THE DIFFRACTION OF A PLANE WAVE BY AN INFINITE SLIT, I*

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Abstract. The diffraction of a normally incident plane wave by an infinite slit of finite width in a perfectly conducting screen is solved using a Lebedev integral transform theorem and the Wiener-Hopf technique. This technique leads to an infinite system of equations to which the method of successive approximation is applied. In particular, an expression for the transmission coefficient (defined as the ratio of the transmitted to the incident power per unit length) is obtained for $a \ll \lambda$ where $2a$ is the slit width and λ is the wave length of the incident wave. There is exact agreement with the known result [3] except for a constant, which has been difficult to determine because of the slow convergence of the series involved; however, a new term has been obtained.

Introduction. The problem of the diffraction of a normally incident plane wave by an infinite slit in a perfectly conducting screen is discussed. The discussion will involve only the electric case, the magnetic case appearing at a later time. An expression for the transmission coefficient is obtained for $a \ll \lambda$ where $2a$ is the slit width and λ is the frequency of the incident wave. There is general agreement with the result of Sommerfeld [1].

The problem was first solved using elliptic cylindrical coordinates by Morse and Rubinstein in 1938 [2], the solution involving an infinite series of Mathieu functions. Here the circular cylindrical coordinate system is chosen to represent the electromagnetic field. In this coordinate system, the boundary value problem is of the two-part variety, and the boundary conditions lead to a dual set of homogeneous integral equations that are solved by the Wiener-Hopf technique.

Statement of the problem. Let the slit of width $2a$ and the screen lie in the xz plane with the z axis coinciding with the axis of the slit; then the slit edges can be specified as $y = 0$, $x = \pm a$. A plane wave of unit intensity with its electric vector parallel to the edge of the slit is normally incident on the slit from the positive y-direction. The problem is independent of z and therefore two-dimensional. Let the z-component of the electric field be denoted by u; then (ϕ measured clockwise from $+y$) choosing circular cylindrical coordinates (ρ, ϕ, z) to represent the field, solutions u_I and u_{II} are defined as follows:

$$u_I = u_0 + u_+ ; \qquad \rho > 0; \qquad |\phi| < \pi/2 \tag{1a}$$

$$u_{II} = u_- ; \qquad \rho > 0; \qquad -\pi < \phi < -\pi/2; \qquad \pi/2 < \phi < \pi \tag{1b}$$

where $u_0 = \exp[+ik(\rho \cos \phi)] - \exp[-ik(\rho \cos \phi)]$, $k = 2\pi/\lambda$, and u_+ and u_- are the scat-

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tered fields due to the presence of the slit. In addition to satisfying $\nabla^2 u + k^2 u = 0$, the known boundary conditions on the scattered fields are

$$u_- = u_+ ; \qquad \phi = \pi/2 \quad (\rho < a);$$

$$u_- = u_+ = 0; \qquad \phi = \pi/2 \quad (\rho > a),$$

$$\left.\frac{\partial u_-}{\partial \phi}\right|_{\phi \to \pi^+/2} - \left.\frac{\partial u_+}{\partial \phi}\right|_{\phi \to \pi^-/2} = -2ik\rho \qquad (\rho < a)$$

together with the radiation condition $\rho^{1/2}(\partial u/\partial \rho) + iku) \to 0$ as $\rho \to \infty$.

Solution. The solution to the problem is represented by an integral of the form $\int_L \mu \Lambda(\mu) H_\mu(k\rho)\, d\mu$, where $H_\mu(k\rho)$ is a Hankel function and L is a contour in the μ complex plane. However, in earlier work [4], it was found convenient to discuss solutions of problems of this type for pure negative imaginary k, i.e., $k = -i\gamma$, $\gamma > 0$, because this puts milder restrictions on the choice of a contour and allows the use of a Lebedev transform theorem [5]. After obtaining the solution, the transformation is made back to real positive k. The scattered fields above and below the slit are

$$u_+ = \int_{-i\infty}^{i\infty} \mu \Lambda(\mu) \cos \mu\phi K_\mu(\gamma\rho)\, d\mu, \qquad -\frac{\pi}{2} < \phi < \frac{\pi}{2}, \tag{2a}$$

$$u_- = \int_{-i\infty}^{i\infty} \mu \Lambda(\mu) \cos \mu(\pi - |\phi|) K_\mu(\gamma\rho)\, d\mu, \qquad \frac{\pi}{2} < \phi < \pi, \qquad -\pi < \phi < -\frac{\pi}{2}, \tag{2b}$$

where $K_\mu(\gamma\rho)$ is the MacDonald function.

