THE AMBIGUOUS TWIST OF LOVE*

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1. Introduction. Since the position field is a fundamental variable of continuum mechanics, this science is endowed with all the richness and complexity of three-dimensional geometry. This geometry leads to some fundamental difficulties both in the foundations of continuum mechanics and in practical applications of it to concrete problems. Here we discuss one such difficulty.

To fix ideas, consider a shaft whose lateral surface is stress-free and whose ends have prescribed configurations. If one end of the shaft is now twisted through an integral multiple of 2π, then the boundary conditions specifying the positions of the material points of the ends remain unchanged. Thus it is unlikely that equilibrium problems for the shaft under these mixed boundary conditions have unique solutions. (This example of nonuniqueness may be added to those of [1, 13, 17].) It is the purpose of this note to investigate this ambiguity and to show how it can be resolved. (No such ambiguity arises for corresponding dynamical problems because the states of the ends can be expected to evolve continuously from their initial states.) Our results have immediate application to the buckling of shafts under prescribed twists (cf. [2, 9, 11]). The techniques we exploit are related to those used to study the stability of closed loops of supercoiled DNA molecules (cf. [4, 7]).

Some of the specific difficulties we confront can be discerned by examining the configurations of a strip of rubber whose ends are held in clamps (see Fig. 1). We regard the position of the left end of the strip as fixed, but for the purpose of effective illustration we allow the right end to translate. This causes no loss of generality since here we wish merely to study the kinematic problem of classifying configurations that can be deformed into each other; we do not examine the existence of equilibrium configurations (although some of the results we obtain are most useful for such studies).

In Fig. 1a we show the reference configuration of the strip, which we may assume to be stress-free. In Fig. 1b we show a configuration obtained from that of Fig. 1a by twisting the right end of the strip through an angle of -4π about the axis joining the left end of the strip to the right. If we now translate the right end of the strip to the left, the configuration of Fig. 1b becomes unstable and pops into a configuration like that shown in Fig. 1c. Obviously the deformation from the configuration of Fig. 1b to that of Fig. 1c is achieved without the strip being cut. The dotted line of Fig. 1c indicates a fictitious continuation of the strip to the right. If part of the strip is made to pass behind and to the right of the right clamp, made to cut the fictitious continuation as in Fig. 1d, and finally brought forward, then the strip

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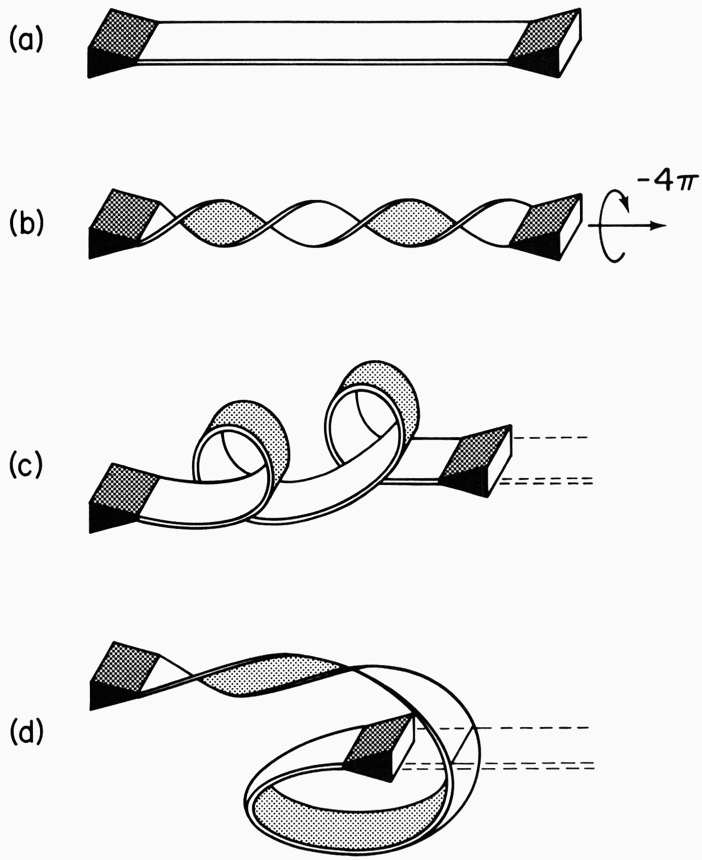

FIG. 1.

pops into the configuration of Fig. 1a. Note that the process illustrated in Fig. 1d requires that whatever supports the right clamp must be released for the strip to cut its fictitious continuation.

This and related simple experiments with rubber bands raise a number of questions:

i) What sort of "boundary conditions" distinguish the state of Fig. 1b from that of Fig. 1a? Can such conditions be stated in a way that promotes the analysis of boundary-value problems?

ii) Can such conditions account for the fact that the configuration of Fig. 1b can be deformed into that of Fig. 1c without the strip being cut?

iii) Can such conditions account for the fact that any configuration of Fig. 1 can be deformed into every other configuration without the strip being cut, provided the clamp at one end can be released from its support?

