A TEST FOR STABILITY OF LINEAR DIFFERENTIAL DELAY EQUATIONS*

By

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Abstract. The changes in the stability of a system of linear differential delay equations resulting from the delay are studied by analyzing the associated eigenvalues of the characteristic equation. A specific contour is mapped by the characteristic equation into the complex plane to give an easy test for stability from an application of the argument principle. When the real part of an eigenvalue is positive, the contour gives bounds on the imaginary part which are important in certain applications to nonlinear problems.

1. Introduction. An important question in studying a system of differential delay equations is how changes in the delay affect stability of the system. Consider the linear differential delay system

$$\dot{x}(t) = Ax(t) + Bx(t - r) \tag{1.1}$$

where $x(t) \in R^n$ and A and B are $n \times n$ matrices. The associated eigenvalues in this system are found by evaluating the characteristic equation

$$\det[A - \lambda I + Be^{-\lambda r}] = 0$$

for λ. For more details see Bellman and Cooke [1] or Hale [3]. When there are two eigenvalues with positive real parts then the differential equation has a two-dimensional unstable manifold. We shall define the system (1.1) with eigenvalues in the right half-plane to be unstable.

By expanding the determinant in the characteristic equation an exponential polynomial

$$F(\lambda, r) = P(\lambda) + Q(\lambda)e^{-\lambda r} = 0 \tag{1.2}$$

is formed where $P(\lambda)$ is a polynomial in λ of degree n and $Q(\lambda)$ is a polynomial of degree less than n. When the system (1.1) is given so that the polynomials $P(\lambda)$ and $Q(\lambda)$ satisfy certain conditions, the stability of the system can be easily determined by mapping the contour C given in Fig. 1.1 where $\mu^* \to +\infty$ with $F(\lambda, r)$ and using the argument principle to determine if any roots λ satisfying (1.2) lie inside the contour in which case the system (1.1) is unstable. The argument principle gives the number of roots of $F(\lambda, r) = 0$ inside C by calculating

$$\frac{1}{2\pi} \Delta_C \arg F(\lambda, r),$$

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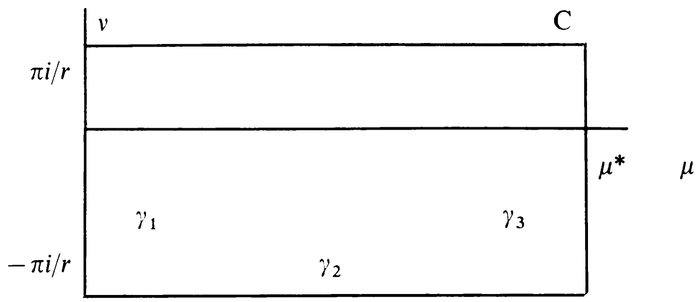

FIG. 1.1 The contour C.

which is equivalent to the number of counterclockwise encirclements of the origin in the image plane (see Churchill et al. [2, p. 298]).

Using this geometric approach we shall determine when the solutions of (1.2) cross the imaginary axis as the delay r varies, i.e., when the system (1.1) has a Hopf bifurcation. Furthermore, the analysis gives bounds on the imaginary parts of the eigenvalues for system (1.1). Finding bounds on the imaginary part of the eigenvalues can be useful in other applications; e.g., see Hale [3, p. 267] or Mahaffy [5] for applications showing the existence of periodic solutions to an oscillating system.

