ON A SECOND-ORDER BOUNDARY-VALUE PROBLEM ARISING IN COMBUSTION THEORY*

BY

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Abstract. We obtain existence and uniqueness results for the boundary-value problem

$$y'' = x^2 - y^2, \quad y(x) \sim \mp x \quad \text{as} \quad x \to \pm\infty.$$

Our main result shows that there are precisely two solutions $y_+(x) > -|x|$ and $y_-(x) < -|x|$. Only the latter is of physical interest in the problem in combustion theory from which the equation arises.

1. Introduction and statement of results. We consider the boundary-value problem

$$y'' = x^2 - y^2, \quad y(x) \to \mp x \quad \text{as} \quad x \to \pm\infty. \tag{1}$$

This problem arises in the treatment of a diffusion flame due to Burke and Schumann [2]. In a recent note of Ludford and Stewart [4] the question of existence and uniqueness of solutions to (1) is raised, and it is mentioned that a numerical solution lying below $y = -|x|$ has been found and is apparently unique. More recently at least one solution lying above $y = -|x|$ was found by Alexander [1]. In this note we establish the following theorem.

THEOREM. There are two solutions $y_+(x)$, $y_-(x)$ to the boundary value problem (1). $y_+(x) > -|x|, \forall x$ and $y_-(x) < -|x|, \forall x$ and $y_+(x)$, $y_-(x)$ satisfy the following:

(a) $y_\pm(x) = y_\pm(-x)$ and $y'_\pm(0) = 0$ (symmetry).
(b) sign $(y'_\pm(x)) = -$ sign (x); $y_+(x)$ and $y_-(x)$ each have a single maximum at $x = 0$.
(c) $y_\pm(x) \sim \mp x + k_\pm a(|x|)$ as $x \to \pm\infty$, where $a(x) \sim (1/2\sqrt{\pi})2^{1/12}x^{1/4}\exp(-(2\sqrt{2}/3)x^{3/2})$ is the decaying solution to the Airy equation $a'' - 2xa = 0$ and $k_\pm$ are constants: $k_+ > 0 > k_-$.

The situation is sketched below in Fig. 1.

As we remark in Sec. 3, the existence of the lower solution $y_-(x)$ can be obtained as a special case of a theorem of Hastings and McLeod [3]. This solution is fairly easily found in the present case and in Sec. 2.2 we provide a proof which differs in some respects from that of Hastings and McLeod. The solution $y_+(x)$ does not follow their theorem and is more awkward to deal with. It is considered in Sec. 2.3. The main problem in dealing with

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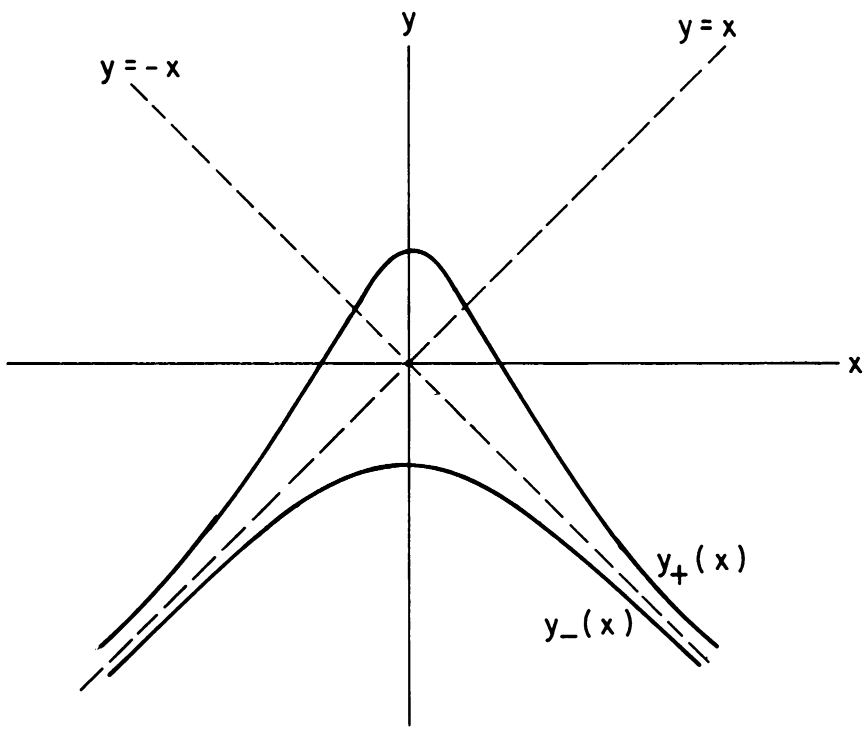