We resolve these questions in this paper. In Sec. 2 we formulate the kinematical problem of defining the amount of twist for three-dimensional bodies. We then define the measure of twist of A.E.H. Love and show that it is both analytically and topologically inadequate for resolving the questions just posed. In Sec. 3 we introduce an alternative measure of twist and associated "boundary conditions" that do not suffer from the defects of Love's twist and are useful for treating deformations of moderately large size. (The applications of these measures certainly encompass all the traditional engineering problems for rods.) In Sec. 4 we describe a general strategy for resolving the questions raised above when the deformations are arbitrarily large. This treatment is based on the concept of linking number. We show that the fairly complicated prescription of boundary conditions for the general case reduces to the much simpler prescription of Sec. 3 when the deformations are only moderately large. We show finally that these prescriptions are analytically suitable for the treatment of boundary-value problems.

2. The twist of Love. Let the region occupied by a body in its reference configuration be the closure $\bar{B}$ of a bounded domain in Euclidean 3-space $\mathbb{E}^3$. Suppose $\bar{B}$ admits a curvilinear coordinate system (x^1, x^2, s) with s ranging over $[0, 1]$, say. In particular, the position of the material point in $\bar{B}$ with coordinates (x^1, x^2, s) is denoted $\mathbf{P}(x^1, x^2, s)$. $\mathbf{P}$ is assumed to be twice continuously differentiable and to satisfy

$$\left(\frac{\partial \mathbf{P}}{\partial x^1} \times \frac{\partial \mathbf{P}}{\partial x^2}\right) \cdot \frac{\partial \mathbf{P}}{\partial s} > 0 \tag{2.1}$$

on its domain of definition $\mathbf{P}^{-1}(\bar{B})$. The surface $\mathbf{P}(\,\cdot\,, \,\cdot\,, s)$ is called the *section s*. We assume that $[0, 1] \ni s \mapsto \mathbf{P}(0, 0, s) \equiv \mathbf{R}(s)$ is a curve in $\bar{B}$. We assume that the *end sections* $\mathbf{P}(\,\cdot\,, \,\cdot\,, 0)$ and $\mathbf{P}(\,\cdot\,, \,\cdot\,, 1)$ contain open sets of the boundary ∂B of B and that these open sets respectively contain $\mathbf{P}(0, 0, 0)$ and $\mathbf{P}(0, 0, 1)$ in their interiors.

We associate with each section s a pair of material vectors $\mathbf{D}_1(s)$ and $\mathbf{D}_2(s)$ such that $\mathbf{D}_1$ and $\mathbf{D}_2$ are continuously differentiable and such that

$$[\mathbf{D}_1(s) \times \mathbf{D}_2(s)] \cdot \mathbf{R}'(s) > 0. \tag{2.2}$$

Here and below the prime denotes the s derivative of the function on which it is placed. $\mathbf{D}_1$ and $\mathbf{D}_2$ may be chosen in any convenient way; e.g., $\mathbf{D}_\alpha(s)$ could be defined to be $\partial\mathbf{P}(0, 0, s)/\partial x^\alpha$ for $\alpha = 1, 2$ or could be taken as some average over a section of $\partial\mathbf{P}(x^1, x^2, s)/\partial x^\alpha$.

Let the position in a deformed configuration of the material point (with coordinates) (x^1, x^2, s) be $\mathbf{p}(x^1, x^2, s)$ and let $\mathbf{r}(s) \equiv \mathbf{p}(0, 0, s)$. Let $\mathbf{d}_1(s)$ and $\mathbf{d}_2(s)$ be the deformed configurations of $\mathbf{D}_1(s)$ and $\mathbf{D}_2(s)$. (E.g., if $\mathbf{D}_\alpha(s) = \partial\mathbf{P}(0, 0, s)/\partial x^\alpha$, then $\mathbf{d}_\alpha(s) = \partial\mathbf{p}(0, 0, s)/\partial x^\alpha$.) The requirement that the deformation preserve orientation is that the Jacobian be positive:

$$\frac{\left(\dfrac{\partial \mathbf{p}}{\partial x^1} \times \dfrac{\partial \mathbf{p}}{\partial x^2}\right) \cdot \dfrac{\partial \mathbf{p}}{\partial s}}{\left(\dfrac{\partial \mathbf{P}}{\partial x^1} \times \dfrac{\partial \mathbf{P}}{\partial x^2}\right) \cdot \dfrac{\partial \mathbf{P}}{\partial s}} > 0. \tag{2.3}$$

Inequalities (2.1) and (2.3) imply that

$$\left[\frac{\partial \mathbf{p}}{\partial x^1}(0, 0, s) \times \frac{\partial \mathbf{p}}{\partial x^2}(0, 0, s)\right] \cdot \mathbf{r}'(s) > 0 \quad \text{for} \quad s \in [0, 1]. \tag{2.4}$$

We assume that $\mathbf{d}_1$ and $\mathbf{d}_2$ are so chosen that (2.1) and (2.3) imply that

$$[\mathbf{d}_1(s) \times \mathbf{d}_2(s)] \cdot \mathbf{r}'(s) > 0 \quad \text{for} \quad s \in [0, 1]. \tag{2.5}$$

We study problems in which the deformed configurations $\mathbf{p}(\cdot, \cdot, 0)$ and $\mathbf{p}(\cdot, \cdot, 1)$ of the end sections are wholly or partly prescribed while no conditions are imposed on the restrictions of $\mathbf{p}$ to the material points on the lateral surface of $\bar{B}$. (The lateral surface consists of all the material points of the boundary ∂B not lying in the end sections. In a well-set problem, the lateral surface should be subjected to traction boundary conditions. The formulation of these conditions is of no concern to us here because we are treating only kinematical questions.)

