INEXTENSIBLE NETWORKS WITH SLACK*

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1. Introduction. Although it is easy to distort a piece of cloth, such distortions usually involve only relatively small stretching of the fibers or threads in the material. Rivlin [1], with rather open networks in mind, formulated a continuum theory in which the cords or fibers in the network are treated as absolutely inextensible, but with no resistance to changes in the angle between two intersecting fibers. Pipkin [2] recast this theory in vector form and discussed various kinds of singularities that solutions can exhibit. Rogers and Pipkin [3] and Rogers [4] have applied the theory to problems involving holes or tears in sheets.

Because Rivlin's theory uses the constraint that no fiber segment can change its length at all, even under compressive loading, solutions in this theory can involve compressive stresses [2]. It is necessary to allow compressive stresses in order to assure existence of solutions, but when compressive stresses are admitted, solutions become highly non-unique. Often there is exactly one solution with non-negative fiber tensions, and this solution is chosen as the correct one. However, as we show in the present paper, there are problems in which no solution has non-negative fiber tensions everywhere.

The purpose of the present paper is to formulate a theory in which fibers can grow shorter but not longer, and can carry tensile but not compressive loads. We also prove a lemma concerning uniqueness of solutions in the new theory.

Since the extended theory presented here is stated in terms of inequalities as well as equations (Sec. 2), it is clear from the outset that the deformation is highly arbitrary in any part of the network in which the fibers are not fully extended. Consequently, the uniqueness lemma is not the usual sort of uniqueness statement. It says, roughly, that the load-carrying part of the network and the deformation of that part are uniquely determined. In Sec. 5 we give some simple examples that show how to use the uniqueness lemma and why it says no more than it does.

The equations of the theory can be integrated explicitly when the general nature of the deformation is known (Sec. 6). For completeness we briefly recapitulate previously published results concerning fully extended regions (Rivlin [1]) and collapsed regions [2]. We also discuss half-slack regions, which are not permissible in Rivlin's theory.

The theory is of such form that any purely tensile solution in Rivlin's theory is also a solution in the new theory, and unique within the new theory, so problems in Rivlin's theory have at most one purely tensile solution. We close with an example showing that some problems have no purely tensile solution within Rivlin's theory.

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We note that although the theory appears to be two-dimensional because we consider deformations of plane sheets, there is no restriction to plane deformations except in the results in Sec. 6d.

2. Networks with slack. We consider a sheet or net that, in its reference state, occupies some region of the X, Y plane. The network is composed of strong fibers that initially lie parallel to the X and Y directions. The sheet is treated as a continuum; every material line X = constant or Y = constant is regarded as a fiber.

In a deformation, the particle initially at $\mathbf{X} = X\mathbf{i} + Y\mathbf{j}$ moves to the place $\mathbf{x}(\mathbf{X}) = x(\mathbf{X})\mathbf{i} + y(\mathbf{X})\mathbf{j} + z(\mathbf{X})\mathbf{k}$. Here $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are unit vectors parallel to the coordinate axes. Both families of fibers undergo the same deformation because the fibers are knotted together at the points where they cross, or, in the case of a closely-woven fabric, held together by friction at these points.

The deformation maps a fiber element $\mathbf{i}dX$ into $\mathbf{x}_{,X}\,dX$, and an element $\mathbf{j}dY$ onto $\mathbf{x}_{,Y}\,dY$. The vectors $\mathbf{x}_{,X}$ and $\mathbf{x}_{,Y}$ are tangential to the deformed fibers. For these vectors we use the notation

$$\mathbf{x}_{,X} = \mathbf{a}, \qquad \mathbf{x}_{,Y} = \mathbf{b}. \tag{2.1}$$

We wish to formulate a theory in which fibers cannot grow longer in tension, but can grow shorter by buckling or crumpling on the microscale, the scale of the actual distance between fibers in the physical network. Just as we do not attempt to describe the actual wavy configuration of a thread in a woven fabric when we say that it is parallel to the X-axis, we do not try to describe the details of fiber buckling in the function $\mathbf{x}(\mathbf{X})$ that defines the deformation.