FIG. 1. The two solutions of Eq. (1).

solutions lying above $y = -|x|$ is of showing that oscillating solutions cannot satisfy the boundary conditions at both $+\infty$ and $-\infty$.

Before starting the analysis, we note that $y'' = x^2 - y^2$ is unchanged if $x \mapsto -x$. Thus solutions are either even ($y(x) = y(-x)$) or occur in pairs. Using the symmetry we need only consider the positive half-space $x \geq 0$ since a symmetric solution of (1) corresponds to a solution to the boundary-value problem with $y'(0) = 0$, $y(x) \sim -x$ as $x \to +\infty$ and a nonsymmetric solution corresponds to a pair of solutions with $y'(0) = \pm\alpha \neq 0$, $y(x) \sim -x$ as $x \to \infty$. It will therefore be convenient to consider the initial-value problem

$$y'' = x^2 - y^2, \quad y(0) = \beta, \quad y'(0) = \alpha. \tag{2}$$

2. Proof of theorem. We prove the theorem with a number of lemmas. We first show that no solution to (1) can cross $y = -|x|$.

2.1 Solutions cannot cross $y = -|x|$.

LEMMA 1. There is no solution to (1) which crosses $y = -|x|$ with $y'(x) < -1$ for $x > 0$ or $y'(x) > 1$ for $x < 0$.

Proof: Consider $x > 0$. Since $y'' < 0$ for $y < -x$ and $y' < 1$ at $y = -x$, y' and hence y must continue to decrease after y crosses $-x$. Hence y cannot be asymptotic to $-x$; in fact $y, y' \to -\infty$ and y ceases to exist at some finite x. Similar arguments apply for $x < 0$. In fact it is easy to see that any solution lying below $y = -|x|$ with $|y'(x)| > 1$ at some point must blow up at finite x.

LEMMA 2. There is no solution to (1) which crosses $y = -|x|$ with $y'(x) > -1$ for $x > 0$ or $y'(x) < 1$ for $x < 0$.

Proof: Again we take $x > 0$. Suppose such a solution $y(x)$ exists; then it lies below $y = |x|$ at $x = 0$, and, from the proof of Lemma 1, if it is not to blow up, then $|y'(0)| \triangleq \alpha_0 < 1$. Let $y(x)$ cross $y = -x$ at $x = x_1$ with $y'(x_1) \triangleq \alpha_1$. We consider the first integral of (1):

$$\frac{y'(x)^2}{2} - \frac{y'(0)^2}{2} = -\frac{y^3(x) - y^3(0)}{3} + x^2 y(x) - 2\int_0^x xy(x)\,dx. \tag{3}$$

Letting $f(x) = y(x) + x$ and evaluating (3) on the interval $(0, x_1)$ we have

$$\frac{\alpha_1^2 - \alpha_0^2}{2} = \frac{y^3(0)}{3} - 2\int_0^{x_1} xf(x)\,dx. \tag{4}$$

Repeating the process on the infinite interval $(0, \infty)$ and using the limiting behavior $(y(x), y'(x)) \sim (-x, 1)$ we have

$$\frac{1 - \alpha_0^2}{2} = \frac{y^3(0)}{3} - 2\int_0^{\infty} xf(x)\,dx. \tag{5}$$

Subtracting (4) from (5) yields

$$\frac{1 - \alpha_1^2}{2} = -2\int_{x_1}^{\infty} xf(x)\,dx. \tag{6}$$

Since $y(x) > -x$ for $x \in (x_1, \infty)$ the integrand is positive; hence the right-hand side of (6) is negative and thus $\alpha_1^2 > 1$. In the above we have used the fact that $y(x)$ cannot recross $y = -x$, which follows from Lemma 1.