We shall define various measures of twist for B in terms of the functions $\mathbf{r}$, $\mathbf{d}_1$, $\mathbf{d}_2$. (These measures will enable us to distinguish between the configurations of Fig. 1.) We thereby reduce the kinematics of twist of a three-dimensional body to that of a Cosserat rod with two directors $\mathbf{d}_1$ and $\mathbf{d}_2$ (cf. [6] or [18]).

The notion of the twist of a rod was introduced by St. Venant [14, 15] and improved by Love [10]. The underlying ideas are related to Cauchy's [5] treatment of average rotation. Love's definition is restricted to the case that $\{\mathbf{d}_1, \mathbf{d}_2, \mathbf{r}'\}$ is orthonormal. Here we give an obvious generalization, based on the work of Ericksen and Truesdell [6], to the case that $\{\mathbf{d}_1, \mathbf{d}_2, \mathbf{r}'\}$ merely satisfies (2.5). As a measure of strain Love's twist has been justly criticized as inadequate by Ericksen and Truesdell. We now show that the prescription of Love's twist is also inadequate for removing ambiguities in the specification of position boundary conditions. We thereafter replace it with other quantities that can accomplish this goal.

Inequality (2.5) implies that the projection of $\mathbf{d}_1$ onto the plane perpendicular to $\mathbf{r}'$ does not vanish. Let $\mathbf{a}_1(s)$ be the unit vector along this projection, let $\mathbf{a}_3 = \mathbf{r}'/|\mathbf{r}'|$, and let $\mathbf{a}_2 = \mathbf{a}_3 \times \mathbf{a}_1$. Then the orthonormality of $\{\mathbf{a}_j\}$ ensures that there is a vector $\mathbf{w}$ such that $\mathbf{a}_j' = \mathbf{w} \times \mathbf{a}_j$.

Suppose that $\mathbf{r}$ is smooth enough to possess a principal tangent, normal, binormal triad $\{\mathbf{a}_3, \mathbf{n}, \mathbf{b}\}$. Then we define ϕ by

$$\mathbf{a}_1(s) = \cos \phi(s)\mathbf{n}(s) + \sin \phi(s)\mathbf{b}(s),$$
$$\mathbf{a}_2(s) = -\sin \phi(s)\mathbf{n}(s) + \cos \phi(s)\mathbf{b}(s). \tag{2.6}$$

The *local twist of Love* of $\mathbf{d}_1$ relative to $\mathbf{r}$ at s is $\phi'(s)$. It is readily shown by the Serret-Frenet formulas that

$$\phi' = \mathbf{w} \cdot \mathbf{a}_3 - \tau \tag{2.7}$$

where τ is the torsion of $\mathbf{r}$. The *total twist of Love* of $\mathbf{d}_1$ relative to $\mathbf{r}'$ is

$$\int_0^1 (\mathbf{w} \cdot \mathbf{a}_3 - \tau)ds. \tag{2.8}$$

This is a functional of $\mathbf{r}$, $\mathbf{d}_1$, $\mathbf{d}_2$ when these functions are smooth enough. We may contemplate prescribing (2.8) or some alternative functional with the aim of effectively distinguishing different classes of configurations meeting the same position boundary conditions on the ends. Such functionals, to be called *twist functionals*, should ideally meet the following roughly stated criteria:

A) The specification of $\mathbf{d}_1(0)/|\mathbf{d}_1(0)|$, $\mathbf{d}_2(0)/|\mathbf{d}_2(0)|$, $\mathbf{d}_1(1) \times \mathbf{d}_2(1)/|\mathbf{d}_1(1) \times \mathbf{d}_2(1)|$, and suitable twist functionals uniquely determines $\mathbf{d}_1(1)/|\mathbf{d}_1(1)|$ and $\mathbf{d}_2(1)/|\mathbf{d}_2(1)|$.

B) Each twist functional is a *null Lagrangian*, i.e., its Euler–Lagrange equation (from the calculus of variations) is identically satisfied. This condition is satisfied if the functional is the integral over [0, 1] of the derivative of a composite function of s. If this condition were not met, then the principle of virtual work would imply that the constraint embodied in the prescription of the functional would have to be maintained by constraint forces or couples distributed along the length of the body, rather than by reactions acting solely on the ends.

C) The collection of all configurations $\{\mathbf{r}, \mathbf{d}_1, \mathbf{d}_2\}$ for which a twist functional has a given value can be readily described.