The fiber element $\mathbf{i}dX$ maps onto $\mathbf{a}dX$, so its lengths before and after the deformation are dX and $|\mathbf{a}|dX$. As a one-sided constraint of inextensibility we postulate that $|\mathbf{a}|$ cannot exceed unity. Similarly, $|\mathbf{b}|$ is not greater than one:

$$|\mathbf{a}| \le 1, \qquad |\mathbf{b}| \le 1. \tag{2.2}$$

Thus, no fiber element can be longer after the deformation than it was originally. When $|\mathbf{a}| = 1$ at some point, we say that the $\mathbf{a}$-fiber is *fully extended* at that point. When $|\mathbf{a}| < 1$, we say that it is *slack* there.

With restriction to cases in which the sheet is loaded by forces around its edges only, the equilibrium state of stress in the deformed sheet is conveniently described in terms of Rivlin's stress potential $\mathbf{F}(\mathbf{X})$ [1]. $\mathbf{F}(\mathbf{X})$ is the force exerted across a curve running from some origin $\mathbf{X}_0$ to $\mathbf{X}$, by the material initially to the right of that material curve, on the material initially to its left. Translational equilibrium requires this force to be path-independent [2]. The force across a curve from $\mathbf{X}_A$ to $\mathbf{X}_B$ is then $\mathbf{F}(\mathbf{X}_B) - \mathbf{F}(\mathbf{X}_A)$.

The force that acts from right to left across a fiber element $\mathbf{j}dY$ (which is along $\mathbf{b}dY$ in the deformed state) is $\mathbf{F}_{,Y}\,dY$. As a constitutive assumption, we postulate that this force is solely due to tensions in fibers of the other family, which lie parallel to the $\mathbf{a}$-direction. Similarly, $-\mathbf{F}_{,X}\,dX$ is the force exerted from above across a fiber element $\mathbf{i}dX$, and we postulate that this must be in the $\mathbf{b}$-direction:

$$\mathbf{F}_{,Y} = T_a\,\mathbf{a}, \qquad -\mathbf{F}_{,X} = T_b\,\mathbf{b}. \tag{2.3}$$

Then T_a and T_b are fiber tensions, measured per unit of undeformed length. In general each of the stress vectors $\mathbf{F}_{,Y}$ and $-\mathbf{F}_{,X}$ can have components in both the $\mathbf{a}$-direction and the

b-direction [2]; the components omitted in (2.3) represent shearing stresses, which we take to be negligible.

We postulate that the fiber tensions T_a and T_b must be non-negative:

$$T_a \geq 0, \qquad T_b \geq 0. \tag{2.4}$$

When T_a is positive to some point, we say that the **a**-fiber is *tense* there. When $T_a = 0$, we say that the fiber is *relaxed*. As a final assumption, we postulate that a slack fiber must be relaxed:

$$\begin{aligned} |\mathbf{a}| < 1 \quad &\text{implies} \quad T_a = 0, \\ |\mathbf{b}| < 1 \quad &\text{implies} \quad T_b = 0. \end{aligned} \tag{2.5}$$

In view of (2.2) and (2.4), this is equivalent to the assertion that a tense fiber must be extended:

$$\begin{aligned} T_a > 0 \quad &\text{implies} \quad |\mathbf{a}| = 1, \\ T_b > 0 \quad &\text{implies} \quad |\mathbf{b}| = 1. \end{aligned} \tag{2.6}$$

The tensions T_a and T_b are reactions to the one-sided constraints (2.2). They are not specified by constitutive equations, but take whatever values equilibrium requires. In particular, they can be infinite. As in other theories involving inextensible fibers, a single fiber can carry a finite load, so that **F** is discontinuous across that fiber and the corresponding tension, from (2.3), has a Dirac delta singularity [2].