With the delay r taken as a parameter and for $r > 0$ the characteristic equation (1.2) has infinitely many solutions λ. Determining if any solution λ has $\operatorname{Re} \lambda > 0$ gives the stability of the system (1.1). When $r = 0$ the system (1.1) reduces to an ordinary differential equation and (1.2) becomes

$$F(\lambda, 0) = P(\lambda) + Q(\lambda) = 0, \tag{1.3}$$

an n-degree polynomial in λ. For this particular case there are several techniques for finding when $\operatorname{Re} \lambda > 0$ where λ solves (1.3). A technique commonly used in engineering control is the Routh-Huriwitz criterion (see Ogata [7] or Lancaster [4]). A matrix is formed from the coefficients of the polynomial. By determining the number of changes in sign of the determinants of the principal minors the number of roots λ with $\operatorname{Re} \lambda > 0$ can be found. It is easily seen that a necessary condition for stability of the ordinary differential equation is that all coefficients of the polynomial (1.3) are positive.

For our results it is assumed that the system (1.1) with $r = 0$ is stable. The questions we address are under what conditions on the system (1.1) do changes in the delay result in instability and what is the form of the region of instability?

2. Main results.

For all of the results below assume the following hypothesis:

(HI): When $r = 0$ all solutions of (1.2) are such that $\operatorname{Re} \lambda < 0$, i.e., $F(\lambda, 0) = 0$ has solutions only in the left half-plane.

First consider the case where $P(\lambda) = \prod_{j=1}^{n} (\lambda + \beta_j)$ and $Q(\lambda) = \alpha > 0$ (a constant), with $\beta_j > 0, j = 1, \ldots, n$. For this case arguments similar to those in Mahaffy [5] can be used to establish the following theorem.

THEOREM 1. Assume (HI). Let $P(\lambda) = \prod_{j=1}^{n} (\lambda + \beta_j)$ and $Q(\lambda) = \alpha$. If $\prod_{j=1}^{n} \beta_j < \alpha$, then there exists a delay r_0 such that for all $r > r_0$,

$$F(\lambda, r) = P(\lambda) + \alpha e^{-\lambda r} = 0 \tag{2.1}$$

has at least two roots with Re $\lambda > 0$, and these roots are such that $|\mathrm{Im}\ \lambda| < \pi/r$. If either $\prod_{j=1}^{n} \beta_j > \alpha$ or $0 \le r < r_0$, then all solutions of (2.1) have Re $\lambda < 0$.

The next result concerns a more general $P(\lambda)$; however, there is not necessarily a single bifurcation point r_0 in this case. Sec. 4 shows an example where the stable region may be separated.

THEOREM 2. Assume (HI). Let $Q(\lambda) = \alpha$ and $P(\lambda)$ be given by the following:

$$P(\lambda) = \prod_{j=1}^{m/2} (\lambda^2 + b_j\lambda + c_j) \prod_{j=m+1}^{n} (\lambda + \beta_j) \tag{2.2}$$

where $b_j, c_j, \beta_j > 0$ and $b_j^2 < 4c_j$. If $\left(\prod_{j=1}^{m/2} c_j\right)\left(\prod_{j=m+1}^{n} \beta_j\right) < \alpha$, then there exists a delay r_1 such that for all $r > r_1$, (2.1) has at least two roots with Re $\lambda > 0$, and these roots are such that $|\mathrm{Im}\ \lambda| < \pi/r$.

When the results are extended to more general $Q(\lambda)$, more restrictive hypotheses must be placed on our theorems in order to use the contour C of Fig. 1.1. An example of such a theorem is given below.

THEOREM 3. Assume $P(\lambda) = \prod_{j=1}^{n} (\lambda + \beta_j)$ and $Q(\lambda) = \prod_{j=1}^{m} (\lambda + \alpha_j)$, where $1 \le m \le n-1$. Assume (HI) and $\prod_{j=1}^{n} \beta_j < \prod_{j=1}^{m} \alpha_j$. If $\sum_{j=1}^{n} 1/\beta_j > \sum_{j=1}^{m} 1/\alpha_j$, then there exists a delay r_0 sufficiently large that for all $r > r_0$, (1.2) has at least two roots with Re $\lambda > 0$, and these roots are such that $|\mathrm{Im}\ \lambda| < \pi/r$.

