QUARTERLY OF APPLIED MATHEMATICS

VOLUME XL JULY · 1982 NUMBER 2

# BREAKDOWN OF SMOOTH SOLUTIONS IN DISSIPATIVE NONLINEAR HYPERBOLIC EQUATIONS*

BY

HARUMI HATTORI

*Rensselaer Polytechnic Institute*

**0. Introduction.** In this paper we study the nonexistence of global smooth solutions of one-dimensional motions for nonlinear viscoelastic fluids and solids by the method of Rozhdestvenskii [1]. This method has been applied to prove the nonexistence of global smooth solutions for the shearing motions in an elastic circular tube in [2].

It is well known that the quasilinear hyperbolic equation

$$v_{tt} = \sigma(v_x)_x \tag{0.1}$$

exhibits the breakdown of smooth solutions in finite time for a certain class of initial data of arbitrary smoothness, no matter how small. This breakdown of smooth solutions is usually associated with the formation of a propagating singular surface often called a shockwave. The absence of some dissipative or damping mechanism in the above equation causes this rather unrealistic result.

Nishida [3] and Slemrod [4] have studied the equation

$$v_{tt} = \sigma(v_x)_x - \alpha v_t \tag{0.2}$$

which includes the effect of first-order linear damping which is not present in (0.1). For (0.2) Nishida showed the existence of a global smooth solution for the small initial data. Slemrod showed the breakdown of smooth solutions for large initial data. His motivation for studying (0.2) was based on his model equation for shearing perturbations of steady shearing flows in a nonlinear, isotropic, incompressible, viscoelastic fluid, in the absence of an applied driving force. In experiments the analysis of the plane Poiseuille flow is more common. In Sec. 1 I shall discuss the plane Poiseuille flow of the above fluid.

MacCamy [6] considered the equation

$$v_{tt} = a(0)\sigma(v_x)_x + \int_0^t \dot{a}(t-z)\sigma(v_x)_x \, dz + f, \tag{0.3}$$

showed the existence of a global smooth solution for small initial data, and conjectured the breakdown of smooth solutions for large initial data. The effect of fading memory for elastic materials causing a dissipative mechanism is included in this model as the stress functional in the stress-strain relation. I shall show the breakdown of smooth solutions in this problem in Sec. 2.

* Received January 21, 1981. This work was done in partial fulfillment of the requirements for a Ph.D. at Rensselaer Polytechnic Institute. The author is grateful to Prof. Marshall Slemrod for his helpful suggestions, discussion and encouragement.

**1. One-dimensional viscoelastic fluid.** We consider the plane Poiseuille flow of nonlinear, isotropic, incompressible, viscoelastic fluid. Two parallel plates are located at $x = 0$ and $x = 1$. We assume the flow is a rectilinear shearing flow, namely

$$v^x = 0, \qquad v^y = v(x, t), \qquad v^z = 0. \tag{1.1}$$

Here $\mathbf{v} = (v^x, v^y, v^z)$ is the velocity of the fluid. Following the arguments of Slemrod [4], we obtain the equation of conservation of linear momentum in the $y$-direction:

$$\rho v_t(x, t) = \sigma\left(\int_0^\infty e^{-\alpha s} v_x(x, t - \tau)\, ds\right)_x - \beta, \tag{1.2}$$

where $\beta$ ($>0$) is the applied driving force which we assume constant. In the following arguments we assume $\rho = 1$. The incompressibility condition div $\mathbf{v} = 0$ is automatically satisfied. On the boundary we require no-slip conditions

$$v(0, t) = v(1, t) = 0. \tag{1.3}$$

Defining $w(x, t)$ and $u(x, t)$ by

$$w(x, t) = \int_0^\infty e^{-\alpha s} v_x(x, t - s)\, ds, \qquad u(x, t) = \int_0^\infty e^{-\alpha s} v_t(x, t - s)\, ds, \tag{1.4}$$

we obtain the following first-order system:

$$w_t = u_x, \qquad u_t = \sigma(w)_x - \alpha u - \beta, \tag{1.5}$$

with boundary conditions

$$u(0, t) = u(1, t) = 0, \tag{1.6}$$

and initial conditions

$$u(x, 0) = u_0(x), \qquad 0 \le x \le 1,$$
$$w(x, 0) = w_0(x). \tag{1.7}$$

The values of $u_0(x)$ and $w_0(x)$ are obtained from their respective definitions by inserting the given velocity history $v_0(x, \tau)$, $-\infty < \tau \le 0$.

We require some conditions on $\sigma$ in order that the system (1.5) be hyperbolic and nonlinear. For the hyperbolicity we require that the matrix

$$\begin{bmatrix} 0 & 1 \\ \sigma' & 0 \end{bmatrix} \tag{1.8}$$

possesses real distinct eigenvalues. This strict hyperbolicity is equivalent to the condition

$$\sigma' > 0. \tag{1.9}$$

For the nonlinearity of the constitutive relation we impose

$$\sigma''(\xi_0) \neq 0 \tag{1.10}$$

for some real number $\xi_0$.

Now we write the system (1.5)–(1.7) in terms of the Riemann invariants. The characteristic curves of the system (1.5) are

$$dx/dt = \pm\sqrt{\sigma'(w)}, \tag{1.11}$$

and the Riemann invariants of the system can be taken as

$$\left.\begin{matrix} r \\ s \end{matrix}\right\} = u \pm \phi(w), \tag{1.12}$$

where

$$\phi(w) = \int_0^w \sqrt{\sigma'(w)}\, dw. \tag{1.13}$$

The transformation given by (1.12) from $(u, w) \in \mathbb{R}^2$ to $(r, s) \in \mathbb{R}^2$ is one-to-one. The Riemann invariants satisfy the diagonal system

$$r_t - \lambda r_x = -\frac{\alpha}{2}(r + s) - \beta, \qquad s_t + \lambda s_x = -\frac{\alpha}{2}(r + s) - \beta, \tag{1.14}$$

where

$$\lambda = \sqrt{\sigma'(r - s)}. \tag{1.15}$$

The initial data $r(x, 0) = r_0(x)$ and $s(x, 0) = s_0(x)$ are taken to be smooth functions. In the following argument from (1.10) we assume

$$\sigma''(\xi) \geq \varepsilon > 0 \quad \text{for} \quad |\xi - \xi_0| \leq \delta. \tag{1.16}$$

(The case where $\sigma''(\xi) \leq -\varepsilon < 0$ is similar.) Then (1.16) is equivalent to the following condition

$$\lambda'(r - s) \geq \varepsilon > 0 \quad \text{for} \quad |r - s - \xi_0| \leq \delta. \tag{1.17}$$

The analysis of the system (1.14) in the absence of applied driving force ($\beta = 0$) has been done by Nishida [3] and Slemrod [4]. Slemrod showed the breakdown of smooth solutions for $t > 0$ in the case where $u_{0x}$ and $w_{0x}$ are sufficiently large. He formulated the initial boundary problem as an initial-value problem on $\mathbb{R}$, and extended Lax's argument [5] to show the breakdown of smooth solutions. The presence of a constant driving force term, however, prevents us from using his argument. Instead we employ the method of Rozhdestvenskii to show the breakdown of smooth solutions in the domain of influence of initial data. The proof is based on studying the two adjacent characteristic curves of the same family. First we assume the existence of global smooth solutions for arbitrary initial data. Then we choose appropriate initial data so that those two characteristic curves intersect (or impinge) and $r(x(t), t)$ has different values on each characteristic curve, hence contradicting the assumption.