3. Boundary-value problems. To pose a boundary-value problem in a fairly general way, let us suppose that the deformation $\mathbf{x}(\mathbf{X})$ is required to take prescribed values $\mathbf{x}_0(\mathbf{X})$ on some portion C_p of the boundary C of the sheet, and that the stress potential $\mathbf{F}(\mathbf{X})$ takes prescribed values $\mathbf{F}_0(\mathbf{X})$ on the remainder of the boundary, C_t:

$$\mathbf{x}(\mathbf{X}) = \mathbf{x}_0(\mathbf{X}) \quad \text{on} \quad C_p, \qquad \mathbf{F}(\mathbf{X}) = \mathbf{F}_0(\mathbf{X}) \quad \text{on} \quad C_t. \tag{3.1}$$

Prescribing **F** corresponds to prescribing boundary tractions as dead loads. The increment $d\mathbf{F}_0$ along a boundary arc $d\mathbf{X}$ is the external force per unit initial length on that arc.

A deformation is *kinematically admissible* if it satisfies the constraints (2.2) and the boundary condition (3.1a). If the boundary condition is such that there is no kinematically admissible deformation, the problem has no solution. We assume that there is at least one kinematically admissible deformation.

A stress function $\mathbf{F}(\mathbf{X})$ is *statistically admissible* if it satisfies the boundary condition (3.1b). We suppose that there is at least one statically admissible stress function, $\mathbf{F}_0(\mathbf{X})$ say, which is of course defined on C_p and on the interior of the sheet as well as on C_t.

A pair $\mathbf{x}(\mathbf{X})$, $\mathbf{F}(\mathbf{X})$ is a *solution* if $\mathbf{x}(\mathbf{X})$ is kinematically admissible, $\mathbf{F}(\mathbf{X})$ is statically admissible, and the equations and inequalities in Sec. 2 are all satisfied. As we shall see, problems often have more than one solution, but the non-uniqueness is of an understandable, physically expected type. The deformation is highly non-unique in slack regions, and tensions are highly non-unique in collapsed regions where the load is supported redundantly.

4. Uniqueness. We now show that if a given problem has a solution, the deformation minimizes the energy of the boundary loads. Then from the conditions characterizing minimum energy, we obtain a useful lemma relevant to uniqueness. We show that if a fiber is tense at some point, in some solution, then it is fully extended at that point in every solution, and its direction there is uniquely determined. It follows that if a fiber is slack at some point, in some solution, the tension in it is zero at that point in every solution.

The energy of the loads is the functional, defined for kinematically admissible deformations $\mathbf{x}(\,\cdot\,)$,

$$E[\mathbf{x}(\,\cdot\,)] = -\int_{C_t} \mathbf{x} \cdot d\mathbf{F}_0 - \int_{C_p} \mathbf{x}_0 \cdot d\mathbf{F}_0. \tag{4.1}$$

The second integral is a constant independent of $\mathbf{x}(\,\cdot\,)$, which is included so that E is the integral of $-x \cdot d\mathbf{F}_0$ around the whole boundary C. The value of this constant depends on the particular statically admissible stress function $\mathbf{F}_0(\mathbf{X})$ used to represent the boundary data. However, if a deformation $\mathbf{x}(\,\cdot\,)$ minimizes E for some choice of $\mathbf{F}_0$, it does so for any other admissible choice as well, because the integral involving $\mathbf{x}(\,\cdot\,)$ in (4.1) depends on values of $\mathbf{F}_0$ only along the boundary C_t where $\mathbf{F}_0$ is prescribed.

By using a statically admissible stress function $\mathbf{F}_0(\mathbf{X})$, defined at interior points as well as on the boundary, we can transform E into an integral over the region occupied by the sheet in its reference state:

$$\begin{aligned} E[x(\,\cdot\,)] &= -\oint_C \mathbf{x} \cdot (\mathbf{F}_{0,X}\,dX + \mathbf{F}_{0,Y}\,dY) \\ &= -\iint [(\mathbf{x} \cdot \mathbf{F}_{0,Y})_{,X} - (\mathbf{x} \cdot \mathbf{F}_{0,X})_{,Y}]\, dX\, dY. \end{aligned} \tag{4.2}$$

