A SHARPENING OF MASLOV'S METHOD OF CHARACTERISTICS TO GIVE THE FULL ASYMPTOTIC SERIES*

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1. Introduction. Maslov's method of characteristics uses a certain Hamiltonian flow to determine the first term in the asymptotic series solution of linear partial differential equations near turning points. In [1] an associated non-Hamiltonian flow was introduced to determine the full asymptotic series of the reduced Helmholtz equation. Here we note the same sharpening of Maslov's method extends to a much more general setting. We describe the sharpening for two independent space variables for all possible positions of the Lagrange manifold. For the case in which the Lagrange manifold cannot be parametrized by either space or momentum variables separately, another auxiliary flow is required to obtain the full asymptotic expansion. We determine this flow, then relate the each position of the Lagrange manifold to either the classical technique or a refined Maslov technique.

2. We assume that the wave-type equation

$$\sum_{j=1}^{2} a_j(\bar{x}) \frac{\partial^2 \psi}{\partial x_j^2} + \sum_{j=1}^{2} b_j(\bar{x}) \frac{\partial^2 \psi}{\partial x_j \partial t} + \sum_{j=1}^{2} c_j(\bar{x}) \frac{\partial \psi}{\partial x_j} + \sum_{n=0}^{2} d_n(\bar{x}) \frac{\partial^n \psi}{\partial t^n} = 0 \tag{1}$$

has an asymptotic solution—for brevity, near turning points of the highest order—of the form

$$\psi(\bar{x}) - \exp\{i\tau t\} \int A(\bar{x}, \bar{p}, \tau) \exp\{i\tau(\bar{x}' \cdot \bar{p} - S(\bar{p}))\} d\bar{p} = O(\tau^{-\infty}) \tag{2}$$

where τ is a large parameter and $S(\bar{p})$ is such that $\bar{x} - \nabla_p S(\bar{p}) = 0$ determines the Lagrange manifold of Maslov near the turning point [2]. Essentially, the Lagrange manifold is the set of points $(\bar{x}, \bar{p})$ at which is concentrated all the contribution to the asymptotic series of the integral. Notice that the integral alone is a full asymptotic solution of the "associated Helmholtz equation."

Carrying the differentiation (1) across the integral (2) determines

$$\begin{aligned} \int d\bar{p} \exp\{i\tau(\bar{x} \cdot \bar{p} - S(\bar{p}))\} \Bigg\{ &(i\tau)^2 \Big[\sum_{j=1} a_j p_j^2 + \sum_j b_j p_j + d_2 \Big] A \\ &+ (i\tau) \Big[\sum_{j=1} 2a_j p_j \frac{\partial A}{\partial x_j} + \sum_j b_j \frac{\partial A}{\partial x_j} + \sum_j c_j p_j A + d_1 A \Big] \\ &+ \Big[\sum_j a_j \frac{\partial^2 A}{\partial x_j^2} + \sum_j c_j \frac{\partial A}{\partial x_j} + d_0 A \Big] \Bigg\} = 0(\tau^{-\infty}). \end{aligned} \tag{3}$$

* Received March 30, 1981.

The coefficient of $(i\tau)^2$ is Maslov's Hamiltonian (and hence is zero) on the manifold [3, Chapter II]. Expanding the Hamiltonian

$$\sum_j a_j(\bar{x})p_j^2 + \sum_j b_j(\bar{x})p_j + d_2(\bar{x}) = \sum_j a_j(\nabla_p S)p_j + \sum_j b_j(\nabla_p S) + d_2(\nabla_p S) + \sum_j \left(x_j - \frac{\partial S}{\partial p_j}\right)D_j = \sum_j \left(x_j - \frac{\partial S}{\partial p_j}\right)D_j$$

where

$$\bar{D} = \int_0^1 \nabla_x H(t(\bar{x} - \nabla_p S(\bar{p})) + \nabla_p S(\bar{p}), \bar{p})dt.$$

with H being Maslov's Hamiltonian. Substituting into (3), noting

$$\int \nabla_p \cdot [\exp\{i\tau(\bar{x} \cdot \bar{p} - S(\bar{p}))\}A\bar{D}\}]d\bar{p}$$

$$= \int \exp\{i\tau(\bar{x} \cdot \bar{p} - S(\bar{p}))\}\{i\tau A(\bar{x} - \nabla_p S(\bar{p})) \cdot \bar{D} + \bar{D} \cdot \nabla_p A + A\nabla_p \cdot \bar{D}\}d\bar{p}, \quad (4)$$

and taking the surface integral over a sufficiently large radius so that it vanishes, (3) becomes

$$\int d\bar{p} \exp\{i\tau(\bar{x} \cdot \bar{p} - S(\bar{p}))\} \left\{(i\tau)\left[-\sum_j D_j \frac{\partial A}{\partial p_j} - \sum_j A \frac{\partial D_j}{\partial p_j} + \sum_j 2a_j p_j \frac{\partial A}{\partial x_j} + \sum_j b_j \frac{\partial A}{\partial x_j} + \sum_j c_j p_j A + d_1 A \right] + \sum_j a_j \frac{\partial^2 A}{\partial x_j^2} + \sum_j c_j \frac{\partial A}{\partial x_j} + d_0 A \right\} = O(\tau^{-\infty}). \quad (4)$$