Before proving Theorem 1.1 we need an a priori estimate.

LEMMA 1.1. Let $|r_0| = \max_{0 \leqslant x \leqslant 1} |r_0(x)|$, $|s_0| = \max_{0 \leqslant x \leqslant 1} |s_0(x)|$. Then in the domain of influence of initial data as long as smooth solutions exist we have

$$|r(x, t)| + |s(x, t)| r_0| + |s_0| + \frac{4\beta}{\alpha}.$$

*Proof.* This proof proceeds in almost the same way as in Slemrod [4]. We introduce the characteristic curves

$$x_1 = x_1(t, \delta) = \delta - \int_0^t \lambda \, d\tau, \qquad \delta \in [0, 1],$$

$$x_2 = x_2(t, \mu) = \mu + \int_0^t \lambda \, d\tau, \qquad \mu \in [0, 1].$$

Then (1.14) becomes

$$\frac{d}{dt}\left\{r(x_1(t, \delta), t) + \frac{\beta}{\alpha}\right\} = -\frac{\alpha}{2}\left\{\left(r(x_1(t, \delta), t) + \frac{\beta}{\alpha}\right) + \left(s(x_1(t, \delta), t) + \frac{\beta}{\alpha}\right)\right\},$$

$$\frac{d}{dt}\left\{s(x_2(t, \mu), t) + \frac{\beta}{\alpha}\right\} = -\frac{\alpha}{2}\left\{\left(r(x_2(t, \mu), t) + \frac{\beta}{\alpha}\right) + \left(s(x_2(t, \mu), t) + \frac{\beta}{\alpha}\right)\right\}.$$

Integrating the above equations along each characteristic curve, we have

$$\begin{aligned} e^{(\alpha/2)t}\left(r(x_1(t, \delta), t) + \frac{\beta}{\alpha}\right) &= r_0(\delta) + \frac{\beta}{\alpha} - \frac{\alpha}{2}\int_0^t e^{(\alpha/2)\tau}\left(s(x_1(\tau, \delta), \tau) + \frac{\beta}{\alpha}\right) d\tau, \\ e^{(\alpha/2)t}\left(s(x_2(t, \mu), t) + \frac{\beta}{\alpha}\right) &= s_0(\mu) + \frac{\beta}{\alpha} - \frac{\alpha}{2}\int_0^t e^{(\alpha/2)\tau}\left(r(x_2(\tau, \mu), \tau) + \frac{\beta}{\alpha}\right) d\tau. \end{aligned} \tag{1.18}$$

Define

$$\max_{x \in G_1(t)} e^{(\alpha/2)t}\left|s(x, t) + \frac{\beta}{\alpha}\right| = S(t),$$

$$\max_{x \in G_1(t)} e^{(\alpha/2)t}\left|r(x, t) + \frac{\beta}{\alpha}\right| = R(t),$$

where $G_1(t)$ is the $x$ line segment which belongs to the domain of influence of initial data at fixed $t$. Then (1.18) implies

$$e^{(\alpha/2)t}\left|r(x_1(t, \delta), t) + \frac{\beta}{\alpha}\right| \le |r_0| + \frac{\beta}{\alpha} + \frac{\alpha}{2}\int_0^t S(\tau) \, d\tau,$$

$$e^{(\alpha/2)t}\left|s(x_2(t, \mu), t) + \frac{\beta}{\alpha}\right| \le |s_0| + \frac{\beta}{\alpha} + \frac{\alpha}{2}\int_0^t R(\tau) \, d\tau.$$

Since for each $t$ we can find $\hat{x}_1, \hat{x}_2$ such that

$$e^{(\alpha/2)t}\left|r(\hat{x}_1, t) + \frac{\beta}{\alpha}\right| = R(t), \qquad e^{(\alpha/2)t}\left|s(\hat{x}_2, t) + \frac{\beta}{\alpha}\right| = S(t),$$

and we can always trace backwards along characteristic curves to find $\delta$, $\mu$ so that $\hat{x}_1 = x_1(t, \delta)$, $\hat{x}_2 = x_2(t, \mu)$ (because $G_1(t)$ for each fixed $t$ belongs to the domain of influence of initial data), we obtain

$$R(t) \le |r_0| + \frac{\beta}{\alpha} + \frac{\alpha}{2}\int_0^t S(\tau) \, d\tau, \qquad S(t) \le |s_0| + \frac{\beta}{\alpha} + \frac{\alpha}{2}\int_0^t R(\tau) \, d\tau.$$

Adding these inequalities we have

$$W(t) \le |r_0| + |s_0| + \frac{2\beta}{\alpha} + \frac{\alpha}{2}\int_0^t W(\tau) \, d\tau,$$

where $W(t) = R(t) + S(t)$. The Gronwall inequality implies

$$W(t) \le \left( |r_0| + |s_0| + \frac{2\beta}{\alpha} \right) e^{(\alpha/2)t},$$

so that

$$\left| r(x, t) + \frac{\beta}{\alpha} \right| + \left| s(x, t) + \frac{\beta}{\alpha} \right| \le |r_0| + |s_0| + \frac{2\beta}{\alpha}$$

Hence, $|r(x, t)| + |s(x, t)| \le |r_0| + |s_0| + (4\beta/\alpha)$.

Now we shall give the following nonexistence proof of smooth solutions.

THEOREM 1.1. Suppose the condition (1.17) is satisfied. Then for appropriate smooth initial data the breakdown of smooth solution in the system (1.14) and (1.15) will occur in finite time.