Now, let T_a^0, T_b^0, $\mathbf{a}_0$, and $\mathbf{b}_0$ be the tensions and direction fields derived from $\mathbf{F}_0$ by using (2.3). That is, T_a^0 is the magnitude of $\mathbf{F}_{0,Y}$, and wherever this magnitude is not zero, $\mathbf{a}_0$ is a unit vector in the direction of $\mathbf{F}_{0,Y}$. Also, let $\mathbf{a}$ and $\mathbf{b}$ be defined in terms of $\mathbf{x}$ by (2.1). Then

$$E[x(\,\cdot\,)] = -\iint (T_a^0 \mathbf{a}_0 \cdot \mathbf{a} + T_b^0 \mathbf{b}_0 \cdot \mathbf{b})\, dX\, dY. \tag{4.3}$$

Because the tensions are non-negative and none of the vectors exceeds unity in magnitude, it is apparent that

$$E[\mathbf{x}(\,\cdot\,)] \geq -\iint (T_a^0 + T_b^0)\, dX\, dY, \tag{4.4}$$

with equality only if $\mathbf{a} = \mathbf{a}_0$ wherever $T_a^0 > 0$ and $\mathbf{b} = \mathbf{b}_0$ wherever $T_b^0 > 0$. Since $\mathbf{a}_0$ and $\mathbf{b}_0$ were derived from an arbitrary admissible stress function rather than a deformation, this choice of $\mathbf{a}$ and $\mathbf{b}$ generally is not kinematically admissible.

However, suppose that the problem specified by the relations in Sec. 2 and the boundary conditions (3.1) has a solution, and let this solution be denoted by $\mathbf{x}_0(\mathbf{X})$, $\mathbf{F}_0(\mathbf{X})$. With this choice of $\mathbf{F}_0$ in (4.1), the choice $\mathbf{a} = \mathbf{a}_0$ and $\mathbf{b} = \mathbf{b}_0$ in (4.3) is kinematically admissible, and E is minimized by $\mathbf{x}_0(\mathbf{X})$. But, as pointed out earlier, if E is minimized by a certain deformation for some particular choice of $\mathbf{F}_0$, it is also minimized by that deformation for any other statically admissible choice of $\mathbf{F}_0$. Consequently, E is minimized by any deformation $\mathbf{x}(\mathbf{X})$ that is part of a solution $\mathbf{x}(\mathbf{X})$, $\mathbf{F}(\mathbf{X})$.

Let T_a^0, T_b^0, $\mathbf{a}_0$ and $\mathbf{b}_0$ be the tensions and direction fields for some solution $\mathbf{x}_0$, $\mathbf{F}_0$, and let $\mathbf{a}$ and $\mathbf{b}$ be the direction fields for a possibly different solution $\mathbf{x}$, $\mathbf{F}$. Then E is minimized by $\mathbf{x}$, so the lower bound (4.4) is achieved. The conditions for equality in (4.4) imply that $\mathbf{a} = \mathbf{a}_0$ wherever $T_a^0 > 0$ and $\mathbf{b} = \mathbf{b}^0$ wherever $T_b^0 > 0$. Thus, the direction of a fiber is uniquely determined at any point at which it is tense in some solution, and it is fully extended there in every solution, even if it is not tense in every solution.

5. Applications of the uniqueness lemma. The *load-carrying part* of the network, in a particular problem, consists of those fiber segments that are tense in some solution of the problem. The uniqueness lemma implies, roughly, that the deformation of the load-carrying part of the network is unique. To illustrate the meaning and use of the lemma, we now consider some rather trivial examples. The results in these examples are unsurprising, but the proofs, based on the uniqueness lemma, are surprisingly easy.

Suppose that an undeformed sheet is clamped along its boundary, so that $\mathbf{x} = \mathbf{X}$ on the whole boundary C. The deformation $\mathbf{x} = \mathbf{X}$ (for all $\mathbf{X}$) is kinematically admissible, with fiber directions $\mathbf{a} = \mathbf{i}$ and $\mathbf{b} = \mathbf{j}$. The fiber tensions are arbitrary non-negative functions $T_a(Y)$ and $T_b(X)$. Each point of each fiber is tense in some solution, so the directions $\mathbf{a} = \mathbf{i}$ and $\mathbf{b} = \mathbf{j}$ are unique, and thus the deformation $\mathbf{x} = \mathbf{X}$ is unique. In this example the whole network belongs to the load-carrying part, even though $T_a = T_b = 0$ is one particular solution.