Requiring

$$-\sum_j D_j \frac{\partial A}{\partial p_j} - \sum_j A \frac{\partial D_j}{\partial p_j} + \sum_j 2a_j p_j \frac{\partial A}{\partial x_j} + \sum_j b_j \frac{\partial A}{\partial x_j} + \sum_j c_j p_j A + d_1 A + \frac{1}{i\tau}\left\{\sum_j \frac{\partial^2 A}{\partial x_j^2} + \sum_j c_j \frac{\partial A}{\partial x_j} + d_0 A\right\} = 0 \quad (5)$$

in a neighborhood of the Lagrange manifold leads to a transport equation if we introduce the flow

$$x_j' = 2a_j(\bar{x})p_j + b_j(\bar{x}), \qquad p_j' = -D_j, \quad (6)$$

where the primes indicate time derivatives [4]. Specifically, (5) holds in such a neighborhood if we allow the asymptotic series

$$A(\bar{x}, \bar{p}, \tau) = \sum_{k=0}^{\infty} A_k(\bar{x}, \bar{p})(i\tau)^{-k} \quad (7)$$

to evolve according to the transport equation

$$A_k' - \left[\sum_j \left(c_j(\bar{x})p_j - \frac{\partial D_j}{\partial p_j}\right) + d_1\right]A_k + \frac{1}{i\tau}\left[\sum_j \left(a_j(\bar{x}) \frac{\partial^2 A_{k-1}}{\partial x_j^2} + c_j(\bar{x}) \frac{\partial A_{k-1}}{\partial x_j}\right) + d_0 A_{k-1}\right] = 0. \quad (8)$$

For example, applying the above to the Tricomi equation

$$y\,\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=0,$$

we obtain the flow

$$x=-\frac{2}{3}\,p_x(0)^3t^3+2p_x(0)p_y(0)t^2+2y(0)p_x(0)t+x(0),\qquad y=-p_x(0)^2t^2+2p_y(0)t+y(0),$$

$$p_x=p_x(0),\qquad p_y=-p_x(0)t^2+2p_y(0).$$

Depending on 'emitter conditions', that is, on the Lagrange manifold, the transport equations have many different solutions; one particularly neat family of solutions is given by $A_k=$ constant for all k, leading to a full asymptotic solution to the Tricomi equation of the form

$$\psi(\bar{x})=\int\exp\{i\tau(xp_x+yp_y+p_y^3/3p_x^2)\}\alpha(p_x,p_y)dp_x\,dp_y$$

where $\alpha(\bar{p})$ is any smooth function with compact support. The solution may be multiplied by a constant asymptotic series.

3. While the above treatment suffices for most wave-type equations, in some cases a mixed coordinate-momentum space Lagrange manifold is required. A simple but useful example is the Helmholtz equation

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+(x^4+y)\,\frac{\partial^2\psi}{\partial t^2}=0, \tag{9}$$

with emitter conditions $(x, y, p_x, p_y)=(\theta, \theta^2, \theta^2, \theta)$. Analogous to (2) we assume an asymptotic solution of the form

$$\psi(\bar{x})-\exp\{i\tau t\}\int A(x,y,p_y,\tau)\exp\{i\tau(yp_y-S(p_y,x))\}dp_y=O(\tau^-) \tag{10}$$

where $A(x, y, p_y, \tau)$ and all its derivatives are bounded, which determines the Lagrange manifold

$$y=\partial S/\partial p_y,\qquad p_x=-\partial S/\partial x. \tag{11}$$

Carrying the differentiation (9) across the integral in (10) obtains

$$\int dp_y\,\exp\{i\tau(yp_y-S(p_y,x))\}\left\{(i\tau)^2\left[p_y^2+\left(\frac{\partial S}{\partial x}\right)^2-(x^4+y)\right]A\right.$$
$$\left.+\,i\tau\left[2p_y\,\frac{\partial A}{\partial y}-2\,\frac{\partial S}{\partial x}\frac{\partial A}{\partial x}-\frac{\partial^2 S}{\partial x^2}\,A\right]+\left[\frac{\partial^2 A}{\partial x^2}+\frac{\partial^2 A}{\partial y^2}\right]\right\}. \tag{12}$$

In this case Maslov's Hamiltonian becomes

$$H=p_y^2+\left(\frac{\partial S}{\partial x}\right)^2-(x^4+y).$$

Expanding, we get

$$p_y^2 + \left(\frac{\partial S}{\partial x}\right)^2 - (x^4 + y) = p_y^2 + \left(\frac{\partial S}{\partial x}\right)^2 - \left(x^4 + \frac{\partial S}{\partial p_y}\right) + \left(y - \frac{\partial S}{\partial p_y}\right)D = \left(y - \frac{\partial S}{\partial p_y}\right)D.$$

where

$$D = \int_0^1 \frac{\partial}{\partial y} H\left(x, t\left(y - \frac{\partial S}{\partial p_y}\right) + \frac{\partial S}{\partial p_y}, \frac{\partial S}{\partial x}, p_y\right)dt.$$