*Proof.* The proof is done by contradiction. We assume there exist global smooth solutions for all choices of initial data. Now, we choose points $(x_1^0, 0)$ and $(x_2^0, 0)$ on the initial line such that $x_1^0 = 0.5$, $x_2^0 = x_1^0 + \gamma$, where $\gamma$ is small positive constant which is determined later in this proof. On the initial line we give the smooth data $s_0(x)$ and $r_0(x)$ which take the values

$$r_0(x_1^0) = \bar{r}_0 - \frac{\Delta r}{2}, \qquad r_0(x_2^0) = \bar{r}_0 + \frac{\Delta r}{2},$$

$$s_0(x) = \bar{s}_0, \qquad x \in [\eta, 1 - \eta], \tag{1.19}$$

where $\bar{r}_0$, $\bar{s}_0$, $\eta\ (>0)$, $\Delta r\ (>0)$ are constants and satisfy

$$|r_0(x) - s_0(x) - \xi_0| \le \frac{\delta}{2}, \qquad x \in [\eta, 1 - \eta],$$

$$r_0(x) - s_0(x) = 0 \quad \text{at} \quad x = 0, 1. \tag{1.20}$$

We define the following values. Set

$$M_0 = |r_0| + |s_0| + \frac{4\beta}{\alpha}, \qquad M_1 = \max_{|r+s| \le M_0} \left| \frac{\alpha}{2}(r + s) + \beta \right|,$$

$$M_2 = \max_{|r+s| \le M_0} |\lambda'(r - s)|, \qquad M_3 = \max_{|r-s| \le M_0} |\lambda(r - s)|. \tag{1.21}$$

We notice that in the domain of influence of initial data we have

$$|r(x, t) - s(x, t)| \le |r(x, t)| + |s(x, t)| \le M_0.$$

We also define the set $G_0$. Set

$$G_0 = \{(r, s): |r - s - \xi_0| \le \delta\},$$

where $\xi_0$ and $\delta$ are defined by (1.17). Since we assume the existence of global smooth solutions for arbitrary initial data, there exists the domain $D = \{(x, t);\ 0 \le x \le 1,\ t \ge 0$ such that $r(x, t)$ and $s(x, t)$ stay in $G_0\}$. We consider the values of $r$ along the $r$-characteristic curves $dx/dt = -\lambda(r - s)$ through $(x_1^0, 0)$ and $(x_2^0, 0)$. We denote the $r$-characteristic curves through $(x_1^0, 0)$ and $(x_2^0, 0)$ by $x_1(t)$ and $x_2(t)$, respectively, and the values of $r$ and $s$ along

$x_1(t)$ and $x_2(t)$ by $r_1(t) = r(x_1(t), t)$, $r_2(t) = r(x_2(t), t)$, $s_1(t) = s(x_1(t), t)$, and $s_2(t) = s(x_2(t), t)$. Since two $r$-characteristic curves $x_1(t)$ and $x_2(t)$ are continuous, if we find small positive $t_1$ such that

$$r_1(t) < r_2(t) \qquad (0 \leq t \leq t_1), \tag{1.22}$$

$$x_2(t_1) \leq x_1(t_1) \tag{1.23}$$

are satisfied in the domain $D$, then, since $x_1^0 < x_2^0$, we can find $t_2 (0 < t_2 \leq t_1)$ such that (1.24) and (1.25) are satisfied:

$$r_1(t_2) < r_2(t_2), \tag{1.24}$$

$$x_2(t_2) = x_1(t_2). \tag{1.25}$$

But this will contradict the fact that we assume the existence of smooth solutions for arbitrary initial data.

For this purpose we first estimate the domain $D$. Consider the $s$-characteristic curve which starts at $x = \eta$ on the initial line and the $r$-characteristic curve which starts at $x = 1 - \eta$ on the initial line. These two characteristic curves stays inside the domain of influence of initial data at least until they cross each other. Since $r$ and $s$ characteristic curves satisfy

$$\frac{dx}{dt} = -\lambda \geq -M_3, \tag{1.26}$$

$$\frac{dx}{dt} = \lambda \leq M_3, \tag{1.27}$$

the domain surrounded by the following two straight lines

$$x - \eta = M_3 t \tag{1.28}$$

$$x - (1 - \eta) = -M_3 t \tag{1.29}$$

and the initial line give an estimate for the domain of influence of the initial data. The above straight lines cross each other at $t = (1/M_3)(\frac{1}{2} - \eta)$, so that $t_0 = (1/2M_3)(\frac{1}{2} - \eta)$ gives a lower bound for the time until which $r$-characteristic curves $x_1(t)$ and $x_2(t)$ stay inside the domain of influence of initial data (see Fig. 1.1). Denote the domain surrounded by (1.28), (1.29), the initial line, and $t = (1/2M_3)(\frac{1}{2} - \eta)$ by $D_0$. Then we see that if $x_1(t)$ and $x_2(t)$ are in $D_0$, $x_1(t)$ and $x_2(t)$ are in the domain of influence of initial data.

From (1.14) we have the following inequality on $s$ along the $s$-characteristic curves $d\tilde{x}/dt = \lambda$:

$$s_0 - M_1 t \leq s(\tilde{x}(t), t) \leq s_0 + M_1 t \tag{1.30}$$

This inequality holds for arbitrary $\tilde{x}(t)$, provided that $\tilde{x}(t)$ is in the domain $D_0$.

Thus, we can replace $\tilde{x}(t)$ by $x_1(t)$ or $x_2(t)$ in (1.30), and estimate the values of $r$ and $s$ along the $r$-characteristic curves $x_1(t)$ and $x_2(t)$. From (1.19) we have the following inequalities for $0 \leq t \leq t_0$. Along $x_1(t)$ we have

$$r_1(t) = \bar{r}_0 - \frac{\Delta r}{2} + \int_0^t \left( -\frac{\alpha}{2}(r + s) - \beta \right) dt, \tag{1.31}$$