Suppose, instead, that zero tractions are prescribed on the whole boundary C. Then $\mathbf{x} = \mathbf{F} = \mathbf{0}$ is a solution in which $\mathbf{a} = \mathbf{b} = \mathbf{0}$ everywhere. Each point of each fiber is slack in this solution, so the uniqueness lemma implies that $T_a = T_b = 0$ in every solution. In this example the tensions are unique, but the deformation is totally indeterminate; every deformation satisfying the constraints (2.2) is a solution.

In the first example, fiber tensions were non-unique because no traction boundary values were given. However, fiber tensions can also be non-unique even in pure traction boundary-value problems, if a collapsed region [2] occurs in the solution. For example, suppose that a square sheet, bounded by the fibers $X = \pm L$ and $Y = \pm L$, is loaded with point forces $\mathbf{f}$ and $-\mathbf{f}$ at the corners (L, L) and $(-L, -L)$, respectively, and the rest of the boundary is traction-free. The resultant-force method [1, 2] suggests that $\mathbf{a} = \mathbf{b} = \mathbf{f}/|\mathbf{f}|$. Let $\mathbf{F} = \mathbf{f}\phi(\mathbf{X})$ where ϕ increases monotonically from zero to one as Y increases from $-L$ to L, and decreases monotonically from one to zero as X increases from $-L$ to L. Then $T_a = |\mathbf{f}|\,\phi_{,Y}$ and $T_b = -\,|\mathbf{f}|\,\phi_{,X}$. Since there is a solution with positive fiber tensions everywhere, the guess about $\mathbf{a}$ and $\mathbf{b}$ is correct and unique, and $\mathbf{x} = (X + Y)\mathbf{a} + \mathbf{c}$. The whole network collapses onto a line segment in the direction of $\mathbf{f}$. With all fibers following the same path, the distribution of the total load $\mathbf{f}$ among these fibers is almost wholly arbitrary.

It can easily happen that the deformation is uniquely determined even in parts of the network that are not load-carrying. For example, suppose that the square sheet in the preceding example is loaded by normal tractions $\pm T_2\mathbf{j}$ along its edges $Y = \pm L$, and by normal tractions $\pm T_1\mathbf{i}$ on the parts $Y \geq 0$ of the edges $X = \pm L$, with no traction on the parts $Y < 0$ of these edges. The undeformed state is a solution with $\mathbf{a} = \mathbf{i}$, $\mathbf{b} = \mathbf{j}$, $T_b = T_2$, and $T_a = T_1$ or 0, depending on the sign of Y. From this solution we find that all fibers X and all fibers Y for $Y \geq 0$ belong to the load-carrying part of the network, so $\mathbf{b} = \mathbf{j}$ everywhere and $\mathbf{a} = \mathbf{i}$ for $Y \geq 0$ in every solution. Then $\mathbf{x} = \mathbf{X} + \mathbf{c}$, and the deformation is unique (up to a translation) even though the fibers Y for $Y < 0$ do not belong to the load-carrying part of the network.

The load-carrying part of the network may consist of disconnected parts, and in such cases the deformation of each part is determined up to a translation, but the rigid translations may be different for each part. For example, suppose that the same square sheet is loaded by tractions $\pm T\mathbf{i}$ on its ends $X = \pm L$, and the edges $Y = \pm L$ are traction-free. One solution is $x = X$, $y = cY$ ($|c| < 1$), $\mathbf{a} = \mathbf{i}$, $\mathbf{b} = c\mathbf{j}$, $T_a = T$, $T_b = 0$. This particular solution shows that each fiber Y = constant belongs to the load-carrying part but no fiber X = constant does. The results $\mathbf{a} = \mathbf{i}$ and $T_b = 0$ remain true in every solution. The most general deformation with $\mathbf{a} = \mathbf{i}$ everywhere is $\mathbf{x} = \mathbf{x}_0(Y) + X\mathbf{i}$, where $|\mathbf{x}_0'(Y)| \leq 1$. In effect, each fiber Y = constant is a maximal connected subset of the load-carrying part of the network, and $\mathbf{x}_0(Y)$ is a rigid translation of that part. Two fibers Y_1 and Y_2 are disconnected in the sense that it is not possible to go from one to the other along a path that follows fibers in the load-carrying part, since the fibers X = constant do not belong to that part.