3. Proofs of the Theorems. The proof of Theorem 1 uses the arguments found in Mahaffy [5]. Consider the contour in Fig. 1.1 and let $\lambda = \mu + i\nu$. From the form of $P(\lambda)$ in Theorem 1 we see that

$$|P(\lambda)| = \prod_{j=1}^{n} (\nu^2 + (\beta_j + \mu)^2)^{1/2},$$

$$\arg P(\lambda) = \sum_{j=1}^{n} \left(\arctan \frac{\nu}{\beta_j + \mu}\right).$$

Along γ_1, $\lambda = -i\nu$, $0 \le \nu \le \pi/r$. Note that while traversing γ_1 in the counterclockwise direction $|P(-i\nu)|$ is monotonically increasing and $\arg P(-i\nu)$ is monotonically decreasing. Geometrically, along γ_1, $P(-i\nu)$ forms the parametric representation of the center of the image arc while $\alpha e^{i\nu r}$ circles halfway around this center maintaining a constant radius of α. As seen in Mahaffy [5], $F(0)$, $P(0)$ and the origin align with $P(0)$ lying between $F(0)$ and the origin. From the above monotonic properties of $P(-i\nu)$ and the fact that $\arg(\alpha e^{i\nu r})$ increases from 0 to π along γ_1 relative to $P(-i\nu)$, i.e., $\arg[F(-i\nu, r) - P(-i\nu)]$ ranges from 0 to π for $0 \le \nu \le \pi/r$, in a similar manner to the analysis in Mahaffy [5] there exists a first ν_0, $0 < \nu_0 < \pi/r$, such that $F(-i\nu_0, r)$, $P(-i\nu_0)$ and the origin align (with the special case of a Hopf bifurcation if $F(-i\nu_0, r) = 0$ for some value of r). We now state as a lemma the following (see [5], Proposition 3.1)

LEMMA 3.1. (i) Suppose $|P(-i\nu_0)| > \alpha$; then $F(\lambda, r)$ does not encircle the origin as λ traverses C, and hence by the argument principle no roots of $F(\lambda, r) = 0$ lie inside C.

(ii) Suppose $|P(-i\nu_0)| < \alpha$; then $F(\lambda, r)$ encircles the origin as λ traverses C. The argument principle can be used to demonstrate that at least two roots of $F(\lambda, r) = 0$ lie inside C.

If $\prod_{j=1}^{n} \beta_j > \alpha$, then since $|P(-i\nu)|$ is monotonically increasing as ν increases and $|P(-i\nu)| > \prod_{j=1}^{n} \beta_j$, it is easily seen from Lemma 3.1(i) that no roots of (2.1) lie inside C independent of the delay r. To extend this argument to show that no roots of (2.1) lie in the right half-plane, continue the argument in [5] along the ν-axis. At $-i\nu_0$ alignment occurred with $F(-iv_0)$ lying between $P(-iv_0)$ and the origin. Then all subsequent alignments between $F(-iv_k)$, $P(-iv_k)$ and the origin as $v \to \infty$ (with v_k being the alignments given in [5] without the restrictions that $v < \pi/r$) have F and P with the same argument relative to the origin. This property continues to hold along C' as we trace a contour C' that encompasses the entire right half-plane. The argument is as in [5] taking the corners of C' to be $\pm(2m-1)\pi i/r$, $\mu^* \pm (2m-1)\pi i/r$ with $m = 1, 2, \ldots$ and let $m \to \infty$ and $\mu^* \to \infty$. From the above arguments $\Delta_{C'} P(\lambda) = \Delta_{C'} F(\lambda, r) = 0$, as $P(\lambda) = 0$ has only the real roots $-\beta_j, j = 1, \ldots, n$.