We now show that in fact $|\alpha_1| < 1$, thus obtaining a contradiction. Note first that clearly $\alpha_1 > -1$ or $y(x)$ could not cross $y = -x$ at all. However, since $y'(0) = \alpha_0 \in (-1, 1)$ and $y'' < 0$ for $x \in (0, x_1)$ we also have $\alpha_1 < 1$. Thus $|\alpha_1| < 1$. A similar argument applies for $x < 0$.

LEMMA 3. There is no C^1 solution to (1) with $y(0) = 0$.

Proof: Suppose $y(x)$ is such a solution and let $y'(0) = \alpha_0$. Using the boundary conditions at $x = \infty$ and the integrals (3) we obtain

$$1 - \alpha_0^2 = -4\int_0^{\infty} xf(x)\,dx \triangleq F, \tag{7}$$

where $f(x) = y(x) + x$ is positive if $\alpha_0 > -1$ and negative if $\alpha_0 < -1$, since in the former case $y > -x$ for all $x \geq 0$ and in the latter $y < -x$ for all $x > 0$. The latter choice yields an immediate contradiction, since $1 - \alpha_0^2 < 0$ while $F > 0$. The former choice implies that $1 - \alpha_0^2 = F < 0$ and thus $\alpha_0 > 1$. But if a full solution of (1) is to exist this implies that the boundary-value problem $y \sim +x$ as $x \to -\infty$ must also be satisfied, or, reflecting about $x = 0$, that the solution based at $(0, -\alpha_0)$ must also be asymptotic to $y = -x$ as $x \to +\infty$. But then for that solution $y'(0) = -\alpha_0 < -1$, and as we have already seen, we obtain a contradiction from (7).

We have now established that solutions to (1) fall into two distinct classes, those lying above $y = -|x|$ (and asymptotic to $y = \mp x$ as $x \to \pm\infty$ from above), and those lying below $y = -|x|$. We now study these two classes.

2.2 Solutions lying below $y = -|x|$.

LEMMA 4. There are no C^1 nonsymmetric solutions of (1) below $y = -|x|$.

Proof: Since $y'' < 0$ for $y < -|x|$ any solution has at most one maximum. Suppose, without loss of generality, that this occurs for $x < 0$, so that the solution is monotonic for $x \geq 0$ with $y'(x) \in (-1, 0)$ and $y'(x) \to -1^+$ as $x \to \infty$. Let $y'(0) = \alpha_0 \in (-1, 0)$. The existence of a full solution implies that the two boundary-value problems

$$\begin{aligned} y'' = x^2 - y^2, \qquad & y(0) = \alpha_0, \qquad y(x) \sim -x \\ & y(0) = -\alpha_0 \quad y(x) \sim -x \end{aligned} \qquad x \to \infty \tag{8}$$

must be simultaneously soluble. Suppose that they are, and denote their solutions by $y_a(x)(y_a'(0) = \alpha_0)$, $y_b(x)y_b'(0) = -\alpha_0)$. We claim that y_a and y_b cannot cross and cannot both be asymptotic to $-x$.

First suppose y_a crosses y_b from above at $x = x_1 > 0$. Then, from the integral (3), we have

$$\frac{y_a'(x_1)^2 - y_b'(x_1)^2}{2} = -2\int_0^{x_1} x(y_a(x) - y_b(x))\, dx. \tag{9}$$

But since $y_a > y_b$ for $x \in (0, x_1)$ the right-hand side is negative, while we require $y_a'(x_1) < y_b'(x_1) < 0$, and hence $y_a'(x_1)^2 > y_b'(x_1)^2$, for crossing to occur. Taking the limit as $x_1 \to \infty$ and using $y_a', y_b' \to -1$ we again obtain a contradiction:

$$0 = -2 \lim_{x\to\infty} \int_0^{x_1} x(y_a(x) - y_b(x))\, dx < 0, \tag{10}$$

since in view of (9) $y_a(x) > y_b(x)$ for all x. Thus y_b and y_a cannot both be asymptotic to $-x$ as $x \to \infty$ and hence no C^1 nonsymmetric solution can exist.

LEMMA 5. The solution $y_0(x)$ with $(y_0(0), y_0'(0)) = (0, 0)$ remains above $y = 0$ for all $x > 0$ and hence remains bounded for all finite x.