If the value of the scattered field in the slit (aperture) is denoted by u_{ap}, then application of the boundary conditions to (2a) and (2b) gives

$$\begin{aligned} \int_{-i\infty}^{i\infty} \mu \Lambda(\mu) \cos \frac{\mu\pi}{2} K_\mu(\gamma\rho)\, d\mu &= u_{ap}, \qquad \rho < a, \\ &= 0, \qquad \rho > a, \end{aligned} \tag{3a}$$

$$\begin{aligned} \int_{-i\infty}^{i\infty} \mu^2 \Lambda(\mu) \sin \frac{\mu\pi}{2} K_\mu(\gamma\rho)\, d\mu &= -\gamma\rho \qquad \rho < a, \\ &= +\frac{1}{2}\left[\frac{\partial u^*}{\partial \phi}\right] \qquad \rho > a, \end{aligned} \tag{3b}$$

where

$$\left[\frac{\partial u^*}{\partial \phi}\right] = \left.\frac{\partial u_-}{\partial \phi}\right|_{\Phi \to \pi^+/2} - \left.\frac{\partial u_+}{\partial \phi}\right|_{\phi \to \pi^-/2}.$$

To study the properties of $\Lambda(\mu)$, use is made of a theorem of Kantorovich and Lebedev [5] which states that if

$$g(\gamma\rho) = \int_L \mu \lambda(\mu) K_\mu(\gamma\rho)\, d\mu \tag{4a}$$

then

$$\frac{\lambda(\mu)}{\sin \pi\mu} = \frac{i}{\pi^2} \int_0^\infty g(\gamma\rho) K_\mu(\gamma\rho) \frac{d\rho}{\rho} \tag{4b}$$

provided both integrals converge, $g(0) = 0$, and $\lambda(\mu)/\sin \pi\mu$ is an even function of μ analytic in a strip of finite width containing the imaginary axis. Application of the Lebedev theorem to (3a) yields:

$$\Lambda(\mu) = \frac{u_{ap}(0)}{\mu\pi i} + \frac{2i \sin(\mu\pi/2)}{\pi^2} \int_0^\infty V(\gamma\rho) K_\mu(\gamma\rho) \frac{d\rho}{\rho} \tag{5a}$$

where $u_{ap}(0)$ is the value of u in the aperture at $\rho = 0$ and

$$\begin{aligned} V(\gamma\rho) &= u_{ap}(\gamma\rho) - u_{ap}(0), & \rho < a, \\ &= -u_{ap}(0), & \rho > a. \end{aligned}$$

Application of the Lebedev theorem to (3b) yields:

$$\Lambda(\mu) = \frac{-2\gamma a \cos \frac{\mu\pi}{2}}{\pi^2 \mu} \left[K_\mu(x) \frac{d s_{0,\mu}(ix)}{dx} - s_{0,\mu}(ix) \frac{dK_\mu(x)}{dx} \right]_{x=\gamma a}$$

$$+ \frac{i \cos \frac{\mu\pi}{2}}{\pi^2 \mu} \int_a^\infty \left[\frac{\partial u^*}{\partial \phi} \right] K_\mu(\gamma\rho) \frac{d\rho}{\rho} \tag{5b}$$

where $s_{0,\mu}(ix)$ is a Lommel function.

By considering (5a, b), it can be shown that the properties of $\Lambda(\mu)$ are:

(1) odd function of μ;
(2) simple pole at $\mu = 0$ with residue $u_{a\rho}(0)/\pi i$;
(3) $|\text{Re } \mu| \to \infty \, |\Lambda(\mu)| \sim |K_\mu(\gamma a)/\mu^2|$;
(4) $|\text{Im } \mu| \to \infty \, |\Lambda(\mu)| \sim |\tau|^{-5/2}; \tau = \text{Im } \mu$.

Consider the homogeneous integral equations

$$\int_{-i\infty}^{i\infty} \mu\Lambda(\mu) \cos \frac{\mu\pi}{2} K_\mu(\gamma\rho) \, d\mu = 0, \qquad \rho > a, \tag{6a}$$

$$\int_{-i\infty}^{i\infty} \mu[\mu\Lambda(\mu) \sin \frac{\mu\pi}{2} - \sigma(\mu)] K_\mu(\gamma\rho) \, d\mu = 0; \qquad \rho < a, \tag{6b}$$

where $\sigma(u)$ is the Lebedev transform of the function

$$\left[\begin{array}{ll} -\gamma\rho; & \rho < a, \\ 0; & \rho > a. \end{array} \right.$$

The integrand in (6b) is an entire function of μ. In order to modify it so that an infinite semicircle may be added to the contour without changing the value of the integral, the substitution

$$K_\mu(\gamma\rho) = \frac{I_{-\mu}(\gamma\rho) - I_\mu(\gamma\rho)}{(2/\pi)\sin \pi\mu}$$

is made in (6b), leading to

$$\int_{-i\infty}^{i\infty} \frac{\mu}{\sin \pi\mu}\left[\mu\Lambda(\mu)\sin\frac{\mu\pi}{2} - \sigma(\mu)\right] I_\mu(\gamma\rho)\, d\mu = 0; \qquad \rho < a. \tag{7}$$