Now suppose $\prod_{j=1}^{n} \beta_j < \alpha$. By assumption when $r = 0$ there are no roots of (2.1) in the right half-plane, which implies that if $\arg P(-iv_1) = -\pi$ from some v_1, then $|P(-iv_1)| > \alpha$. ($|P(-iv_1)| < \alpha$ implies that the origin lies between P and F at v_1 but v_1 is the first alignment when $r = 0$, so Lemma 3.1 (ii) gives existence of roots of $F(\lambda, 0) = 0$ in the right half-plane, a contradiction.) So by continuity of $|P(-iv)|$ there exists a v^* such that $|P(-iv^*)| = \alpha$ with $-\pi < \arg P(-iv^*) < 0$. (Note that if $\arg |P(-iv)| > -\pi$ for all v then $\lim_{v\to\infty} |P(-iv)| = \infty$ can be used.) Now solve $r_0 v^* = \pi + \arg P(-iv^*)$ or

$$r_0 = \frac{\pi + \arg P(-iv^*)}{v^*},$$

which gives the value r_0 such that

$$P(-iv^*) + \alpha \exp(ir_0 v^*) = 0.$$

With this value r_0 the image of C under F passes through the origin, implying that two eigenvalues λ are $\pm iv^*$. (Note that $v^* < \pi/r_0$ and v^* is equivalent to the v_0 above for alignment. It easily follows by a argument similar to the one above that no other roots of (2.1) with $r = r_0$ have $\operatorname{Re} \lambda \geq 0$.)

Suppose $0 < r < r_0$; then the monotonicity of $\arg P(-iv)$ and a comparison of $\arg P(-iv)$ to $\arg[F(-iv, r) - P(-iv)]$ imply that alignment occurs at some $v_0 > v^*$. Alignment occurs when $\arg[F(-iv, r) - P(-iv)] \;\; -\arg P(-iv) = \pi$ or equivalently $rv - \arg P(-iv) = \pi$. By the monotone properties of $|P(-iv)|$, $|P(-iv)| > \alpha$; thus by Lemma 3.1(i) no roots of (2.1) lie in the right half-plane.

If $r > r_0$ a similar argument gives $v_0 < v^*$ which by the monotone property of $|P(-iv)|$ and Lemma 3.1(ii) shows the existence of at least two roots of (2.1) inside C. Furthermore, as the roots are inside C it follows that $|\operatorname{Im} \lambda| < \pi/r$, completing the proof of Theorem 1.

In Theorem 2, $P(\lambda)$ is a general nth-order polynomial with all roots in the left half-plane. Let $\lambda = \mu + iv$; then

$$|P(\lambda)| = \prod_{j=1}^{m/2} [(\mu^2 - v^2 + b_j\mu + c_j)^2 + (2\mu v + b_j v)^2]^{1/2} \prod_{j=m+1}^{n} [(\mu + \beta_j)^2 + v^2]^{1/2}$$

and

$$\arg P(\lambda) = \sum_{j=1}^{m/2} \arctan\left(\frac{2\mu v + b_j v}{\mu^2 - v^2 + b_j\mu + c_j}\right) + \sum_{j=m+1}^{n} \arctan\left(\frac{v}{\mu + \beta_j}\right)$$

with $0 \leq \arctan x < \pi$. From the contour C in Fig. 1.1 along γ_1, $\lambda = -iv$, $0 \leq v \leq \pi/r$, so the above equations become

$$|P(-iv)| = \prod_{j=1}^{m/2} [(c_j - v^2)^2 + (b_j v)^2]^{1/2} \prod_{j=m+1}^{n} (v^2 + \beta_j^2)^{1/2} \tag{3.1}$$

and

$$\arg P(-iv) = \sum_{j=1}^{m/2} \arctan\ -b_j v/(c_j - v^2) + \sum_{j=m+1}^{n} \arctan(-v/\beta_j), \tag{3.2}$$

with $0 \leq \arctan x < \pi$. As in Theorem 1, arg $P(-iv)$ is monotonically decreasing as λ traverses γ_1 in the counterclockwise direction; however, it is no longer true that $|P(-iv)|$ increases monotonically. The simple bifurcation result in Theorem 1 followed from the monotonic property of $|P(-iv)|$. The example in Sec. 4 shows the more complex nature of the bifurcations in the present case.

