frac{1}{2}\lambda\right)_k u^{\lambda-\nu-2k-1}.$$

If $p + 1/2 \leq \lambda$ and $n = 1$, the condition (3.1) is not satisfied. In this case we can take $n = 0$ so that $f_{n-1}(t)$ defined by (2.3) as well as the first sum in (3.2) contains no terms. Thus, by (3.2),

$$\begin{aligned} x^{-\lambda-1} g(x) = {} & \sum_{k=0}^{p-1} 2^k x^{-k-1} \left(\frac{1}{2} + \frac{1}{2}\nu - \frac{1}{2}\lambda\right)_k J_{\nu+k+1}(x) \\ & + 2^p \left(\frac{1}{2} + \frac{1}{2}\nu - \frac{1}{2}\lambda\right)_p \int_0^1 (xt)^{-p} J_{\nu+p}(xt) t^\lambda\, dt. \end{aligned} \tag{6.2}$$

We should mention here that the expansion (3.2) is valid even if we take $n = 1$. But since the condition (3.1) is not satisfied, we must use (3.6) with the understanding that $E_2(x, \nu, p)$ cannot be expressed as the integral (3.8). If $p + 1/2 > \lambda$ and $n = 1$, the condition (3.1) is satisfied; $R_1(t) = 0$ for $0 < t \le 1$ and we obtain

$$\begin{aligned} x^{-\lambda-1}g(x) &= 2^{\lambda} x^{-1-\lambda}\Gamma\left(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\lambda\right)\Big/\Gamma\left(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\lambda\right) \\ &\quad+\sum_{k=0}^{p-1} 2^k x^{-k-1}\left(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\lambda\right)_k J_{\nu+k+1}(x) \\ &\quad- 2^p\left(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\lambda\right)_p \int_1^\infty (xt)^{-p}J_{\nu+p}(xt)t^{\lambda}\,dt. \end{aligned} \tag{6.3}$$

An upper bound for the last integral can be obtained by using integration by parts twice:

$$\int_1^\infty t^{\lambda-p}J_{\nu+p}(xt)\,dt = -x^{-1}J_{\nu+p+1}(x) + \eta(x) \tag{6.4}$$

where

$$|\eta(x)| \le x^{-2}(2p+\nu+1-\lambda)\{1+(2p+\nu+3-\lambda)/(1+p-\lambda)\}. \tag{6.5}$$

The expansions (6.2) and (6.3) can also be obtained by using repeated integration by parts. This technique is used in [9] and [10].

Example 2. Let

$$E(\alpha, \lambda) = \int_0^1 e^{\alpha t}t^{\lambda}J_0(xt)\,dt. \tag{6.6}$$

Integrals of this type appear in the study of currents in an aerial parallel to the earth and in certain problems in acoustics [4; 6]. Expansions for $E(\pi i, 0)$ and $E(\pi i, 1)$ for large x are given by Pidduck [8]. We consider $E(\alpha, 1)$ where α is any given complex number. Let $f(t) = t\,e^{\alpha t}$ and $n = p = 3$. The condition (3.1) is satisfied,

$$f_2(t) = t + \alpha t^2 + (\alpha^2/2)t^3$$

and $\dot{R}_3(t) = O(t^4)$ as $t \to 0+$. By Theorem 1, for $\nu = 0$,

$$\begin{aligned} E(\alpha, 1) &= x^{-1}J_1(x)e^{\alpha} - x^{-2}J_2(x)\alpha e^{\alpha} - \alpha x^{-3} \\ &\quad+ x^{-3}J_3(x)\alpha(\alpha-1)e^{\alpha} + E_1 - E_2, \end{aligned} \tag{6.7}$$

where E_1 and E_2 are defined by (3.7) and (3.8) respectively. Since

$$\mathscr{D}^3(t^{-1}f_2(t)) = 3\alpha\, t^{-5},$$

an upper bound for $|E_2|$ can be obtained by using (6.4) and (6.5); thus

$$|E_2| = \left|3\alpha x^{-3}\int_1^\infty t^{-1}J_3(xt)\,dt\right| \le 3|\alpha|\,x^{-4}(1+23/x). \tag{6.8}$$

Again, since $R_3(t) = f(t) - f_2(t)$,

$$t^7\mathscr{D}^3(t^{-1}R_3(t)) = -\alpha^3t^4/2 + \sum_{n=4}^{\infty}(n-1)(n-3)\alpha^{n+1}t^{n+2}/n!. \tag{6.9}$$

Therefore,

$$E_1 = 2^{-1}(\alpha/x)^3 P_1(x) - x^{-3}\sum_{n=4}^{\infty}(n-1)(n-3)\alpha^{n+1}P_{n-1}(x)/n!, \tag{6.10}$$

where

$$P_n(x) = \int_0^1 t^n J_3(xt)dt. \tag{6.11}$$

To obtain an upper bound for $|P_n(x)|$, we use (5.3) and the fact that $|J_m(y)| \leq 1$ for $0 \leq y < \infty$, $m = 0, 1, \ldots$. When $n = 1, 3$, we use integration by parts and when $n > 4$, we use the mean value theorem and then (5.3). Thus for $n = 1$, $|P_n(x)| \leq 4/x$ and for $n \geq 3$, $|P_n(x)| \leq 2/x$; consequently,

$$|E_1| \leq 2|\alpha|^3 e^{|\alpha|}/x^4. \tag{6.12}$$

Finally we give an example to emphasize that the computation of good uniform error bounds may not be easy even when such bounds exist.

Example 3. Let

$$g(x, \alpha) = \int_0^\alpha tK_0(at)J_0(xt)\, dt \tag{6.13}$$

where K_0 is the modified Bessel function of the third kind,

$$K_0(x) = \sum_{n=0}^{\infty} 2^{-2n}(n!)^{-2}x^{2n}(\psi(n+1) + \log 2 - \log x).$$

By Theorem 2, as $x \to \infty$,

$$g(x, \infty) \sim \sum_{k=0}^{\infty}(-1)^k a^{2k} x^{-2k-2}, \tag{6.14}$$

and by Theorem 1,

$$g(x, 1) \sim \sum_{k=0}^{\infty}(-1)^k a^{2k} x^{-2k-2} + \sum_{k=0}^{\infty} a^k K_k(a) x^{-k-1} J_{k+1}(x). \tag{6.15}$$

It is known that $g(x, \infty) = (x^2 + a^2)^{-1}$ [15, p. 410]. Therefore, if we terminate the expansion (6.14) after a finite number of terms, we can estimate the error directly. Since,

$$g(x, 1) = g(x, \infty) - \int_1^\infty tK_0(at)J_0(xt)\, dt, \tag{6.16}$$

the corresponding error estimate in the asymptotic approximation of $g(x, 1)$ can be obtained readily from (6.16) by using integration by parts. On the other hand, even for small values of k, $\mathscr{D}^k(t^{-1}R_n(t))$ becomes quite complicated and it is difficult to obtain any reasonable estimates for the error without the aid of a computer.

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