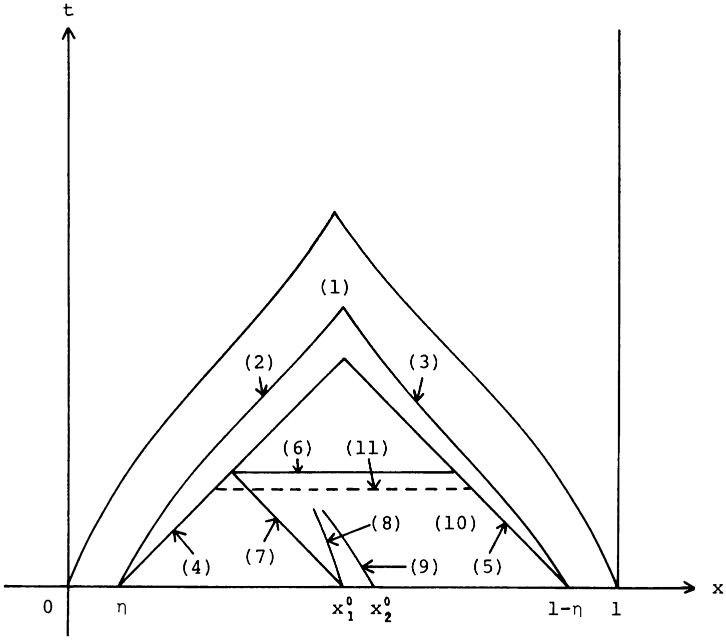


FIG. 1. (1) The domain of influence of initial data; (2) The $s$-characteristic curve which starts at $(\eta, 0)$; (3) The $r$-characteristic curve which starts at $(1-\eta, 0)$; (4) The straight line $x = M_3 t + \eta$; (5) The straight line $x = -M_3 t + (1-\eta)$; (6) The straight line $t = (1/2M_3)(\frac{1}{2} - \eta)$; (7) The straight line $x = M_3 t + 0.5$; (8) The $r$-characteristic curve $x_1(t)$; (9) The $r$-characteristic curve $x_2(t)$; (10) The domain $D_0$; (11) The straight line $t = \min((1/2M_3)(\frac{1}{2} - \eta), (\delta/4M_1))$. If $(1/2M_3)(\frac{1}{2} - \eta) \le (\delta/4M_1)$, $D_0 = D_1$. If $(\delta/4M_1) \le (1/2M_3)(\frac{1}{2} - \eta)$, $D_1 < D_0$.

$$\bar{r}_0 - \frac{\Delta r}{2} - M_1 t \le r_1(t) \le \bar{r}_0 - \frac{\Delta r}{2} + M_1 t, \tag{1.32}$$

$$\bar{s}_0 - M_1 t \le s_1(t) \le \bar{s}_0 + M_1 t, \tag{1.33}$$

$$x_1(t) = x_1^0 - \int_0^t \lambda(r_1 - s_1)\, dt. \tag{1.34}$$

And along $x_2(t)$ we see that

$$\bar{r}_0 + \frac{\Delta r}{2} - M_1 t \le r_2(t) \le \bar{r}_0 + \frac{\Delta r}{2} + M_1 t, \tag{1.35}$$

$$\bar{s}_0 - M_1 t \le s_2(t) \le \bar{s}_0 + M_1 t, \tag{1.36}$$

$$x_2(t) = x_2^0 - \int_0^t \lambda(r_2 - s_2)\, dt. \tag{1.37}$$

From (1.32), (1.33), (1.35), and (1.36) we have the following inequality:

$$\bar{r}_0 - \bar{s}_0 - \frac{\Delta r}{2} - 2M_1 t - \xi_0 \le r_i(t) - s_i(t) - \xi_0$$
$$\le \bar{r}_0 - \bar{s}_0 + \frac{\Delta r}{2} + 2M_1 t - \xi_0 \qquad (i = 1, 2). \tag{1.38}$$

Since $|\bar{r}_0 \pm (\Delta r/2) - \bar{s}_0 - \xi_0| \le (\delta/2)$, if $t \le (\delta/4M_1)$,

$$-\delta \le \bar{r}_0 - \bar{s}_0 - \frac{\Delta r}{2} - 2M_1 t - \xi_0 \le r_i(t) - s_i(t) - \xi_0$$

$$\le \bar{r}_0 - \bar{s}_0 + \frac{\Delta r}{2} + 2M_1 t - \xi_0 \le \delta \qquad (i = 1, 2).$$

So we see that $|r_i(t) - s_i(t) - \xi_0| \le \delta\,(i = 1, 2)$ if $t \le (\delta/4M_1)$. Set

$$t_0 = \min\left(\frac{1}{2M_3}\left(\frac{1}{2} - \eta\right), \frac{\delta}{4M_1}\right).$$

Then if we define $D_1$ to be the domain surrounded by the initial line, (1.28), (1.29), and $t = t_0$, we see that $D_1 < D$.

Now we will pick $t_1$ such that it is less than $t_0$ and satisfies (1.22) and (1.23). From (1.32) and (1.35) the condition (1.22) is satisfied if the following inequality holds:

$$2M_1 t < \Delta r. \tag{1.39}$$

To find a sufficient condition for (1.23) we subtract (1.34) from (1.37), then we have

$$x_2(t) - x_1(t) = x_2^0 - x_1^0 - \int_0^t \{\lambda(r_2 - s_2) - \lambda(r_1 - s_1)\}\, dt$$

$$= x_2^0 - x_1^0 - \int_0^t \lambda'(\xi)(r_2 - r_1 + s_1 - s_2)\, dt,$$

where $\xi$ is between $(r_2 - s_2)$ and $(r_1 - s_1)$, and $0 \le t \le \bar{t}_0$. If $(r_2 - s_2)$ and $(r_1 - s_1)$ satisfy $|r_i - s_i - \xi_0| \le \delta\,(i = 1, 2)$, $\xi$ satisfies $|\xi - \xi_0| \le \delta$. The condition (1.23) is equivalent to

$$\int_0^t \lambda'(\xi)(r_2 - r_1 + s_1 - s_2)\, dt \ge x_2^0 - x_1^0 = \gamma$$

for $t = t_1$. We evaluate the above integral. From (1.17), (1.19), (1.32), and (1.35),

$$\lambda'(\xi)(r_2 - r_1) \ge \varepsilon(\Delta r - 2M_1 t),$$

and from (1.19), (1.33), and (1.36)

$$\lambda'(\xi)(s_1 - s_2) \ge 2M_1 M_2 t;$$

thus,

$$\int_0^t \{\lambda'(\xi)(r_2 - r_1) + \lambda'(\xi)(s_1 - s_2)\}\, dt \ge \int_0^t \{\varepsilon(\Delta r - 2M_1 t) - 2M_1 M_2 t\}\, dt$$

$$= \varepsilon\, \Delta r t - (\varepsilon M_1 + M_1 M_2)t^2$$

for $0 \le t \le \bar{t}_0$. Hence if the inequality

$$\varepsilon\, \Delta r t - M_1(\varepsilon + M_2)t^2 \ge \gamma \tag{1.40}$$

is satisfied at $t = t_1$, it will give a sufficient condition for (1.23).