D) Two configurations giving the twist functionals the same set of values can be deformed into each other. (The notion of deformation must be made precise. This requirement is obviously closely related to criterion C. It also bears a deep relationship to criterion B in that it says that twist functionals are deformation invariants while criterion B implies that the first variations (or Gâteaux differentials) of the twist functionals vanish identically when conditions at $s = 0, 1$ are suitably fixed. This suggests that twist functionals are constants on arcwise-connected subsets of their domains. This is just another way of saying that the twist functionals are deformation invariants.)

E) The twist functionals have a mathematical form suitable for available methods of analysis of boundary-value problems of nonlinear elasticity. In particular, their forms must be useful for the characterization of configurations as extrema of energy functionals because such characterizations are important in the study of stability. We interpret this requirement narrowly by limiting our attention to problems whose configurations are completely determined by $\{\mathbf{r}, \mathbf{d}_1, \mathbf{d}_2\}$. In this case we specifically require that the domain of each twist functional be a subset of $\{\mathbf{r}, \mathbf{d}_1, \mathbf{d}_2\}$ that belongs to each Sobolev space $W_p^1(0, 1)$ with $p > 1$ and that the twist functional be weakly continuous with respect to $W_p^1(0, 1)$ for each $p > 1$. (This condition is very closely related to condition *B*. See [3].)

To avoid immersing ourselves in technical details not central to our goals, we shall not provide proofs of our assertions about the weak continuity of twist functionals. These may be constructed from the following facts (cf. [12], e.g.). The space $W_p^1(0, 1)$ of vector-valued functions $\mathbf{u}$ consists of functions $\mathbf{u}$ in $L_p(0, 1)$ whose (distributional) derivatives $\mathbf{u}'$ are also in $L_p(0, 1)$. The norm $\|\cdot\|$ on $W_p^1(0, 1)$ is defined by

$$\|\mathbf{u}\|^p = \int_0^1 (\mathbf{u} \cdot \mathbf{u} + \mathbf{u}' \cdot \mathbf{u}')^{p/2}\, ds.$$

If $\mathbf{u}$ is in $W_p^1(0, 1)$, then $\mathbf{u}$ is continuous on [0, 1]. A sequence $\{\mathbf{v}_k\}$ in $W_p^1(0, 1)$ *converges weakly* to $\mathbf{v}$ if

$$\int_0^1 [(\mathbf{v}_k - \mathbf{v}) \cdot \mathbf{x} + (\mathbf{v}_k' - \mathbf{v}') \cdot \mathbf{y}]\, ds \to 0 \quad \text{as} \quad k \to \infty$$

for all $\mathbf{x}, \mathbf{y} \in L_{p*}(0, 1)$ where $p^* = p/(p-1)$. If $\mathbf{v}_k$ converges weakly to $\mathbf{v}$ in $W_p^1(0, 1)$, then $\{\mathbf{v}_k\}$ is bounded in $W_p^1(0, 1)$ and $\mathbf{v}_k$ converges uniformly to $\mathbf{v}$ on [0, 1]. A functional $\mathbf{u} \mapsto \Psi[\mathbf{u}]$ is *weakly continuous* on $W_p^1(0, 1)$ if $\Psi[\mathbf{v}_k] \to \Psi[\mathbf{v}]$ for each weakly convergent sequence $\{\mathbf{v}_k\}$ in $W_p^1(0, 1)$. If Ψ has the form

$$\Psi[\mathbf{u}] = \int_0^1 [\mathbf{f}(\mathbf{u}(s), s) \cdot \mathbf{u}'(s) + \mathbf{g}(\mathbf{u}(s), s)]ds \tag{2.9}$$

where $\mathbf{f}$ and $\mathbf{g}$ are continuous, then Ψ is weakly continuous on $W_p^1(0, 1)$ for each $p > 1$.

The identity (2.7) ensures that (2.8) meets criterion B. But (2.8) does not meet criterion A because the total twist depends upon the principal triad, which is not known a priori. Thus it does not meet criterion C, either. Since configurations of Fig. 1b and Fig. 1c have different values of (2.8), criterion D is not satisfied. The functional (2.8) does not satisfy criterion E for the simple reason that (2.8) is not even defined on an open subset of $W_p^1(0, 1)$, because τ depends on the third derivative of $\mathbf{r}$. Thus (2.8) is inadequate for our purposes. Note that

(2.7) implies that (2.8) equals $\phi(1) - \phi(0) \bmod 2\pi$. But $\phi(1) - \phi(0)$ depends on the values of the principal triad at the ends and this information is not prescribed in any standard boundary conditions. Since criterion B holds, the twist of Love is an invariant in the sense discussed in the parenthetical remarks following the statement of criterion D. But these remarks show that it is a useless invariant. (It does, however, have some value for ring-like bodies in which case $\mathbf{r}$ is a simple closed curve; cf. [4, 7].) We now construct a more satisfactory twist functional.