Now substituting into (12) and noting

$$\begin{aligned}\int \exp\{i\tau(yp_y - S)\}A\left(y - \frac{\partial S}{\partial p_y}\right)D\,dp_y &= \int \frac{\partial}{\partial p_y}[\exp\{i\tau(yp_y - S)\}AD]dp_y - \int \exp\{i\tau(yp_y - S)\}\left[D\frac{\partial A}{\partial p_y} + A\frac{\partial D}{\partial p_y}\right]dp_y \\ &= -\int \exp\{i\tau(yp_y - S)\}\left[D\frac{\partial A}{\partial p_y} + A\frac{\partial D}{\partial p_y}\right]dp_y,\end{aligned} \tag{13}$$

(12) becomes

$$\int dp_y \exp\{i\tau(yp_y - S(p_y, x))\} \cdot \left[i\tau\left(-D\frac{\partial A}{\partial p_y} - A\frac{\partial D}{\partial p_y} + 2p_y\frac{\partial A}{\partial y} - 2\frac{\partial S}{\partial x}\frac{\partial A}{\partial x} - \frac{\partial^2 S}{\partial x^2}A\right) + \frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2}\right].$$

Then introducing the flow

$$x' = -2\frac{\partial S}{\partial x}, \qquad y' = 2p_y, \qquad p_x' = -4x^3, \qquad p_y' = -D, \tag{14}$$

leads to the transport equation

$$\frac{dA_k}{dt} - A_k\frac{\partial D}{\partial p_y} - \frac{\partial^2 S}{\partial x^2}A_k + \frac{1}{i\tau}\left(\frac{\partial^2 A_{k-1}}{\partial x^2} + \frac{\partial^2 A_{k-1}}{\partial y^2}\right) = 0. \tag{15}$$

More generally, for wave-type equations (1) requiring a Lagrange manifold as in (11), the flow

$$\begin{gathered} x_1' = b_1(\bar{x}) - (2a_1(\bar{x}) + c_1(\bar{x}))\frac{\partial S}{\partial x_1}, \qquad y_1' = b_2(\bar{x}) - (2a_2(\bar{x}) + c_2(\bar{x}))p_2, \\ p_1' = -\partial H/\partial x_1, \qquad p_2' = -D \end{gathered} \tag{16}$$

determines the transport equation

$$\frac{dA}{dt} - A\frac{\partial D}{\partial p_2} - a_1(\bar{x})\frac{\partial^2 S}{\partial x_1^2}A + d_1(\bar{x})A + \frac{1}{i\tau}\left\{\sum_j\left[a_j(\bar{x})\frac{\partial^2 A}{\partial x_j^2} + c_j(\bar{x})\frac{\partial A}{\partial x_j}\right] + d_0(\bar{x})A\right\} = 0. \tag{17}$$

For those cases requiring the Lagrange manifold

$$x_1 = \partial S/\partial p_1, \qquad p_2 = -\partial S/\partial x_2,$$

e.g., emitter conditions $(x, y, p_x, p_y) = (\theta, \theta^2, \theta, \theta^2)$, the subscripts on the coordinates and momenta in (16) and (17) are interchanged.

4. For a given linear hyperbolic differential equation, different 'emitter conditions' generate different Lagrange manifolds. While the classical technique often suffices, near turning points of the Lagrange manifold some version of Maslov's technique is required. For a particular problem, let L be the Lagrange manifold and $(x_1^0, y_1^0, p_1^0, p_2^0) \in L$ be a turning point of the manifold. Because L is a smooth two-dimensional manifold, there must exist two smooth functions, e.g., f and g, such that $z_1 = f(z_3, z_4)$ and $z_2 = g(z_3, z_4)$, where z_1, z_2, z_3, z_4 is a suitable permutation of x_1, x_2, p_1, p_2. Since L is a Lagrange manifold, the form $dx_1 \wedge dp_1 + dx_2 \wedge dp_2$ restricted to the tangent space of L must vanish. Thus we obtain a finite list of cases, namely:

1) $p_1 = f(\bar{x})$, $p_2 = g(\bar{x})$. In this case $f(\bar{x})dx_1 + g(\bar{x})dx_2$ is exact and the point is not a turning point—in the vicinity the classical technique applies [2, 3].

2) $x_1 = f(\bar{p})$, $x_2 = g(\bar{p})$. In this case $f(\bar{p})dp_1 + g(\bar{p})dp_2$ is exact. In this case we may apply the procedure of Sec. 2.

3) $x_1 = f(x_2, p_1)$, $p_2 = g(x_2, p_1)$. In this case $f(x_2, p_1)dp_1 - g(x_2, p_1)dx_2$ is exact. In this case we may apply the procedure of Sec. 3.

4) $x_1 = f(x_1, p_1)$, $p_2 = g(x_1, p_1)$. In this case the Poisson bracket $\{f, g\} = -1$. It follows from a straight forward calculation that already (1), (2), or (3) holds.

Thus the extended Maslov technique applies to all possible cases of the second-order linear hyperbolic equation in two space and one time variables.

References

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