Set $t_0' = \min(t_0, \Delta r/2M_1)$; then we can find $t_1$ and $\gamma$ which satisfy (1.36). If $t_0' > \varepsilon\, \Delta r/2M_1(\varepsilon + M_2)$, set $t_1 = \varepsilon\, \Delta r/2M_1(\varepsilon + M_2)$ and choose $\gamma$ such that $0 < \gamma \le \varepsilon\, \Delta r/4M_1)\varepsilon + M_2)$. If $t_0' \le \varepsilon\, \Delta r/2M_1(\varepsilon + M_2)$, set $t_1 = t_0'$ and choose $\gamma$ such that $0 < \gamma \le (\varepsilon\, \Delta r t_1$

$-M_1(\varepsilon + M_2)t_1^2)$. Since the a priori estimate of Lemma 1.1 depends only on the maximum absolute values $|r_0|$ and $|s_0|$ of initial data, we can change initial data, namely $\gamma$, without changing $|r_0|$ and $|s_0|$.

**2. One-dimensional nonlinear viscoelasticity.** In this section we consider the one-dimensional motion of an elastic bar. The effect of fading memory appears in the constitutive relation in a different manner from that of the previous section. Suppose $x + v(x, t)$ is the position of a section at time $t$ which is at position $x$ in the unstretched configuration. Then $v$ is the displacement. MacCamy [6] and Dafermos and Nohel [7] considered the constitutive relation

$$T = \sigma(v_x) + \int_0^t a'(t-\tau)\sigma(v_x(x, \tau))\, d\tau, \tag{2.1}$$

where $T$ is the stress, and analysed the problem

$$v_{tt} = \sigma(v_x)_x + \int_0^t a'(t-\tau)\sigma(v_x)_x\, d\tau + g(x, t) \tag{2.2}$$

with the fixed boundary conditions

$$v(0, t) = v(1, t) \equiv 0, \tag{2.3}$$

and with the initial conditions

$$v(x, 0) = v_0(x), \qquad v_t(x, 0) = v_1(x). \tag{2.4}$$

We require the same conditions for initial data and $g(x, t)$ as in MacCamy [6]:

$$v_i(0) = v_i(1) = v_i''(0) = v_i''(1) = 0 \qquad i = 0, 1, \tag{2.5}$$

$$g(0, t) = g(1, t) = g_{xx}(0, t) = g_{xx}(1, t) = 0. \tag{2.6}$$

These conditions allow us to extend $v_0$, $v_1$, and $g$ smoothly to periodic functions on $-\infty < x < \infty$, so that the above initial boundary value problem becomes a pure initial value problem. In both papers the existence of a unique smooth solution for small initial data was shown (although by different methods). Dafermos and Nohel changed (2.2) into a more convenient form, namely,

$$v_{tt}(x, t) + k(0)v_t(x, t) = \sigma'(v_x(x, t))v_{xx}(x, t) - \int_0^t k'(t-\tau)v_t(x, \tau)\, d\tau + \Phi(x, t). \tag{2.7}$$

Here, $\Phi(x, t) = g(x, t) + k(t)v_1(x) + \int_0^t k'(t-\tau)g(x, \tau)\, d\tau$, and $k(t)$ is related to $a(t)$ via the equation

$$k(t) + \int_0^t a'(t-\tau)k(\tau)\, d\tau = -a'(t), \qquad 0 \leq t < \infty.$$

In the following argument for simplicity we assume $\Phi(x, t)$ and $k'(t)$ are bounded, and set

$$\Phi_0 = \sup_{0 \leq t < \infty,\, 0 \leq x \leq 1} |\Phi(x, t)|, \qquad k_0 = \sup_{0 \leq t < \infty} |k'(t)|.$$

By defining $w = v_x$ and $u = v_t$, we write (2.5) as a first-order system

$$w_t = u_x, \qquad u_t = \sigma'(w)w_x + \Phi - k(0)u - \int_0^t k'(t-\tau)u(x, \tau)\, d\tau, \tag{2.8}$$

with initial conditions

$$w(x, 0) = v_{0_x}(x), \qquad u(x, 0) = v_1(x).$$

Here $w(x, 0)$ and $u(x, 0)$ are even and odd functions respectively and satisfy

$$u(x, 0) = u_{xx}(x, 0) = w_x(x, 0) = 0 \quad \text{at} \quad x = 0, \pm 1, \pm 2, \ldots .$$

As in the previous section we require $\sigma' > 0$ and

$$\sigma''(\xi_0) \neq 0 \tag{2.9}$$

for some real number $\xi_0$. We write (2.6) in the Riemann-invariant form. Define

$$\left.\begin{matrix} r \\ s \end{matrix}\right\} + u \pm \int_0^w \sqrt{\sigma'(w)}\, dw, \tag{2.10}$$

$$\lambda = \sqrt{\sigma'(r - s)}. \tag{2.11}$$

Then (2.8) becomes

$$r_t - \lambda r_x = -\frac{k(0)}{2}(r + s) + \Phi - \int_0^t k'(t - \tau)\left(\frac{r(x, \tau) + s(x, \tau)}{2}\right) d\tau,$$

$$s_t + \lambda s_x = -\frac{k(0)}{2}(r + s) + \Phi - \int_0^t k'(t - \tau)\left(\frac{r(x, \tau) + s(x, \tau)}{2}\right) d\tau. \tag{2.12}$$

Initial conditions are taken to be smooth and periodic and satisfy

$$\begin{aligned} &r(x, 0) + s(x, 0) = 0, \qquad (r(x, 0) + s(x, 0))_{xx} = 0, \\ &(r(x, 0) - s(x, 0))_x = 0, \qquad x = 0, \quad \pm 1, \ \pm 2, \ldots, \end{aligned} \tag{2.13}$$

so that they are compatible with (2.5). From (2.9), as in the previous section, we assume

$$\lambda'(r - s) \geq \varepsilon > 0 \quad \text{for} \quad |r - s - \xi_0| \leq \delta. \tag{2.14}$$

(The case where $\lambda' \leq -\varepsilon < 0$ is similar.)

We give an a priori estimate to prove Theorem 2.1.

LEMMA 2.1. Suppose $|r_0| = \sup_{x \in \mathbb{R}} |r_0(x)|$, $|s_0| = \sup_{x \in \mathbb{R}} |s_0(x)|$. Then as long as smooth solutions exist we have

$$|r(x, t)| + |s(x, t)| \leq \frac{1}{2}\left(|r_0| + |s_0| + \frac{4\Phi_0}{\alpha}\right)(2 + \alpha t + k_0 t^2)\exp\left(\frac{k_0}{2}\right) t^2,$$

where $\alpha = k(0)\,(>0)$.