Proof: Since the solution is symmetric we only consider $x \geq 0$. Suppose that $y_0(x)$ first crosses (or has a minimum on) $y = 0$ at $x = x_1$. Using the integral (3) we obtain

$$\frac{y_0'(x_1)^2}{2} = -2\int_0^{x_1} x y_0(x)\, dx. \tag{11}$$

It is clear that $y_0(x) > 0$ for $x \in (0, x_1)$, since $y_0'' > 0$ initially. Thus the right-hand side is strictly negative and we obtain a contradiction. It follows that $y_0(x) > 0$ for all $x > 0$ (and hence $x < 0$ also), and that $y_0(x)$ remains finite for all finite x, since $y'' < 0$ for $y > |x|$ and thus y' must decrease in that region until y recrosses $y = |x|$ and reenter the region $y \in (-|x|, |x|)$. In fact $y_0(x)$ oscillates about $y = +|x|$ as $x \to \pm\infty$.

Lemma 4 shows that, for $y < -|x|$, we need only consider symmetric solutions ($y_0'(0) = 0$). We next show that there is such a solution to the initial-value problem (2) with $y(0) = \beta_0 < 0$, which ceases to exist at finite $|x|$.

LEMMA 6. There is a constant $\beta_- = -(3/2)^{1/3}$ such that, if $y_\beta(0) = \beta < \beta_-$, $y'_\beta(0) = 0$ then the solution $y_\beta(x)$ ceases to exist at finite x.

Proof: Again we take $x \geq 0$. Let $y_\beta(0) = \beta$. Integrating Eq. (2) from 0 to x_1 directly we have

$$y'_\beta(x_1) = x_1^3/3 - \int_0^{x_1} y_\beta^2(x)\, dx. \tag{12}$$

Since $y''_\beta < 0$ for $y_\beta < |x|$, y'_β decreases and thus $y_\beta(x) < \beta$ for $x \in (0, -\beta]$. Using $y_\beta^2(x) > \beta^2$ and setting $x_1 = -\beta$, (12) yields

$$y'_\beta(-\beta) < 2\beta^3/3. \tag{13}$$

Choosing $\beta \leq \beta_- = -(3/2)^{1/3}$, we have $y'_\beta(x) < -1$ and $y_\beta(x) < -x$ at $x = -\beta$. Now since $y''_\beta(x) < 0$, y_β and y'_β will continue to decrease and in fact they both go to $-\infty$ at finite x. Clearly this behavior persists for all y_β, with $y_\beta(0) < \beta_-$.

PROPOSITION 7. (Existence). There is at least one symmetric solution $y < -|x|$ to the boundary-value problem (1).

Proof: Lemmas 5 and 6 show that, for $y_0(0) = 0$, $y_0(x)$ remains bounded for all x, while for $y_\beta(0) < \beta_-$, $y_\beta(x)$ ceases to exist at finite x. The continuous dependence of solutions on initial conditions implies that there must be at least one member of the family $\{y_\alpha(x) \mid (y_\alpha(0), y'_\alpha(0)) = (\alpha, 0), \alpha \in (\beta_-, 0)\}$ such that y_α neither blows up at finite x nor crosses $y = -x$ from below and subsequently oscillates about $y = +x$ (recall Lemma 2). Such a solution can only be asymptotic to $y = -x$ from below.

LEMMA 8. (Uniqueness). The solution $y_\alpha(x)$ of Proposition 7 is unique.

Proof: Suppose that a second such solution y_β with $y_\beta(0) = \beta \neq \alpha$ exists. We claim that y_β can neither cross y_α nor be asymptotic to it (and hence also to $-x$) as $x \to \infty$. For simplicity and without loss of generality, assume $\beta < \alpha < 0$. Suppose y_β crosses y_α from below at $x = x_1$; then, from the integral (12),

$$y'_\beta(x_1) - y'_\alpha(x_1) = \int_0^{x_1} (y_\alpha^2(x) - y_\beta^2(x))\, dx. \tag{14}$$

Since $y_\beta < y_\alpha < 0$ for $x \in (0, x_1)$ the integral is negative, while we require $y'_\alpha(x_1) < y'_\beta(x_1) (< 0)$ for crossing to occur. Thus $y_\beta(x) < y_\alpha(x)$ for all x. Next suppose y_β, $y_\alpha \sim -x$ as $x \to \infty$ and take the limit in (14) to obtain

$$0 = \lim_{x \to \infty} \int_0^{x} (y_\alpha^2(x) - y_\beta^2(x))\, dx. \tag{15}$$

Again the integral is strictly negative and we obtain a contradiction.