3. A twist functional for moderately large deformations. Suppose that $\mathbf{R} \in C^3([0, 1])$ and that $\mathbf{R}'(s) \neq \mathbf{O}$ for s in $[0, 1]$. (This is an innocuous assumption because it pertains only to the reference state.) Let $\mathbf{E}_1(s)$ and $\mathbf{E}_2(s)$ be a principal normal and binormal to $\mathbf{R}$ at $\mathbf{R}(s)$ and let $\mathbf{E}_3 = \mathbf{E}_1 \times \mathbf{E}_2 = \mathbf{R}'/|\mathbf{R}'|$. We assume that $\mathbf{E}_1$ and $\mathbf{E}_2$ are continuously differentiable. (If $\mathbf{R}$ is straight over an open s-interval, then we have considerable freedom in assigning $\mathbf{E}_1$ and $\mathbf{E}_2$. Matters could be arranged so that $\mathbf{E}_\alpha = \mathbf{D}_\alpha$ for $\alpha = 1, 2$.) The unit vector lying along the projection of $\mathbf{d}_1$ onto the plane perpendicular to $\mathbf{R}'$ is

$$\mathbf{e}_1 = \frac{\mathbf{d}_1 - (\mathbf{d}_1 \cdot \mathbf{E}_3)\mathbf{E}_3}{\sqrt{\mathbf{d}_1 \cdot \mathbf{d}_1 - (\mathbf{d}_1 \cdot \mathbf{E}_3)^2}}, \tag{3.1}$$

which is well-defined if $\mathbf{d}_1$ is nowhere parallel to $\mathbf{E}_3$. (If $\mathbf{D}_\alpha = \mathbf{E}_\alpha$, this condition is assured by the requirement that $\mathbf{d}_1$ nowhere rotate from the plane of $\mathbf{E}_1$ and $\mathbf{E}_2$ by an angle $\geq \pi/2$. This bound, allowing moderately large rotations, is suitable for all traditional engineering applications.) Then $\mathbf{e}_1$ has the form

$$\mathbf{e}_1 = \cos \chi_1 \mathbf{E}_1 + \sin \chi_1 \mathbf{E}_2. \tag{3.2}$$

The *local twist of* $\mathbf{d}_1$ *relative to* $\mathbf{R}$ at s is $\chi_1'(s)$. Since

$$\mathbf{e}_1' = \chi_1'(\mathbf{E}_3 \times \mathbf{e}_1) + \cos \chi_1 \mathbf{E}_1' + \sin \chi_1 \mathbf{E}_2', \tag{3.3}$$

the Serret–Frenet formulas yield

$$\chi_1' = (\mathbf{E}_3 \times \mathbf{e}_1) \cdot \mathbf{e}_1' - T \tag{3.4}$$

where T is the torsion of $\mathbf{R}$. We likewise define the twist χ_2' of $\mathbf{d}_2$ relative to $\mathbf{R}$ by replacing the subscript 1 whenever it appears in (3.10)–(3.14) by 2.

Now if $\mathbf{d}_1(0)$ is prescribed (equal to $\mathbf{D}_1(0)$), or more generally, if $\mathbf{e}_1(0)$ is prescribed, then we know $\chi_1(0) \bmod 2\pi$. Without loss of generality we take $\chi_1(0)$ to lie in $[0, 2\pi)$. If $\mathbf{e}_1(1)$ is also prescribed, then we only know $\chi_1(1) \bmod 2\pi$. We remove this ambiguity in $\chi_1(1)$ by prescribing the *total twist of* $\mathbf{d}_1$ *relative* to $\mathbf{R}$, namely

$$\int_0^1 \chi_1' \, ds = \int_0^1 (\mathbf{E}_3 \times \mathbf{e}_1) \cdot \mathbf{e}_1' \, ds - \int_0^s T \, ds \tag{3.5}$$

where $\mathbf{e}_1$ is given by (3.1).

We note that the total twists of $\mathbf{d}_1$ and $\mathbf{d}_2$ relative to $\mathbf{R}$ are functionals satisfying criterion A. The definition of the (local) twist of $\mathbf{d}_\alpha$ with respect to $\mathbf{R}$ as χ_α' ensures that criterion B is also met. Criterion C is also satisfied: configurations for which (2.14) is prescribed (and well-defined) are those for which $\mathbf{e}_1$ winds about (*the known curve*) $\mathbf{R}$ the prescribed number of times. Criterion D is satisfied only in the restricted sense that the deformations of $\mathbf{d}_1$ from its reference state can only be moderately large when the total twist of $\mathbf{d}_1$ about $\mathbf{R}$ is prescribed. (When $\mathbf{D}_\alpha = \mathbf{E}_\alpha$, such deformations are defined by the requirement that $\mathbf{d}_1$ subtend an angle $<\pi/2$ with respect to the $(\mathbf{E}_1, \mathbf{E}_2)$-plane.) Then all such deformations are characterized by the number of times $\mathbf{e}_1$ winds about $\mathbf{R}$. Criterion E is satisfied because (3.5) has the form (2.9).

We note that similar considerations are valid when the configurations of the end sections are not fully prescribed. E.g., suppose that the ends are hinged along $\mathbf{D}_1(0)$ and $\mathbf{D}_1(1)$ so that $\mathbf{d}_1(0)/|\mathbf{d}_1(0)|$ and $\mathbf{d}_1(1)/|\mathbf{d}_1(1)|$ are prescribed. Ambiguity in the number of times $\mathbf{e}_1$ winds about $\mathbf{R}$ is removed by the specification of (3.5).