To prove Theorem 2 a technique similar to the proof of Theorem 1 can be used. (HI) implies that whenever arg $P(i\hat{v}) = (2n-1)\pi$ for n an integer, then $|P(-i\hat{v})| > \alpha$. If this is not the case, then for some $\hat{v}$ where $|P(-i\hat{v})| \leq \alpha$, the origin lies between $P(-i\hat{v})$ and $F(-i\hat{v}, 0)$ ($F(-i\hat{v}, 0)$ at the origin for equality) and the orientation of F compared to the orientation of P relative to the origin is different. In particular, as P does not encircle the origin as a contour C' enclosing the right half-plane is traversed, then $F(\lambda, 0)$ must encircle the origin as λ traverses C', contradicting (HI). Combining this result with the monotonic property of arg $P(-iv)$ and the hypothesis that $\alpha > \left(\prod_{j=1}^{m/2} c_j\right)\left(\prod_{j=m+1}^{n} \beta_j\right)$, there exist $v_1 < v_2 < \ldots < v_k$ such that $0 > \arg P(-iv_k) > -\pi$ and $|P(-iv_j)| = \alpha, j = 1, \ldots, k$. (It is conjectured that $1 \leq k \leq 3$.) $F(\lambda, r_j)$ passes through the origin whenever the delay r_j is such that

$$r_j = \frac{\pi + \arg P(-iv_j)}{v_j}, j = 1, \ldots, k. \tag{3.3}$$

Note that $r_1 > r_2 > \cdots > r_k$ by the monotonicity of v_j and arg $P(-iv_j)$.

If $r > r_1$, then an argument similar to the one in the proof of Theorem 1 gives $v_0 < v_1$ where v_0 is the first alignment of F, P and the origin. v_1 is the smallest value v for which $|P(-iv)| = \alpha$ and by assumption $|P(0)| < \alpha$. By continuity of $|P(-iv)|$ it follows that for $v_0 < v_1$, $|P(-iv_0)| < \alpha$. Hence the origin lies between $F(-iv_0, r)$ and $P(-iv_0)$. As in the proof of Theorem 1 or in [5], this alignment condition gives $F(\lambda, r)$ a different orientation relative to the origin when compared to $P(\lambda)$, resulting in the encirclement of the origin at least twice (from symmetry) in the image plane under $F(\lambda, r)$ as λ traverses the contour C. Thus for $r > r_1$ at least two roots of $F(\lambda, r) = 0$ lie inside the contour C, i.e., at least two roots λ have Re $\lambda > 0$ and $|\text{Im } \lambda| < \pi/r$.

Because $|P(-iv)|$ is not monotonic it is possible that eigenvalues cross the imaginary axis for values of the delay $r < r_1$. From (HI) and by continuity it follows that for some interval $0 \leq r < r_*$ the system (1.1) is stable. This is easily shown by the fact that $|P(-iv)| > \alpha$ for v in some neighborhood of v_n with arg $P(-iv_n) = (2n+1)\pi$, $n = -1, -2, \ldots, -k$; hence for sufficiently small delays $F(\lambda, r)$ has the same orientation as $P(\lambda)$ relative to origin as λ traverses any contour C' in the right half-plane; i.e., there are no roots of $F(\lambda, r) = 0$ in the right half-plane. If $r_* = r_1$ there is the bifurcation only at r_1; however, if $r_* < r_1$ there may be disconnected regions of stability (see Sec. 4) or it is possible that there are roots of $F(\lambda, r) = 0, r < r_1$ with Re $\lambda > 0$ and $|\text{Im } \lambda| > \pi/r$.