We now turn to the solutions lying above $y = -|x|$.

2.3 Solutions lying above $y = -|x|$.

LEMMA 9. There is a constant $\beta_+ > 0$ such that, if $y_\beta(0) = \beta > \beta_+$, $y'_\beta(0) = 0$, the solution $y_\beta(x)$ crosses $y = -|x|$ and ceases to exist at finite $|x|$.

Proof: The idea of the proof is similar to that of Lemma 6, but since it involves rather lengthy computations and estimates, we relegate it to the Appendix.

PROPOSITION 10. (Existence). There exists at least one symmetric solution to (1) satisfying $y(x) > -|x|$.

Proof: As in Proposition 7, this is a consequence of Lemmas 5 and 9 and the continuous dependence of solutions upon initial conditions.

LEMMA 11. No solution to (1) possessing a minimum exists.

Proof: Consider $x \geq 0$. We use the transformation $y = g - x$ to rewrite the equation as

$$g'' = 2xg - g^2, \tag{16}$$

the boundary-value problem becoming $g'(0) = +1, g(x) \to 0^+$ as $x \to +\infty$. If a solution $y(x)$ exists possessing a minimum at some $x \geq 0$ ($y(x) \in (-x, x)$) then it is easy to see that this must be followed by a maximum ($y(x) > x$) and a second minimum. Maxima and minima are separated by points at which y crosses $+x$. It follows that if y possess two successive minima then g also possess two successive minima. Let these occur at x_1 and x_2 (possibly $x_2 = \infty$). Suppose that $g(x_2) \leq g(x_1)$ and denote the point $x \in (x_1, x_2]$ at which $g(x) = g(x_1)$ by x_3. Then $g'(x_3) \leq 0$. We use the first integral of (16) evaluated between x_1 and x_3 to obtain

$$\frac{g'^2(x_3)}{2} = (x_3 - x_1)g^2(x_1) - \int_{x_1}^{x_3} g^2(x)\, dx. \tag{17}$$

But, since $g(x) > g(x_1) > 0$ for all $x \in (x_1, x_3)$, the integral is greater in magnitude than the expression $(x_3 - x_1)g^2(x_1)$ and thus the right-hand side of (17) is strictly negative and we obtain a contradiction. Hence successive minima of g must lie at greater and greater distances from $g = 0$, showing that the solution $y = g - x$ cannot approach $y = -x$ as $x \to \infty$.

LEMMA 12. There are no C^1 nonsymmetric solutions to (1) with $y > -|x|$.

Proof: Given Lemma 11, the proof is in essence identical to that of Lemma 4.

LEMMA 13. The symmetric solution of Proposition 10 is unique.

Proof: Let $y_\alpha(x)$, with $y_\alpha(0) = \alpha > 0$, $y'_\alpha(0) = 0$, be the symmetric solution with greatest $y_\alpha(0)$. Thus any solution $y_\gamma(x)$ with $y_\gamma(0) = \gamma > \alpha$ crosses y_α and $y = -|x|$ from above and subsequently $y_\gamma \to -\infty$ at finite x. Consider a solution $y_\beta(x)$ with $y_\beta(0) = \beta$, $\beta \in (0, \alpha)$. If $y_\beta \sim -x$ there are two possibilities: either it remains below $y_\alpha(x)$ for all x or crosses it from below and is then subsequently asymptotic to $-x$.