4. A family of twist functionals based on the linking number. If $\mathbf{d}_1(0)$ and $\mathbf{d}_1(1)$ are prescribed with $\mathbf{d}_2(0)$ and $\mathbf{d}_2(1)$ left free of kinematic restraint, the body B may be regarded as hinged at 0 and 1. In this case we remove the ambiguity inherent in the specification of $\mathbf{d}_1(0)$ and $\mathbf{d}_1(1)$ by prescribing a suitable twist functional associated with $\mathbf{d}_1$. If $\mathbf{d}_1$ and $\mathbf{d}_2$ are prescribed at both ends, then the body may be regarded as welded to rigid supports at its ends. To remove the ambiguity in this case, it still suffices to prescribe just a twist functional for $\mathbf{d}_1$ in addition to the values of $\mathbf{d}_1$ and $\mathbf{d}_2$ at the ends, for this information clearly yields the twist functional for $\mathbf{d}_2$. Thus we can reduce our analysis to one based on the study of the two curves $\mathbf{r}$ and $\mathbf{r} + \mathbf{d}_1$ with $\mathbf{r}'(s) \times \mathbf{d}_1(s) \neq \mathbf{O}$ for $s \in [0, 1]$ as a consequence of (2.5). We now drop the subscript 1 from $\mathbf{d}$.

If $\mathbf{r}$ and $\mathbf{q}$ are simple *closed* curves in $\mathbb{E}^3$ that have no point common, then the number of times $\mathbf{q}$ winds around $\mathbf{r}$ is given by the linking number $\lambda(\mathbf{r}, \mathbf{q})$, originally defined by Gauss [8], which is an integer invariant under all deformations of $\mathbf{r}$ and $\mathbf{q}$ for which the deformed images of $\mathbf{r}$ and $\mathbf{q}$ remain simple and never touch. In particular, if $\mathbf{r}$ is spanned by an oriented immersed surface Σ, then the linking number is, to within a sign, the number of times $\mathbf{q}$ pierces Σ from one side minus the number of times it pierces Σ from the opposite side (provided due account is taken of places where $\mathbf{q}$ touches but does not cross Σ). Cf. [16]. Another intuitively simple characterization is given by [4]. Thus λ would be a suitable twist functional for toroidal bodies. We wish to adapt it for our purposes. If $[0, L] \ni s \mapsto \mathbf{r}(s), \mathbf{q}(s) \in \mathbb{E}^3$ are absolutely continuous, closed, nonintersecting curves, then $\lambda(\mathbf{r}, \mathbf{q})$ is also given by an explicit analytic formula:

$$\lambda(\mathbf{r}, \mathbf{q}) \equiv \frac{1}{4\pi} \int_0^L \int_0^L \frac{\mathbf{q}(s) - \mathbf{r}(t)}{|\mathbf{q}(s) - \mathbf{r}(t)|^3} \cdot [\mathbf{q}'(s) \times \mathbf{r}'(t)]\, ds\, dt. \tag{4.1}$$

(The linking number can also be defined as the topological degree of $S^1 \times S^1 \ni (s, t) \mapsto \mathbf{q}(s) - \mathbf{r}(t) \in \mathbb{E}^3 \backslash \{\mathbf{O}\}$, where S^1 represents the circle of radius $L/2\pi$. Thus λ makes sense for continuous curves $\mathbf{r}$ and $\mathbf{q}$.)

We now construct a twist functional useful for very large deformations. Let us first suppose that $\mathbf{r}$ and $\mathbf{d}$ are prescribed at $s = 0, 1$. Let $L > 1$, $\varepsilon > 0$, let $\boldsymbol{\rho}: [1, L] \to \mathbb{E}^3$ be a fixed, absolutely continuous, simple curve with $\boldsymbol{\rho}(1) = \mathbf{r}(1)$ and $\boldsymbol{\rho}(L) = \mathbf{r}(0)$, and let $\boldsymbol{\delta}: [1, L] \to \mathbb{E}^3$ be a fixed, absolutely continuous curve with $\boldsymbol{\delta}(1) = \mathbf{d}(1)$ and $\boldsymbol{\delta}(L) = \mathbf{d}(0)$. We set

$$\begin{aligned} (\bar{\mathbf{r}}(s), \bar{\mathbf{d}}(s)) &\equiv (\mathbf{r}(s), \mathbf{d}(s)) \quad \text{for} \quad s \in [0, 1], \\ &\equiv (\boldsymbol{\rho}(s), \boldsymbol{\delta}(s)) \quad \text{for} \quad s \in [1, L]. \end{aligned} \tag{4.2}$$