6. Integration of the basic equations. The equations in Sec. 2 can be integrated explicitly, but the form of the solution in a given region depends on its qualitative nature there. Regions can be classified kinematically as follows:

a) *Slack* regions, in which $|\mathbf{a}| < 1$ and $|\mathbf{b}| < 1$.
b) *Half-slack* regions, in which one family of fibers is extended but the other family is slack.
c) *Collapsed* regions, where $\mathbf{a}$ and $\mathbf{b}$ are parallel at every point.
d) *Fully extended* regions, in which $\mathbf{a}$ and $\mathbf{b}$ are non-parallel unit vectors.

6a. Slack regions. A region that is slack in some solution is relaxed ($T_a = T_b = 0$) in every solution, but the deformation is highly arbitrary. If $\mathbf{x}(\mathbf{X})$ is slack in some region and $\mathbf{u}(\mathbf{X})$ is a displacement field that vanishes on the boundary of that region, $\mathbf{x}(\mathbf{X}) + k\mathbf{u}(\mathbf{X})$ is still slack for all sufficiently small values of k.

A slack region contiguous to the edge of the sheet may be fully extended in another solution of the same problem, but it is necessarily still a relaxed region. In this connection, a theorem of Rogers [3] concerning relaxed regions is relevant: relaxed regions are bounded by fibers or by the boundary of the sheet. This was proved only for fully extended regions, but the result is general.

6b. Half-slack regions. Consider a half-slack region in which $|\mathbf{a}| = 1$ and $|\mathbf{b}| < 1$. Then $T_b = 0$ in every solution, and it follows from (2.3) that $\mathbf{F} = \mathbf{F}(Y)$ and then that $T_a\mathbf{a} = \mathbf{F}'(Y)$. If the fiber Y belongs to the load-carrying part of the network, so that $T_a > 0$ in some solution, it follows that $\mathbf{a} = \mathbf{a}(Y)$ and $T_a = T_a(Y)$. Thus each fiber Y = constant is *straight* in the deformed state, and the tension in it is constant along it.

If the fibers Y = constant do not belong to the load-carrying part of the network, they may nevertheless be required to be fully extended because the positions of their ends are specified. However, this would occur only if the end positions also require the fibers to be straight, since otherwise the fibers could be slackened. Then the solution has the same form as for load-carrying fibers.

If the extended fibers are neither load-carrying nor kinematically required to be straight, they can be slackened, so the region becomes a slack region. We dismiss this case.

In the non-trivial cases for which $\mathbf{a} = \mathbf{a}(Y)$, the deformation has the form

$$\mathbf{x} = X\mathbf{a}(Y) + \mathbf{x}_0(Y), \tag{6.1}$$

and the fiber directions for the second family are

$$\mathbf{b} = X\mathbf{a}'(Y) + \mathbf{x}_0'(Y). \tag{6.2}$$

Unless the **a**-fibers are parallel, so that $\mathbf{a}'(Y) = \mathbf{0}$, the requirement that $|\mathbf{b}| < 1$ limits the extent of the half-slack region in the X-direction. For example, suppose that $|\mathbf{b}| = 1$ on $X = 0$ but $|\mathbf{b}| < 1$ for small positive values of X. Then $|\mathbf{b}| = 1$ again when $X = -2\mathbf{x}_0' \cdot \mathbf{a}'/\mathbf{a}' \cdot \mathbf{a}'$, a value which is positive under the stated hypotheses. The neighboring fibers Y and $Y + dY$ have in the meantime crossed at a point half-way between the two places where $|\mathbf{b}| = 1$.

6c. Collapsed regions. The deformation in a collapsed region is of the form $\mathbf{x} = \mathbf{f}(p)$, where $p = p(\mathbf{X})$. All particles on a locus p = constant go to the same point $\mathbf{f}(p)$ in the deformed body, and the region maps onto a curve, the *collapse curve.* The fiber directions $\mathbf{a} = \mathbf{f}'p_{,X}$ and $\mathbf{b} = \mathbf{f}'p_{,Y}$ are parallel to the collapse curve and to one another. Conversely, if the **a** and **b** directions are parallel throughout some region, it is a collapsed region.