We consider the equation

$$z'' = -z(z + 2y_\alpha(x)); \qquad z(0) = \beta - \alpha < 0, \qquad z'(0) = 0 \tag{18}$$

for the difference $z = y_\beta - y_\alpha$. Since $z + 2y_\alpha = y_\alpha + y_\beta > 0$ initially we have $z'' > 0$ and z' becomes positive and hence $z(x)$ increases. This behavior persists until some point $x = x_0$, where $z'' = 0$ and z' reaches its maximum, at which point either $z = 0$ (y_β crosses y_α) or $z + 2y_\alpha = 0$ ($y_\beta = -y_\alpha$). Suppose that the latter occurs, so that y_β remains below y_α. There will then be a second point $x_b > x_a$ such that $y_\beta(x_b) < y_\alpha(x_b)$ and $y'_\beta(x_b) = y'_\alpha(x_b)$ ($z' = 0$) with $y''_\beta = x_b^2 - y_\beta^2 < y''_\alpha = x_b^2 - y_\alpha^2$ ($z'' < 0$). Thus z' will become negative and, since $z'' < 0$ for all $x > x_b$, if y_β remains below y_α, z' will remain negative and thus z will decrease for $x > x_b$, becoming more negative, until its magnitude is sufficiently large that $y_\beta = y_\alpha + z < -x$. Thus y_β cannot in fact be asymptotic to $-x$, since it must cross $-x$ (cf. Lemmas 1–2).

Now suppose that y_β does in fact cross y_α from below at some finite $x = x_1$ (in fact the analysis above implies that this *must* be the case). We claim that, at x_1, $y'_\alpha(x_1) < y'_\beta(x_1) < 0$, since $y'_\beta(x) < 0$ for all x if $y_\beta \sim -x$, for such solutions can have no minima by Lemma 11. Hence $y'^2_\alpha(x_1) > y'^2_\beta(x_1)$. To complete the proof we show that y_β cannot be asymptotic to $-x$ if it crosses y_α once at $x = x_1$ with $y'^2_\alpha(x_1) - y'^2_\beta(x_1) > 0$. Using the integral (3), subtracting, and taking the limit we obtain

$$0 = \frac{y'^2_\alpha(x_1) - y'^2_\beta(x_1)}{2} + 2 \lim_{x\to\infty} \int_{x_1}^{x} x(y_\beta(x) - y_\alpha(x))\, dx \tag{19}$$

$$0 = \frac{y'^2_\alpha(x_1) - y'^2_\beta(x_1)}{2} + 2 \lim_{x\to\infty} \int_{x_1}^{x} x(y_\beta(x) - y(x))\, dx \tag{19}$$

where we have used the limiting behavior, $y_\alpha, y_\beta \sim -x$ and the fact that $y_\alpha(x_1) = y_\beta(x_1)$. But $y_\beta > y_\alpha$ for $x > x_1$ and thus the right-hand side of (19) is strictly positive and we obtain a contradiction. It remains to check that y_β cannot recross y_α. To see this, suppose that it does, at $x = x_2$, where $y'_\beta(x_2) < y'_\alpha(x_2) < 0$, so that $y'^2_\beta(x_2) > y'^2_\alpha(x_2)$. Using the integral (3) once more we find

$$y'^2_\beta(x_2) - y'^2_\alpha(x_2) = y'^2_\beta(x_1) - y'^2_\alpha(x_1) + 4 \int_{x_1}^{x_2} x(y_\alpha(x) - y_\beta(x))\, dx \tag{20}$$

which again leads to a contradiction.

Thus we have shown that y_β cannot remain below y_α for all x, but must cross it at some point, after which is must remain above y_α.

2.4 Asymptotic properties. We have now proved all but part (c) of the theorem. To obtain the asymptotic estimates, we let $y = g - x$ and consider the linearized system

$$g'' - 2xg = 0; \qquad g(x) \to 0 \quad \text{as} \quad x \to \infty \tag{21}$$

arising from Eq. (16). A similar equation is obtained for $x \to -\infty$, since our solution is symmetric, and we shall only consider $x > 0$. Letting $\bar{x} = 2^{1/3}x$ and $\ddot{g}$ denote $d^2g/d\bar{x}^2$, we obtain the standard Airy equation

$$\ddot{g} - \bar{x}g = 0, \tag{22}$$

the solution of which, as $\bar{x} \to \infty$, is given by a linear combination of the functions

$$A_i(\bar{x}) \sim \tfrac{1}{2}\pi^{-1/2}\bar{x}^{+1/4} \exp(-2/3\bar{x}^{3/2}), \tag{23a}$$

$$B_i(\bar{x}) \sim \pi^{-1/2}\bar{x}^{-1/4} \exp(+2/3\bar{x}^{3/2}). \tag{23b}$$