To prove Theorem 3 the relative positions of $P(-iv)$, $F(-iv, r)$ and the origin are determined in a manner similar to the techniques found above. To obtain alignment of $P(-iv)$, $F(-iv, r)$ and the origin the following condition must hold:

$$\arg\,[F(-iv^*, r) - P(-iv^*)] - \arg P(-iv^*) = \pi, \tag{3.4}$$

where $\arg[F(-iv^*, r) - P(-iv^*)] = v^*r + \arg Q(-iv^*)$ with $0 < v^* < \pi/r$. Clearly if $\arg Q(-iv) < \arg P(-iv)$ for $0 < v < \pi/r$, then this condition cannot hold.

Along γ_1, $\arg Q(-iv) = \sum_{j=1}^{m} \arctan(-v/\alpha_j)$ and $\arg P(-iv) = \sum_{j=1}^{n} \arctan(-v/\beta_j)$. Considering the Maclaurin series expansion for the inverse tangent function,

$$\arctan x = x - x^3/3 + x^5/5 - \ldots, \; |x| < 1,$$

along with the hypothesis that $\sum_{j=1}^{n} (1/\beta_j) > \sum_{j=1}^{m} (1/\alpha_j)$, then if $r \geq r_1$ for some r_1 sufficiently large, $0 < v < \pi/r$ and the higher-order terms in the expansion can be ignored. It follows that $\arg Q(-iv) > \arg P(-iv)$ for $0 < v < \pi/r$, $r \geq r_1$. Therefore there exists a v^* such that (3.4) holds with $0 < v^* < \pi/r$.

From the condition $\prod_{j=1}^{n} \beta_j < \prod_{j=1}^{m} \alpha_j$ it follows that for $r \geq r_2$ where r_2 is sufficiently large, $|Q(-iv)| > |P(-iv)|$, $0 < v < \pi/r$. Let $r_0 = \max\{r_1, r_2\}$; then for $r \geq r_0$, $P(-iv^*)$, $F(-iv^*, r)$ and the origin align with the origin between $P(-iv^*)$, and $F(-iv^*, r)$ for some $0 < v^* < \pi/r$ (where v^* depends on r). The remainder of the proof is similar to the proofs of the previous theorems, giving at least two encirclements of the origin by $F(\lambda, r)$ as λ traverses C. Thus $F(\lambda, r) = 0$ has at least two roots with $\operatorname{Re} \lambda > 0$ and $|\operatorname{Im} \lambda| < \pi/r$.

4. Example. In this section a specific example will be used to illustrate how the contour C can be used for stability analysis. The example chosen has a disconnected region of stability with respect to the parameter r. Though the technique below is illustrated on a specific problem, it can be extended to more general problems. The example was derived by a geometric interpretation of the function $|P(-iv)|$ as being the product of the distances from the point $-iv$ on the imaginary axis to the solutions of $P(\lambda) = 0$ in the complex plane. The solutions of $P(\lambda) = 0$ lie in the left half-plane. If they are near the imaginary axis a minimum of $|P(-iv)|$ will occur for $-iv$ near the solution.

The example that will be studied is the delay system (1.1) where

$$A = \begin{pmatrix} -3/2 & 0 & 5 \\ 4 & 0 & -19 \\ 0 & 1/4 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 & 3.9 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The resulting characteristic equation is given by

$$F(\lambda, r) = \lambda^3 + (3/2)\lambda^2 + (19/4)\lambda + (17/8) + 3.9e^{-\lambda r} = 0$$

where $P(\lambda) = (\lambda + 1/2)((\lambda + 1/2)^2 + 4)$ and $\alpha = 3.9$.