Let $\mathscr{G}_1(\boldsymbol{\rho}, \boldsymbol{\delta})$ be the set of all $(\mathbf{r}, \mathbf{d})$ for which $\bar{\mathbf{r}}$ is a simple, absolutely continuous, closed curve and $\bar{\mathbf{d}}$ is an absolutely continuous, closed curve, and for which there exists an $\eta > 0$ such that $[0, L] \times [0, \eta] \ni (s, x) \mapsto \bar{\mathbf{r}} + x\bar{\mathbf{d}}$ is one-to-one. If $(\mathbf{r}, \mathbf{d}) \in \mathscr{G}_1(\boldsymbol{\rho}, \boldsymbol{\delta})$, then we can use (4.1) to define $\lambda(\bar{\mathbf{r}}, \bar{\mathbf{r}} + x\bar{\mathbf{d}}) \equiv \Lambda(\mathbf{r}, \mathbf{d}; \boldsymbol{\rho}, \boldsymbol{\delta})$. If $\mathbf{r}(0)$ or $\mathbf{r}(1)$ are not prescribed, then by defining $(\boldsymbol{\rho}, \boldsymbol{\delta})$

as above we obtain a family of such $(\boldsymbol{\rho}, \boldsymbol{\delta})$ parametrized by the variable values of $\mathbf{r}(0)$ and $\mathbf{r}(1)$. More generally, let $\mathscr{G}_0(\boldsymbol{\rho}, \boldsymbol{\delta})$ be defined like $\mathscr{G}_1(\boldsymbol{\rho}, \mathbf{d})$ except that $\bar{\mathbf{r}}$ and $\bar{\mathbf{d}}$ need not be absolutely continuous. If $(\mathbf{r}, \mathbf{d}) \in \mathscr{G}_0$, then we can define $\Lambda(\mathbf{r}, \mathbf{d}; \boldsymbol{\rho}, \boldsymbol{\delta})$ by using the degree-theoretic definition of linking numbers. Let us call $(\mathbf{r}, \mathbf{d})$ a *strip*. We say that a strip $(\mathbf{r}_0, \mathbf{d}_0)$ can be *deformed* in a subset $\mathscr{G}$ of strips into a strip $(\mathbf{r}_1, \mathbf{d}_1)$ if there is a one-parameter family of strips $\{(\mathbf{r}_\alpha, \mathbf{d}_\alpha), \alpha \in [0, 1]\}$ with $[0, 1] \ni \alpha \mapsto (\mathbf{r}_\alpha, \mathbf{d}_\alpha) \in \mathscr{G}$ continuous. (The norm on $\mathscr{G}$ is that of uniform convergence, $\alpha \mapsto (\mathbf{r}_\alpha, \mathbf{d}_\alpha)$ is a *homotopy* from $(\mathbf{r}_0, \mathbf{d}_0)$ to $(\mathbf{r}_1, \mathbf{d}_1)$.) We say that $(\bar{\mathbf{r}}, \bar{\mathbf{d}})$ is *unknotted* if $\bar{\mathbf{r}}$ can be deformed to a circle in the space of all simple, closed curves. We also consider more general deformations of such strips in which $\mathbf{r}$ is allowed to cut through $\boldsymbol{\rho}$ a finite number of times. Such a deformation is said to *cut the support*. Using the degree-theoretic definition of linking number and its known properties (cf. [16], e.g.), we readily obtain the following results.

THEOREM 4.3 (i) $\Lambda(\,\cdot\,, \cdot\,; \boldsymbol{\rho}, \boldsymbol{\delta})$ is an integer-valued continuous function on $\mathscr{G}_0(\boldsymbol{\rho}, \boldsymbol{\delta})$. Thus it is constant on each component (maximal connected subset) of $\mathscr{G}_0(\boldsymbol{\rho}, \boldsymbol{\delta})$.

(ii) If $(\bar{\mathbf{r}}, \bar{\mathbf{d}})$ and $(\bar{\mathbf{r}}^*, \bar{\mathbf{d}}^*)$ are unknotted and if $\Lambda(r, d; \boldsymbol{\rho}, \boldsymbol{\delta}) = \Lambda(\mathbf{r}^*, \mathbf{d}^*; \boldsymbol{\rho}, \boldsymbol{\delta})$, then $(\mathbf{r}, \mathbf{d})$ and $(\mathbf{r}^*, \mathbf{d}^*)$ lie in the same component of $\mathscr{G}_0(\boldsymbol{\rho}, \boldsymbol{\delta})$ and can accordingly be deformed into each other (in $\mathscr{G}_0(\boldsymbol{\rho}, \boldsymbol{\delta})$).

(iii) Let $(\mathbf{r}, \mathbf{d})$ and $(\mathbf{r}^*, \mathbf{d}^*) \in \mathscr{G}_0(\boldsymbol{\rho}, \boldsymbol{\delta})$. $(\mathbf{r}, \mathbf{d})$ can be deformed into $(\mathbf{r}^*, \mathbf{d}^*)$ by deformations cutting the support if and only if $\Lambda(\mathbf{r}, \mathbf{d}; \boldsymbol{\rho}, \boldsymbol{\delta}) = \Lambda(\mathbf{r}^*, \mathbf{d}^*; \boldsymbol{\rho}, \boldsymbol{\delta}) \bmod 2$.