Taking the decaying exponential $A_i(\bar{x})$, for we are interested only in bounded solutions, we obtain

$$g(x) \sim \frac{1}{2\sqrt{\pi}}\, 2^{1/12} x^{1/4} \exp\left(-\frac{2\sqrt{2}}{3}\, x^{3/2} \right). \tag{24}$$

To complete the proof we note that $y = g - x$ and that the upper solution $y_+(x)$ approaches $y = -|x|$ from above while the lower approaches it from below, which gives the appropriate signs for the constants k_+, k_-.

3. Remarks. In a recent paper Hastings and McLeod [1980] study a similar problem:

$$y'' = xy + 2y|y|^{\alpha}; \quad y(x) \sim 0 \quad \text{as} \quad x \to +\infty, \quad y(x) \sim (-\tfrac{1}{2}x)^{1/\alpha} \quad \text{as} \quad x \to -\infty, \tag{25}$$

where α is a strictly positive real number. They prove existence and uniqueness for a positive solution $y > 0$, $y' < 0$ and discuss its asymptotic properties. If we let $y = -x - g$ then Eq. (1) becomes

$$g'' = 2xg + g^2; \quad g(x) \sim 0 \quad \text{as} \quad x \to +\infty, \quad g(x) \sim -2x \quad \text{as} \quad x \to -\infty, \tag{26}$$

which, after a further scaling transformation, is a special case of (25) with $\alpha = 1$. However, the Hastings–McLeod result yields only the solution with $g > 0$ ($y < -|x|$).

Techniques similar to those in the present paper have also been applied to a boundary-value problem involving the first Painlevé transcendent, which arises in studies of heat transfer in the earth's mantle. The problem is defined on the positive half-line

$$y'' = y^2 - x, \quad y(0) = 0, \quad y(x) \sim +\sqrt{x} \quad \text{as} \quad x \to \infty \tag{27}$$

and again one obtains two solutions. In this case one is monotonic and the other has a single minimum (cf. Spence and Holmes [5]).

4. Appendix: Proof of Lemma 9. As before, we need consider only the region $x \geq 0$, since the solution is symmetric. Now any solution based at $y(0) = \beta > 0$, $y'(0) = 0$ decreases until it crosses $y = +x$ at some point $x = x_1$ with slope $y'(x_1) = \gamma < 0$. The slope γ may be estimated by noting that

$$y'(x_1) = \gamma = x_1^3/3 - \int_0^{x_1} y^2(x)\, dx, \tag{28}$$

(cf. Eq. (12)). Certainly $y^2(x) > x_1^2$ for $x \in (0, x_1)$, and thus we have

$$\gamma < x_1^3/3 - \int_0^{x_1} x_1^2\, dx = -2x_1^3/3. \tag{29}$$

We now wish to show that, if $|\gamma|$ is sufficiently large, then the solutions must subsequently cross $y = -x$ from above and hence cease to exist at finite x (cf. the proof of Lemma 1). Suppose $y(x)$ crosses $y = -x$ at $x = x_3$; then clearly $y(x)$ lies below the line $y = \delta(x - x_c)$ and above $y = \gamma(x - x_b)$ for $x \in (x_1, x_2)$, where x_b, x_c and x_2 are the points indicated in Fig. 2. Note that $x_3 < x_2$, and we select x_2 such that $y'(x_2) \leq \delta$. This ensures that $y(x)$ cannot cross the line $y = \delta(x - x_c)$. We note that $\gamma < \delta < 0$.