*Proof.* We introduce the characteristic curves in the same way as in Lemma 1.1:

$$x_1 = x_1(\tau, \gamma) = \gamma - \int_0^\tau \lambda\, d\mu, \qquad \gamma \in \mathbb{R},$$

$$x_2 = x_2(\tau, \eta) = \eta + \int_0^\tau \lambda\, d\mu, \qquad \eta \in \mathbb{R}.$$

Then (2.12) becomes

$$\frac{d}{d\tau} r(x_1(\tau, \gamma), \tau) = -\frac{\alpha}{2}\{r(x_1(\tau, \gamma), \tau) + s(x_1(\tau, \gamma), \tau)\} + \Phi(x_1(\tau, \gamma), \tau) - \int_0^\tau \frac{k(\tau-\mu)}{2}(r(x_1(\tau, \gamma), \mu) + s(x_1(\tau, \gamma), \mu))\, d\mu,$$

$$\frac{d}{d\tau} s(x_2(\tau, \eta), \tau) = -\frac{\alpha}{2}\{r(x_2(\tau, \eta), \tau) + s(x_2(\tau, \eta), \tau)\} + \Phi(x_2(\tau, \gamma), \tau) - \int_0^\tau \frac{k(\tau-\mu)}{2}(r(x_2(\tau, \eta), \mu) + s(x_2(\tau, \eta), \mu))\, d\mu.$$

Integrating the above equations along each characteristic curve we have

$$e^{(\alpha/2)\tau} r(x_1)\tau, \gamma), \tau) = r(\gamma) - \frac{\alpha}{2}\int_0^\tau e^{(\alpha/2)\mu} s(x_1(\mu, \gamma), \mu)\, d\mu + \int_0^\tau e^{(\alpha/2)\mu}\Phi(x_1(\mu, \gamma), \mu)\, d\mu - \int_0^\tau e^{(\alpha/2)\mu}\int_0^\mu \frac{k(\mu-p)}{2}(r(x_1(\mu, \gamma), p) + s(x_1(\mu, \gamma), p))\, dp\, d\mu,$$

$$e^{(\alpha/2)\tau} s(x_2(\tau, \eta), \tau) = s(\eta) - \frac{\alpha}{2}\int_0^\tau e^{(\alpha/2)\mu} r(x_2(\mu, \eta), \mu)\, d\mu + \int_0^\tau e^{(\alpha/2)\mu}\Phi(x_2(\mu, \eta), \mu)\, d\mu - \int_0^\tau e^{(\alpha/2)\mu}\int_0^\mu \frac{k(\mu-p)}{2}(r(x_2(\mu, \eta), p) + s(x_2(\mu, \eta), p))\, dp\, d\mu.$$

Since we assume the existence of smooth solutions, by means of the mean value theorem for integrals we have

$$e^{(\alpha/2)\tau} r(x_1(\tau, \gamma), \tau) = r(\gamma) = \frac{\alpha}{2}\int_0^\tau e^{(\alpha/2)\mu} s(x_1(\mu, \gamma), \mu)\, d\mu + \int_0^\tau e^{(\alpha/2)\mu}\Phi(x_1(\mu, \gamma), \mu)\, d\mu - \int_0^\tau e^{(\alpha/2)\mu}\mu\, \frac{k(\mu-\xi_1(\mu))}{2}(r(x_1(\mu, \gamma), \xi_1(\mu)) + s(x_1(\mu, \gamma), \xi_1(\mu))\, d\mu$$

$$e^{(\alpha/2)\tau} s(x_2\tau, \eta), \tau) = s(\eta) - \frac{\alpha}{2}\int_0^\tau e^{(\alpha/2)\mu} r(x_2(\mu, \eta), \mu)\, d\mu + \int_0^\tau e^{(\alpha/2)\mu}\Phi(x_2(\mu, \eta), \mu)\, d\mu - \int_0^\tau e^{(\alpha/2)\mu}\mu\, \frac{k(\mu-\xi_2(\mu))}{2}(r(x_2(\mu, \eta), \xi_2(\mu)) + s(x_2(\mu, \eta), \xi_2(\mu))\, d\mu,$$

where $0 \le \xi_1(\mu) \le \mu, 0 \le \xi_2(\mu) \le \mu$. Define

$$\bar{r}(x_1(t, \gamma), t) = \max_{0 \leqslant \tau \leqslant t} e^{(\alpha/2)\tau}\left| r(x_1(\tau, \gamma), \tau)\right|,$$

$$\bar{s}(x_2(t, \eta), t) = \max_{0 \leqslant \tau \leqslant t} e^{(\alpha/2)\tau}\left| r(x_2(\tau, \eta), \tau)\right|.$$

Then we obtain

$$\bar{r}(x_1(t, \gamma), t) = \max_{0 \leqslant \tau \leqslant t}\left| r(\gamma) - \frac{\alpha}{2}\int_0^\tau e^{(\alpha/2)\mu} s(x_1(\mu, \gamma), \mu)\, d\mu + \int_0^\tau \Phi(x_1(\mu, \gamma), \mu)\, d\mu - \int_0^\tau e^{(\alpha/2)\mu}\mu\, \frac{k(\mu-\xi_1(\mu))}{2}(r(x_1(\mu, \gamma), \xi_1(\mu)) + s(x_1(\mu, \gamma), \xi_1(\mu))\, d\mu\right|,$$

$$\leq |r_0| + \max_{0 \leqslant \tau \leqslant t} \left\{ \frac{\alpha}{2} \int_0^\tau e^{(\alpha/2)\mu} |s(x_1(\mu, \gamma), \mu)|\, d\mu + \int_0^\tau e^{(\alpha/2)\mu} |\Phi(x_1(\mu, \gamma), \mu)|\, d\mu \right.$$

$$\left. + \int_0^\tau e^{(\alpha/2)\mu} \mu \frac{|k(\mu - \xi_1(\mu))|}{2} \{ |r(x_1(\mu, \gamma), \xi_1(\mu))| + |s(x_1(\mu, \gamma), \xi_1(\mu))| \}\, d\mu \right\}$$

$$\leq |r_0| + \frac{\alpha}{2} \int_0^t e^{(\alpha/2)\mu} |s(x_1(\mu, \gamma), \mu)|\, d\mu + \Phi_0 \int_0^t e^{(\alpha/2)\mu}\, d\mu$$

$$+ \frac{k_0}{2} \int_0^t \mu\, e^{(\alpha/2)\mu} \{ |r(x_1(\mu, \gamma), \xi_1(\mu))| + |s(x_1(\mu, \gamma), \xi_1(\mu))| \}\, d\mu,$$

$$\bar{s}(x_2(t, \eta), t) \leq |s_0| + \frac{\alpha}{2} \int_0^t e^{(\alpha/2)\mu} |r(x_2(\mu, \eta), \mu)|\, d\mu + \Phi_0 \int_0^t e^{(\alpha/2)\mu}\, d\mu$$

$$+ \frac{k_0}{2} \int_0^t \mu\, e^{(\alpha/2)\mu} \{ |r(x_2(\mu, \eta), \xi_2(\mu))| + |s(x_2(\mu, \eta), \xi_2(\mu))| \}\, d\mu.$$