First the locations of the bifurcations are found by solving $|P(-iv_j)| = \alpha$. This can be done numerically by calculating $|P(-iv)|$ as v increases from zero. If the location of the roots of $P(\lambda) = 0$ are known then consider $|P(-iv)|$ for $0 \leq v \leq \max_{\lambda_j} |\operatorname{Im} \lambda_j| = M$ with $P(\lambda_j) = 0$. All local minima of $|P(-iv)|$ must occur in this range, as seen from Eq. (2.2). For $v > M$, $|P(-iv)|$ is monotonically increasing. By using this information local minima were found for the above example at $v = 0$ with $|P(0)| = 2.125$ and $v \doteq 1.75$ with $|P(-1.75i)| = 3.849$.

By assumption, $\alpha > 2.125$. If α were less than 3.849, then $|P(-iv)| = \alpha$ at only one value v and there would be a single bifurcation. There is also a local maximum at $v \doteq 1.32$ with $|P(-1.32i)| = 4.00$. If α were greater than 4.00 again there would be a single bifurcation.

When $\alpha = 3.9$ a more complicated bifurcation behavior is observed. $|P(-iv_j)| = 3.9$ for $v_1 \doteq 1.09$, $v_2 \doteq 1.60$, and $v_3 \doteq 1.87$. At these values of v_j $\arg P(-iv_j)$ has values approximately equal to -1.48, -2.03, and -2.49 radians respectively. Eq. (3.3) gives delays

$r_1 = 1.52$, $r_2 = .7$, and $r_3 = .35$. At each of the values r_j, $j = 1, 2, 3$, a Hopf bifurcation occurs. Hence the delay system (1.1) with the above choices of A and B is stable for delays in the intervals $[0, .35)$ and $(.7, 1.52)$ and unstable for delays in the intervals $(.35, .7)$ and $(1.52, \infty)$.

The mapping $F(\lambda, r)$ was plotted for several values of r as λ traversed the contour C in Figure 1.1 with $\mu^* = 10$. The figures below were drawn with a Calcomp plotting routine showing both $P(\lambda)$ and $F(\lambda, r)$. For better visual clarity the y-axis scale is twice that of the

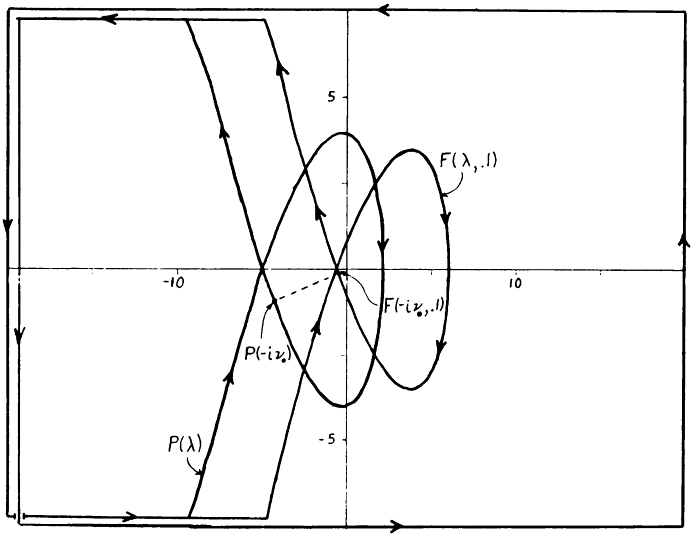

FIG. 4.2(a). $r = .1$.