Simple computations show that

$$x_b = (\gamma - 1)x_1/\gamma, \quad x_c = 2x_1x_2/(x_1 + x_2), \quad \delta = (x_1 + x_2)/(x_1 - x_2). \tag{30}$$

We also note that $x_b - x_1 = -x_1/\gamma$ and $x_2 - x_c = -x_2/\delta$. We now estimate the slope of $y(x)$ at $x = x_2$:

$$y'(x_2) = \gamma + \left(\frac{x_2^3 - x_1^3}{3}\right) - \int_{x_1}^{x_2} y^2(x)\, dx. \tag{31}$$

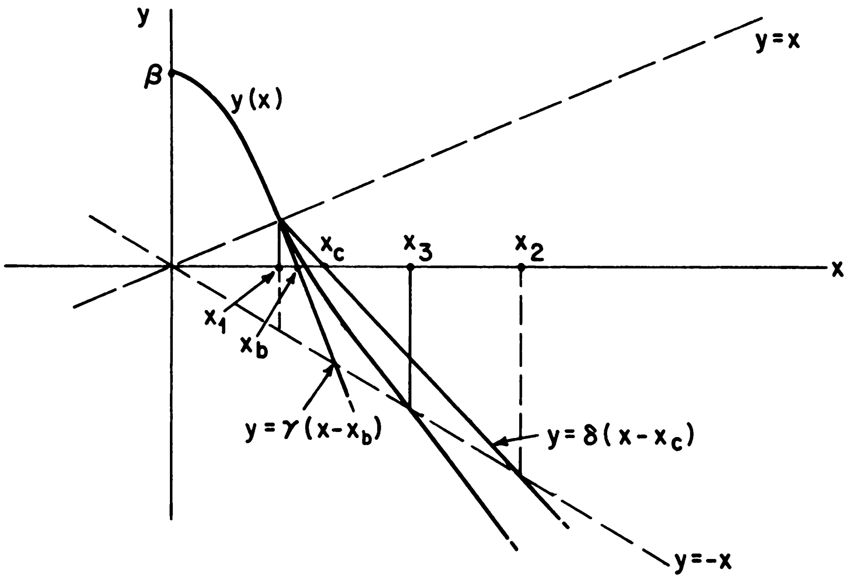

FIG. 2. Proof of Lemma 9.

But clearly

$$\int_{x_1}^{x_2} y^2(x)\,dx > \int_{x_1}^{x_b} \gamma^2(x-x_b)^2\,dx + \int_{x_c}^{x_2} \delta^2(x-x_c)^2\,dx$$

$$= \int_0^{x_b-x_1} \gamma^2\xi^2\,d\xi + \int_0^{x_2-x_c} \delta^2 y^2\,dy$$

$$= -\tfrac{1}{3}(x_1^3/\gamma + x_2^3/\delta). \tag{32}$$

Thus we have

$$y'(x_2) < \gamma + \frac{x_2^3}{3}(1 + 1/\delta) + \frac{x_1^3}{3}\left(\frac{1}{\gamma} - 1\right), \tag{33}$$

or, using (29),

$$y'(x_2) < \frac{x_2^3}{3}\left(1 + \frac{1}{\delta}\right) + x_1^3\left(\frac{1}{3\gamma} - 1\right). \tag{34}$$

(Recall that $\gamma < \delta < 0$.) Now, as noted above, a sufficient condition for $y(x)$ to lie between the two straight lines as claimed (and hence to cross $y = -x$ at $x = x_3$) is that $y'(x_2) \le \delta$ (we clearly also require $\delta < -1$, or $y = \delta(x - x_c)$ does not cross $y = -x$ as shown). Using

$$x_2 = \left(\frac{\delta - 1}{\delta + 1}\right)x_1 \tag{35}$$

in (34), our requirement becomes

$$y'(x_2) < \frac{x_1^3}{3}\left\{\left(\frac{\delta - 1}{\delta + 1}\right)^3 \left(\frac{\delta + 1}{\delta}\right) + \frac{1}{\gamma} - 3\right\} \leq \delta. \tag{36}$$

Now we can pick β sufficiently large to ensure that the first crossing point x_1, and hence γ, is as large as we wish. So pick $|\gamma| > 2x_1^3/3 > |\delta| \gg 1$, in which case the bracketed expression approaches the limit

$$\frac{x_1^3}{3}\{\quad\} \to -2\,\frac{x_1^3}{3}. \tag{37}$$

and, since we have

$$\gamma < -\frac{2x_1^3}{3} < \delta, \tag{38}$$

we obtain the desired result. Thus $y(x)$ must be as shown in Fig. 2, and hence must cross $y = -x$ at some $x_3 \in (x_1, x_2)$ and subsequently cease to exist.

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