Define

$$R(t) = \sup_{x \in \mathbb{R},\, 0 \leqslant \tau \leqslant t} e^{(\alpha/2)\tau} |r(x, \tau)|, \qquad S(t) = \sup_{x \in \mathbb{R},\, 0 \leqslant \tau \leqslant t} e^{(\alpha/2)\tau} |s(x, \tau)|.$$

Then the above inequalities imply that

$$\bar{r}(x_1(t, \gamma), t) \leq |r_0| + \frac{\alpha}{2} \int_0^t S(\mu)\, d\mu + \frac{2}{\alpha} \Phi_0\, e^{(\alpha/2)t} + \frac{k_0}{2} \int_0^t \mu (R(\mu) + S(\mu)\, d\mu,$$

$$\bar{s}(x_2(t, \eta), t) \leq |s_0| + \frac{\alpha}{2} \int_0^t R(\mu)\, d\mu + \frac{2}{\alpha} \Phi_0\, e^{(\alpha/2)t} + \frac{k_0}{2} \int_0^t \mu (R(\mu) + S(\mu)\, d\mu.$$

Since $r$, $s$ are periodic in $x$, for each $t$ there are characteristic curves $\hat{x}_1(t, \hat{\gamma})$, $\hat{x}_2(t, \hat{\eta})$ so that

$$\bar{r}(\hat{x}_1(t, \hat{\gamma}), t) = R(t), \qquad \bar{s}(\hat{x}_2(t, \hat{\eta}), t) = S(t).$$

If we choose this $\hat{\gamma}$, $\hat{\eta}$ for each $t$, we obtain

$$R(t) \leq |r_0| + \frac{2}{\alpha} \Phi_0\, e^{(\alpha/2)t} + \frac{\alpha}{2} \int_0^t S(\mu)\, d\mu + \frac{k_0}{2} \int_0^t \mu (R(\mu) + S(\mu))\, d\mu$$

$$S(t) \leq |s_0| + \frac{2}{\alpha} \Phi_0\, e^{(\alpha/2)t} + \frac{\alpha}{2} \int_0^t R(\mu)\, d\mu + \frac{k_0}{2} \int_0^t \mu (R(\mu) + S(\mu))\, d\mu.$$

Adding these two inequalities, we have

$$W(t) \leq |r_0| + |s_0| + \frac{4}{\alpha} \Phi_0\, e^{(\alpha/2)t} + \int_0^t \left( \frac{\alpha}{2} + k_0 \mu \right) W(\mu)\, d\mu,$$

where $W(t) = R(t) + S(t)$. Using the generalized Gronwall inequality we obtain

$$W(t) \leq |r_0| + |s_0| + \frac{4}{\alpha} \Phi e^{(\alpha/2)t} + \exp\left( \frac{\alpha}{2} t + \frac{k}{2} t^2 \right)$$

$$\cdot \int_0^t \left( \frac{\alpha}{2} + k\mu \right) \left( |r_0| + |s_0| + \frac{4}{\alpha} \Phi e^{(\alpha/2)\mu} \right) \exp\left( -\frac{\alpha}{2} \mu - \frac{k}{2} \mu^2 \right) d\mu,$$

$$\leq |r_0| + |s_0| + \frac{4}{\alpha} \Phi_0\, e^{(\alpha/2)t} + \tfrac{1}{2}(\alpha t + k_0 t^2) \left( |r_0| + |s_0| + \frac{4}{\alpha} \Phi_0 \right) \exp\left( \frac{\alpha}{2} t + \frac{k_0}{2} t^2 \right).$$

Hence we have

$$\begin{aligned} |r(x, t)| + |s(x, t)| &\le (|r_0| + |s_0|)e^{-(\alpha/2)t} \\ &\quad + \frac{4}{\alpha}\Phi_0 + \frac{1}{2}\left(|r_0| + |s_0| + \frac{4}{\alpha}\Phi_0\right)(\alpha t + k_0 t^2)\exp\left(\frac{k_0}{2}t^2\right), \\ &\le \frac{1}{2}\left(|r_0| + |s_0| + \frac{4}{\alpha}\Phi_0\right)(2 + \alpha t + k_0 t^2)\exp\left(\frac{k_0}{2}t^2\right). \end{aligned}$$

Now we state the theorem.

THEOREM 2.1. Suppose $\sigma''$ satisfies (2.9); then the breakdown of smooth solution in (2.10) and (2.11) will take place in finite time for certain smooth initial data.

*Proof.* The proof of the breakdown of smooth solution is basically the same as in Theorem 1.1 except that we have to handle the integral term. We assume the existence of global smooth solutions for arbitrary initial data.

As in Theorem 1.1, we choose points $(x_1^0, 0)$ and $(x_2^0, 0)$ on the initial line such that $x_1^0 = 0.5$, $x_2^0 = x_1^0 + \gamma$, and we give the smooth initial data $r_0(x)$ and $s_0(x)$ which take the values

$$\begin{aligned} &r_0(x_1^0) = \bar{r}_0 - \frac{\Delta r}{2}, \qquad r_0(x_2^0) = \bar{r}_0 + \frac{\Delta r}{2}, \\ &s_0(x) = \bar{s}_0, \qquad x \in [\eta, 1-\eta], \end{aligned} \tag{2.15}$$

and satisfy

$$\begin{aligned} &|r_0(x) - s_0(x) - \xi_0| \le \frac{\delta}{2}, \qquad x \in [\eta, 1-\eta], \\ &r(x, 0) + s(x, 0) = 0, \qquad (r(x, 0) + s(x, 0))_{xx} = 0, \\ &(r(x, 0) - s(x, 0))_x = 0, \qquad x = 0, \pm 1, \pm 2, \ldots, \end{aligned} \tag{2.16}$$

where $\bar{r}_0$, $\bar{s}_0$, and $\Delta r$ ($>0$) are constant, and $\eta$ is a small positive constant. We can extend the above initial data smoothly and periodically to $-\infty < x < \infty$, so that Lemma 2.1 applies.