We leave aside cases in which the collapsed region is slack or half-slack, since such regions have already been discussed. If both **a** and **b** are unit vectors, then $\mathbf{a} = \mathbf{b}$ or $\mathbf{a} = -\mathbf{b}$. With (2.1), these are differential equations which imply that

$$\mathbf{x} = \mathbf{f}(X + Y) \quad \text{or} \quad \mathbf{x} = \mathbf{f}(X - Y). \tag{6.3}$$

A region may be collapsed because of kinematic boundary conditions, but in such cases the fibers can be slackened unless the boundary conditions also require the collapse curve to be a straight line. If the collapsed region belongs to the load-carrying part of the network, the equilibrium conditions (2.3) require the collapse curve to be straight [2]. Thus, in any case in which the problem *requires* a collapsed region, the collapse curve is a straight line, so the deformation has the form

$$\mathbf{x} = (X + Y)\mathbf{a} + \mathbf{c} \quad \text{or} \quad \mathbf{x} = (X - Y)\mathbf{a} + \mathbf{c}, \tag{6.4}$$

in which **a** is a constant unit vector and **c** is a constant. The stress potential has the form $\mathbf{F} = \mathbf{a}\phi(\mathbf{X})$ + constant, where ϕ is monotonic in X and in Y but is otherwise largely arbitrary. An example of this kind was discussed in Sec. 5.

6d. Fully extended regions. From (2.1) it follows that in any deformation, wherever **a** and **b** are smooth they satisfy

$$\mathbf{a}_{,Y} = \mathbf{x}_{,XY} = \mathbf{b}_{,X}. \tag{6.5}$$

In a fully extended region, where $\mathbf{a} \cdot \mathbf{a} = \mathbf{b} \cdot \mathbf{b} = 1$,

$$\mathbf{a} \cdot \mathbf{a}_{,Y} = \mathbf{b} \cdot \mathbf{b}_{,X} = 0. \tag{6.6}$$

From (6.5) and (6.6) together, it follows that $\mathbf{a}_{,Y}$ is orthogonal to both **a** and **b**, which are non-parallel by hypothesis in fully extended regions. Thus,

$$\mathbf{a}_{,Y} = \mathbf{b}_{,X} = \phi(\mathbf{X})\mathbf{a} \times \mathbf{b}. \tag{6.7}$$

Rivlin's theory [1] is, in effect, the theory of plane deformations of fully extended regions. In a plane deformation the deformed sheet lies in the plane $\mathbf{k} \cdot \mathbf{x} = 0$, say, so **a**, **b**, and all of their derivatives are also orthogonal to **k**. This implies that ϕ in (6.7) is zero, so

$$\mathbf{a} = \mathbf{a}(X) \quad \text{and} \quad \mathbf{b} = \mathbf{b}(Y). \tag{6.8}$$

Then the deformation has the form

$$\mathbf{x} = \mathbf{f}(X) + \mathbf{g}(Y); \tag{6.9}$$

the fibers of a given family are congruent in the deformed state.

With (6.8), it follows from (2.3) that

$$\mathbf{k} \times \mathbf{a} \cdot \mathbf{F} = M(X) \quad \text{and} \quad \mathbf{k} \times \mathbf{b} \cdot \mathbf{F} = N(Y), \tag{6.10}$$

whence

$$F = J^{-1}[M(X)\mathbf{b}(Y) - N(Y)\mathbf{a}(X)], \tag{6.11}$$

where

$$J = \mathbf{k} \cdot \mathbf{a} \times \mathbf{b}. \tag{6.12}$$

Further details of the derivation are given by Pipkin [2] and, in its original form, by Rivlin [1].

Pipkin [2] discussed the conditions satisfied at folds in fully extended regions, and showed in particular that in a solution with no compressive stresses, the tensions T_a and T_b must vanish at any point (X, Y) such that the fibers $X =$ constant and $Y =$ constant both cross the fold. Consequently, folds require no special attention within the present theory. If a fold cuts across both families of fibers, it is in a relaxed region, and such regions can usually be made slack.