Let us observe that $(\boldsymbol{\rho}, \boldsymbol{\delta})$ corresponds to the fictitious continuation of the strip in Figs. 1c and 1d. Statements (i) and (ii) explain the deformation leading from Fig. 1b to Fig. 1c. Each configuration has the same value of Λ. The deformation leading from Fig. 1d to Fig. 1a is explained by Statement (iii). By using the invariance of the linking number under deformations and the characterization of the linking number as the algebraic number of times $\bar{\bar{\mathbf{r}}} + x\bar{\bar{\mathbf{d}}}$ pierces a surface spanning $\bar{\bar{\mathbf{r}}}$, we can readily see that the specification of the twist functional of Sec. 3 is equivalent to the specification of $\Lambda(\mathbf{r}, \mathbf{d}; \boldsymbol{\rho}, \boldsymbol{\delta})$ for a large number of $(\boldsymbol{\rho}, \boldsymbol{\delta})$. We finally note that the dependence of $\Lambda(\mathbf{r}, \mathbf{d}; \boldsymbol{\rho}, \boldsymbol{\delta})$ on $(\boldsymbol{\rho}, \boldsymbol{\delta})$, which at first sight seems to be a defect of our formulation, is both unavoidable and geometrically natural: the pair $(\boldsymbol{\rho}, \boldsymbol{\delta})$ actually describes the way the body is supported. Each different system of supports necessarily allows a different class of deformations. The dependence upon the system of supports can be observed in Figs. 1c and 1d by replacing the fictitious continuation shown there with others going in different directions.

Theorem 4.3 and the following remarks show that Λ satisfies Criteria A, C, D. Since Λ is a deformation invariant it automatically satisfies Criterion B as well. (See the discussion following the statement of Criterion D.) To show that Λ satisfies Criterion E on $W_p^1(0, 1) \cap \mathscr{G}_0(\boldsymbol{\rho}, \boldsymbol{\delta})$ we may apply the methods of analysis described in the paragraph containing (2.9) to the functional defined by (4.1).

5. Comments. If the directions of $\mathbf{d}_1(1)$ and $\mathbf{d}_2(1)$ are fixed, the end $s = 1$ may be regarded as being welded to the plane spanned by $\mathbf{d}_1$ and $\mathbf{d}_2$. In this case the end has no rotational degrees of freedom. If the direction of $\mathbf{d}_1(1)$ is fixed but that of $\mathbf{d}_2(1)$ is free, then the end $s = 1$ may be regarded as being hinged about $\mathbf{d}_1(1)$. In this case the end has one rotational degree of freedom. We have shown how to prescribe twist for these cases in Sec. 3 and 4. If $\mathbf{d}_1(1)$ is confined to a fixed plane with the direction of $\mathbf{d}_2(1)$ free, then the end may be regarded as being supported in a Cardan joint with outer gimbal fixed. (See [2] for a specific statement of boundary conditions.) In this case the end has two rotational degrees of freedom. The twist introduced in Sec. 3 is perfectly adapted to describe the twist of the fixed plane about $\mathbf{D}_3(1)$, provided the deformation is only moderately large in the sense that $\mathbf{d}_1$ nowhere rotates from the plane of $\mathbf{E}_1$ and $\mathbf{E}_2$ by an angle $\geq \pi/2$. We have constructed a twist functional based on the linking number that allows this $\pi/2$ to be replaced with π. (The demonstration that this twist functional has the requisite properties requires the use of some technical machinery from algebraic topology at a couple of stages.) We cannot expect such a twist functional to be valid for deformations of arbitrary size because a complete unwinding of a severely twisted state can be achieved when $\mathbf{d}_1$ is allowed to undergo large rotations in its plane. If the directions of $\mathbf{d}_1(1)$ and $\mathbf{d}_2(1)$ are each free, then the end may be regarded as supported by a ball-and-socket joint (in which case it has three rotational degrees of freedom) and no kind of twist can be prescribed.

The motivation for Criterion E of Sec. 2 is that the weak continuity of the twist functional would play a fundamental role in a variational characterization of given problem of elasticity. Such variational characterizations are important in the analysis of stability. It is not evident from our development that our twist functionals of Sec. 3 and 4 make sense when the position field $\mathbf{p}$ of Sec. 2 belongs to a Sobolev space $W_q^1(B)$ of the sort used in the study of variational problems of elasticity (cf. [3]). Indeed, there are serious technical difficulties in the definition of the Jacobian (2.3) when $\mathbf{p}$ is in $W_q^1(B)$ (cf. [3]). We can finesse this important issue by making the following observations (cf. [1]). The twist functionals are perfectly well-defined for each rod theory of an infinite hierarchy of increasing complexity. The properties of the twist functionals support a proof of the existence of minimizers for a large class of variational problems of nonlinearly elastic rods with the twist functional fixed. These minimizers are classical solutions of the equilibrium equations. The solutions of the rod problems have a subsequence generating approximations to the three-dimensional theory that converges weakly to the minimizer for the three-dimensional theory (known to exist by the work of [3]). Thus we can indirectly show that our twist functionals are genuinely useful for three-dimensional problems.

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