Now the integrand in (7) is still entire, and its decay on a right half plane is such as to allow the addition of an infinite semicircle without changing the value of the integral. Consider (6a) and modify the integrand so as to be able to add an infinite semicircle on a left half plane without changing the value of the integral. In the present form this cannot be done. Therefore, let

$$\mu\Lambda(\mu)\cos\frac{\mu\pi}{2} = \theta(\mu) + \theta(-\mu) \tag{8}$$

where $\theta(\mu)$, obtained from (5a), is

$$\begin{aligned}\theta(\mu) &= \frac{i\mu}{2\pi}\int_0^a [u_{ap}(\gamma\rho) - u_{ap}(0)] I_{-\mu}(\gamma\rho)\frac{d\rho}{\rho} \\ &+ \frac{\mu i u_{ap}(0)\left(\frac{\gamma a}{2}\right)}{\pi}\left[I_\mu(\gamma a)\frac{d\mathcal{J}_{-1,\mu}(-i\gamma a)}{d\gamma\alpha} - \mathcal{J}_{-1,\mu}(-i\gamma a)\frac{d}{d\gamma a}I_{-\mu}(\gamma a)\right]\end{aligned} \tag{9}$$

Substituting (8) into (6a) gives

$$\int_{-i\infty}^{i\infty} \theta(\mu) K_\mu(\gamma\rho)\, d\mu = 0; \qquad \rho > a. \tag{10}$$

Now the integrand in (10) is analytic to the left of Re $\mu = +2$, and its decay on a left half plane is such as to allow the addition of an infinite semicircle without changing the value of the integral.

By applying the Wiener-Hopf technique beginning with (7) and writing the integrand in terms of a plus function, i.e., a function that is analytic on a right half plane with algebraic decay in all directions on that half plane, the following equation results:

$$\frac{(\gamma a/2)^\mu}{\mu\Gamma(\mu)}\left[\frac{\mu\sin(\mu\pi/2)\Lambda(\mu) - \sigma(\mu)}{\sin\mu\pi}\right] = g^+(\mu). \tag{11}$$

By writing $\theta(\mu)$ and $\sigma(\mu)$ in terms of "plus" and "minus" functions and substituting into (11), one obtains

$$g^+(\mu) + \frac{(\gamma a/2)^{2\mu} h^-(-\mu)}{2\Gamma(\mu)\Gamma(1+\mu)\cos^2(\mu\pi/2)} + \frac{(\gamma a/2)^{2\mu} q_2^+(\mu)}{\Gamma^2(\mu+1)\sin\mu\pi} = \frac{\tan(\mu\pi/2)}{\pi} h^-(\mu) - \frac{q_1^-(\mu)}{\pi\mu}, \tag{12}$$

where

$$h^-(\mu) = \frac{(\gamma a/2)^\mu \Gamma(1-\mu)\theta(\mu)}{\mu}, \tag{13}$$

$$q_1^-(\mu) = -\Gamma(1-\mu)\left(\frac{\gamma a}{2}\right)^{\mu+1}\left[I_{-\mu}(\gamma a)\frac{d}{d\gamma a}\mathcal{J}_{0,\mu}(i\gamma a) - \mathcal{J}_{0,\mu}(i\gamma a)\frac{d}{d\gamma a}I_{-\mu}(\gamma a)\right], \tag{14}$$

and $q_2^+(\mu) = -q_1^-(-\mu)$. In (12), the left side is to be a plus function and the right side a minus function with a common strip of overlap. To accomplish this the appropriate partial fraction series is subtracted off from each term. It turns out that the left side can be made analytic to the right of Re $\mu = -1$ and the right side analytic to the left of Re $\mu = 0$. Now, since each side is analytic on its respective half plane with algebraic decay in all directions on the half plane and the half planes have a common strip of overlap, then by Liouville's theorem, each side is equal to a constant, namely zero.