Both here and in [2], we have not been sufficiently careful about the definition of a fold. In the sense used here, a fold is a curve across which **a** or **b** (or both) is discontinuous and J, defined in (6.12), changes sign. It is possible for J to change sign across a curve without any discontinuity in **a** or **b**. In that case we call the locus $J = 0$ an *envelope* rather than a fold. On an envelope, **a** and **b** are parallel to one another and parallel to the envelope. The expression (6.11) for **F** remains valid at an envelope, in the sense that it can be evaluated by using l'Hospital's rule. Consequently, it is not necessary to take the possibility of envelopes into account when seeking solutions.

7. Purely tensile solutions in Rivlin's theory. In Rivlin's theory, the deformation must be fully extended everywhere but fiber tensions need not be non-negative everywhere. Because of the possibility of folds, solutions are highly non-unique. However, experience indicates that most problems have exactly one purely tensile solution, i.e. one with non-negative fiber tensions. In the context of the present theory, the uniqueness of the purely tensile solution is explained by the uniqueness lemma. Any purely tensile solution in Rivlin's theory is a solution in the present theory and so it is unique, among purely tensile solutions, to the extent implied by the uniqueness lemma and the further requirement that all regions are fully extended.

A difficulty arises in Rivlin's theory when no purely tensile solution is found. Under the false conjecture that there is always at least one purely tensile solution, a great deal of ingenuity might be wasted in a futile attempt to find it. Such questions can often be settled by seeking a solution within the present extended theory instead. If a solution that is not fully extended is found, and the uniqueness lemma implies that there is no fully extended, purely tensile solution, then in Rivlin's theory no purely tensile solution exists.

Let us illustrate this by an example contrived to make the analysis easy. A sheet is clamped along the edge $X = 0$, so that $\mathbf{x} = \mathbf{j}Y$ there. The edges $Y = \pm H$ are traction-free. Tractions $T\mathbf{u}(Y)$ per unit initial length are applied as dead loads on the edge $X = L$. Here **u** and the associated vector $\mathbf{v}$ are defined by

$$\mathbf{u} = \mathbf{i}\cos(cY) - \mathbf{j}\sin(cY),$$
$$\mathbf{v} = \mathbf{i}\sin(cY) + \mathbf{j}\cos(cY). \tag{7.1}$$

The boundary values of the stress potential implied by these tractions can be taken to be

$$\mathbf{F}_0(L, Y) = (T/c)\mathbf{v}(Y), \qquad \mathbf{F}_0(X, \pm H) = (T/c)\mathbf{v}(\pm H). \tag{7.2}$$

In Rivlin's theory the conjecture that $\mathbf{x} = \mathbf{X}$ everywhere leads to a solution in which tensions are positive everywhere (if $cH \leq \pi/2$) except in the boundary fiber $X = L$. This fiber is singular, and carries a finite compressive load.

Within the present theory, a half-slack solution is easily obtained. The stress potential is $\mathbf{F}(Y) = \mathbf{F}_0(L, Y)$, so $\mathbf{F}'(Y) = T\mathbf{u}(Y)$ and thus $T_a = T$ and $\mathbf{a}(Y) = \mathbf{u}(Y)$. Because all fibers Y are load-carrying, this field $\mathbf{a}$ is unique, and with the condition on $X = 0$, the deformation is uniquely determined as

$$\mathbf{x} = Y\mathbf{j} + X\mathbf{u}(Y). \tag{7.3}$$

All this is true with the proviso that $|\mathbf{b}|$ must not exceed unity. From (7.3) and (7.1),

$$\mathbf{b} = \mathbf{j} - cX\mathbf{v}(Y). \tag{7.4}$$

The condition $|\mathbf{b}| \leq 1$ is satisfied throughout the sheet if $cH < 2\cos(cL)$, which is true if cH and cL are small.

Since the deformation is unique and $\mathbf{b}$ is not a unit vector everywhere, we conclude that within Rivlin's theory no purely tensile solution exists.

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