By making use of a priori estimate we define

$$M_0 = \frac{1}{2}\left(|r_0| + |s_0| + \frac{4}{\alpha}\Phi_0\right)(2 + \alpha T + k_0 T^2)\exp\left(\frac{k_0}{2}T^2\right),$$

where $T$ is a positive number. We also define the following values:

$$\begin{aligned} &M_1 = \max_{|r+s| \le M_0} \left|-\frac{\alpha}{2}(r+s)\right| + \Phi_0, \qquad M_2 = \max_{|r-s| \le M_0} |\lambda'(r-s)|, \\ &M_3 = \max_{|r-s| \le M_0} |\lambda(r-s)|, \qquad M_4 = \max_{|r-s| \le M_0} \frac{k_0}{2}|r+s|, \end{aligned} \tag{2.17}$$

and the set

$$G_0 = \{(r, s): |r - s - \xi_0| \le \delta\},$$

where $\xi_0$ and $\delta$ are defined by (2.14). We choose $T$ so that it satisfies

$$\frac{1}{M_3}\left(\frac{1}{2}-\eta\right) < T,$$

which is always possible. (The number $(1/M_3)(\frac{1}{2}-\eta)$ is obtained in the same was as in Theorem 1.1).

Although this is a pure initial value problem we restrict our argument to between $0 \le x \le 1$. We proceed in the same way as in Theorem 1.1. In the domain $D_0$ (obtained in the same way as in Theorem 1.1) we obtain the following inequalities along $r$-characteristic curves $x_1(t)$ and $x_2(t)$ $(x_1(0) = x_1^0 < x_2(0) = x_2^0)$. Along $x_1(t)$ we have

$$r_1(t) = \bar{r}_0 - \frac{\Delta r}{2} + \int_0^t \left\{-\frac{\alpha}{2}(r+s) + \Phi\right\} dt - \int_0^t \int_0^t k(t-\tau)\left(\frac{r(x,\tau)+s(x,\tau)}{2}\right) d\tau\, dt, \tag{2.18}$$

$$\bar{r}_0 - \frac{\Delta r}{2} - M_1 t - \frac{M_4}{2} t^2 \le r_1(t) \le \bar{r}_0 - \frac{\Delta r}{2} + M_1 t + \frac{M_4}{2} t^2, \tag{2.19}$$

$$\bar{s}_0 - M_1 t - \frac{M_4}{2} t^2 \le s_1(t) \le \bar{s}_0 + M_1 t + \frac{M_4}{2} t^2, \tag{2.20}$$

$$x_1(t) = x_1^0 - \int_0^t \lambda(r_1 - s_1)\, dt. \tag{2.21}$$

And along $x_2(t)$ we see that

$$\bar{r}_0 + \frac{\Delta r}{2} - M_1 t - \frac{M_4}{2} t^2 \le r_2(t) \le \bar{r}_0 + \frac{\Delta r}{2} + M_1 t + \frac{M_4}{2} t^2, \tag{2.22}$$

$$\bar{s}_0 - M_1 t - \frac{M_4}{2} t^2 \le s_2(t) \le \bar{s}_0 + M_1 t + \frac{M_4}{2} t^2, \tag{2.23}$$

$$x_2(t) = x_2^0 - \int_0^t \lambda(r_2 - s_2)\, dt. \tag{2.24}$$

The above inequalities correspond to (1.31)–(1.34) and (1.35)–(1.37). From (2.19), (2.20), (2.22) and (2.23) we have

$$\bar{r}_0 - \bar{s}_0 - \frac{\Delta r}{2} - 2M_1 t - M_4 t^2 - \xi_0 \le r_i(t) - s_i(t) - \xi_0 \le \bar{r}_0 - \bar{s}_0 + \frac{\Delta r}{2} + 2M_1 t + M_4 t^2 - \xi_0 \qquad (i = 1, 2). \tag{2.25}$$

Since $|\bar{r}_0 \pm (\Delta r/2) - \bar{s}_0 - \xi_0| \le (\delta/2)$, if

$$2M_1 t + M_4 t^2 \le \frac{\delta}{2}, \tag{2.26}$$

$|r_i(t) - s_i(t) - \xi_0| \le \delta$ is satisfied for $i = 1, 2$. This means if $t < (1/2M_3)(\frac{1}{2}-\eta)$ and (2.26) are satisfied, $r$, $s$ on $r$-characteristic curves $x_1(t)$, $x_2(t)$ stay in $G_0$. From (2.19) and (2.22), if

$$2M_1 t + M_4 t^2 < \Delta r \tag{2.27}$$

(1.22) is satisfied. And from (2.21) and (2.24), if the inequality

$$\varepsilon\, \Delta r t - M_1(\varepsilon + M_2)t^2 - \tfrac{1}{3}M_4(\varepsilon + M_2)t^3 \geq x_2^0 - x_1^0 = \gamma \tag{2.28}$$

is satisfied for $t = t_1$, (1.23) is satisfied. The above inequalities (2.25), (2.27), (2.28) correspond to (1.38), (1.39) and (1.40). Of course we can easily find $t_1$ ($<(1/2M_3)(\frac{1}{2} - \eta)$) and $\gamma$ such that they satisfy (2.26), (2.27), and (2.28).

From (2.28) we see that the integral term in (2.12) gives the third-order correction in time to the original method of Rozhdestvenskii.

## References


[1] B. L. Rozhdestvenskii, *On the discontinuity of solutions of quasilinear equations*, Amer. Math. Soc. Transl. (2) **101**, 239–250 (1973)

[2] R. L. Fosdick and G. P. MacSithigh, *Shearing motions and the formation of shocks in an elastic circular tube*, Quart. Appl. Math. **38** (1980)

[3] T. Nishida, *Global smooth solutions for the second-order quasilinear wave equations with the first-order dissipation* (unpublished)

[4] M. Slemrod, *Instability of steady shearing flow in a non-linear viscoelastic fluid*, Arch. Rat. Mech. Anal. **68**, 211–255 (1978)

[5] P. D. Lax, *Development of singularities of solutions of nonlinear hyperbolic partial differential equations*, J. Math. Phys. **5**, 611–613 (1964)

[6] R. C. MacCamy, *A model for one-dimensional nonlinear viscoelasticity*, Quart. Appl. Math. **25**, 21–33 (1977)

[7] C. M. Dafermos and J. A. Nohel, *Energy methods for nonlinear hyperbolic Volterra integrodifferential equations*, Univ. of Wisconsin M.R.C. Technical Summary Report No. 1876