Since $h(-2m-1)$, $m = 0, 1, 2, \ldots$ will be needed in the expressions for the solution the transmission coefficient, the minus side of (12) can be written (dropping the superscript on h and q)

$$-\frac{2}{\pi^2}h'(-2m-1)+\frac{1}{\pi^2}\sum_{\substack{n=0\\ n\neq m}}^{\infty}\frac{h(-2n-1)}{n-m}-\frac{1}{2\pi^2}\sum_{n=0}^{\infty}\frac{(\gamma a/2)^{4n+2}h(-2n-1)}{(2n)!(2n+1)!(m+n+1)^2}$$
$$+\frac{2}{\pi^2}\sum_{n=0}^{\infty}\frac{(\gamma a/2)^{4n+2}h(-2n-1)}{(2n)!(2n+1)!(m+n+1)}\left[\log\frac{\beta a}{a}-\psi(2n+1)-\frac{1}{4n+2}\right]$$
$$-\frac{1}{\pi^2}\sum_{n=0}^{\infty}\frac{(\gamma a/2)^{4n+2}h(-2n-1)}{(2n)!(2n+1)!(m+n+1)}=\frac{q(2m+1)}{\pi(2m+1)}-\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n q(n)(\gamma a/2)^{2n}}{(n!)^2(2m+n+1)} \tag{15}$$

where $\log\beta = -\psi(1) =$ Euler's constant. The solution u_- is expressed in terms of the $h(-2m-1)$ by considering (2b) and writing the integral representation as a residue series. To accomplish this, it's necessary to use (8). Inserting (8) into (2b) gives

$$u_- = 2\int_{-i\infty}^{i\infty}\theta(\mu)\cos[\mu(\pi-|\phi|)]K_\mu(\gamma\rho)\Big/\cos\frac{\mu\pi}{2}\,d\mu.$$

Since $\theta(\mu)$ is analytic to the left of Re $\mu = +2$ and behaves as $|I_{-\mu}(\gamma a)|$, an infinite semicircle can be added on a left half plane. The poles of the integrand are the simple poles of $\cos \mu\pi/2$ which occur at $\mu = \pm(2p+1)$, $p = 0, 1, 2, \ldots$. Therefore, applying the Cauchy residue theorem gives

$$u_- = -8i\sum_{n=0}^{\infty}\theta(-2n-1)K_{2n+1}(\gamma\rho)(-1)^n\cos(2n+1)|\phi|.$$

This can be rewritten in terms of $h(-2n-1)$ and $H^{(2)}_{2n+1}(k\rho)$ as

$$u_- = 4\pi\sum_{n=0}^{\infty}(-1)^n(ka/2)^{2n+1}h(-2n-1)$$
$$\cdot\cos[(2n+1)|\phi|]H^{(2)}_{2n+1}(k\rho)/\Gamma(2n+1). \tag{16}$$

Transmission coefficient. In terms of the complex Poynting vector, the transmission coefficient τ can be written

$$\tau = -\frac{1}{ak}\int_{\pi/2}^{\pi}\mathrm{Re}\left[i\tilde{u}_-\frac{\partial u_-}{\partial\rho}\right]\rho\,d\phi. \tag{17}$$

When (16) is substituted into (17) and the integral is evaluated for large ρ, the following expression is obtained:

$$\tau = 2\pi^2 ka\left[|h(-1)|^2+\frac{k^4a^4}{64}|h(-3)|^2+\cdots\right]. \tag{18}$$

By using (9) and (13), the infinite system (15) is solved by the method of successive approximations. When the expressions for $h(-2j-1)$ are substituted into (18), the transmission coefficient becomes

$$\tau = \frac{\pi^2 k^3 a^3}{32} - \frac{\pi^2 k^5 a^5}{64}\left(\log\frac{\beta ka}{2} - \frac{1}{2}\right) + \frac{3\pi^2 k^7 a^7}{512}\left(\log\frac{\beta ka}{2} - \frac{1}{2}\right)^2 + \cdots.$$

Summary. A Lebedev transform, when applied to the problem of normal incidence of a plane wave on an infinite slit, yields two homogeneous integral equations which are solved by the Wiener-Hopf technique, resulting in an infinite system of linear algebraic equations. The method of successive approximation was applied to these equations. It was possible to obtain the first few terms in the expression for the transmission coefficient. There is exact agreement between this result and the known result [3] except for a constant; however, a new term has been obtained.

References

[1] A. Sommerfeld, *Vorlesungen über theoretische Physik*, vol. 4, Wiesbaden, Dieterich, 1950

[2] P. M. Morse and P. J. Rubenstein, *The diffraction of waves by ribbons and by slits*, Phys. Rev. **54**, 895–898 (1938)

[3] C. J. Bouwkamp, *Diffraction theory*, Reports on Progress in Physics **17**, 35–100 (1954)

[4] F. Oberhettinger, *Diffraction of waves by a wedge*, Comm. Pure and Appl. Math **7**, 551–563 (1954)

[5] N. N. Lebedev, *Sur une formule d'inversion*, Comptes Rendus (Doklady) de l'Academie des Sciences de l'URSS **(M8)52**, 655–658 (1946)

[6] P. B. Bailey and G. E. Barr, *Diffraction by a slit or strip*, J. Math. Physics **10**, 1906–1913 (1969)