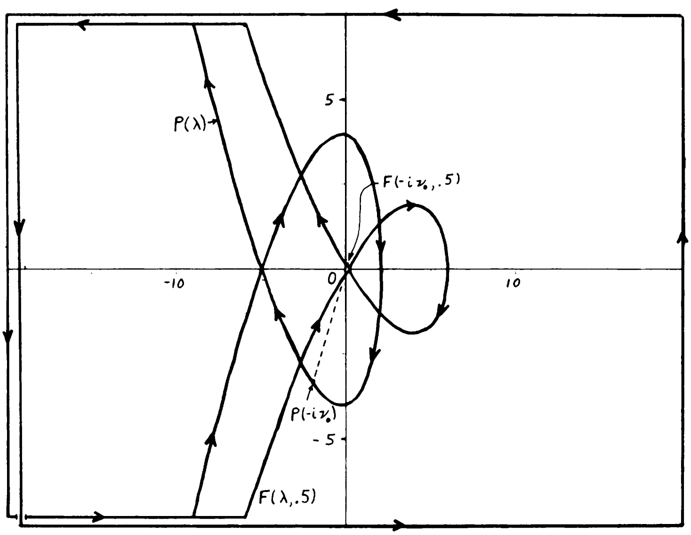

FIG. 4.2(b). $r = .5$.

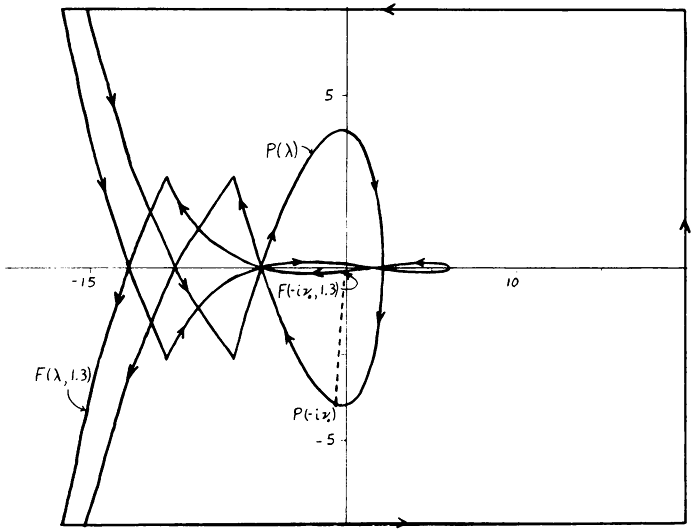

FIG. 4.2(c). $r = 1.3$.

x-axis. The plots show only the behavior of the image graphs near the origin. The directions of the image graphs away from the origin are shown on the perimeters of Figs. 4.2(a–e). Included in the figures is the position of the first alignment of $P(-iv)$, $F(-iv, r)$ and the origin. This alignment is derived from the numerical output of the program also used to define the plot.

Figs. 4.2(a–e) show the effects of increasing r. In Figs. 4.2(a, c), $F(\lambda, r)$ does not encircle the origin as λ traverses C; hence the system (1.1) with the above matrices A and B is stable for these values of r. In Figs. 4.2(b, e), $F(\lambda, r)$ encircles the origin twice as λ traverses C;

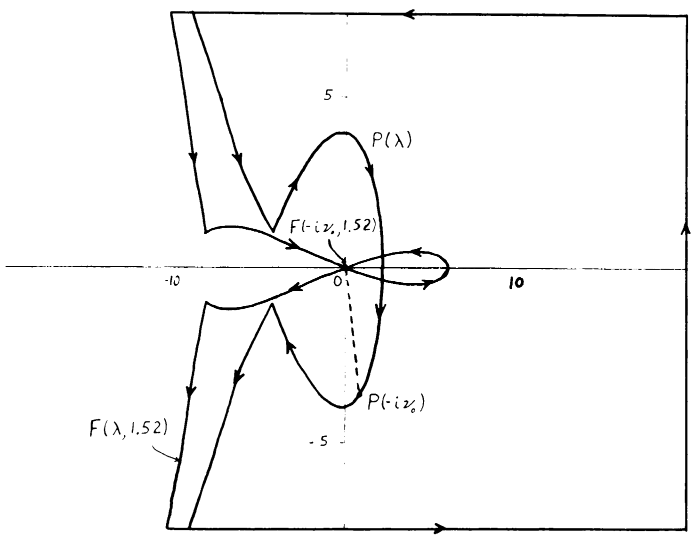

FIG. 4.2(d). $r = 1.52$.

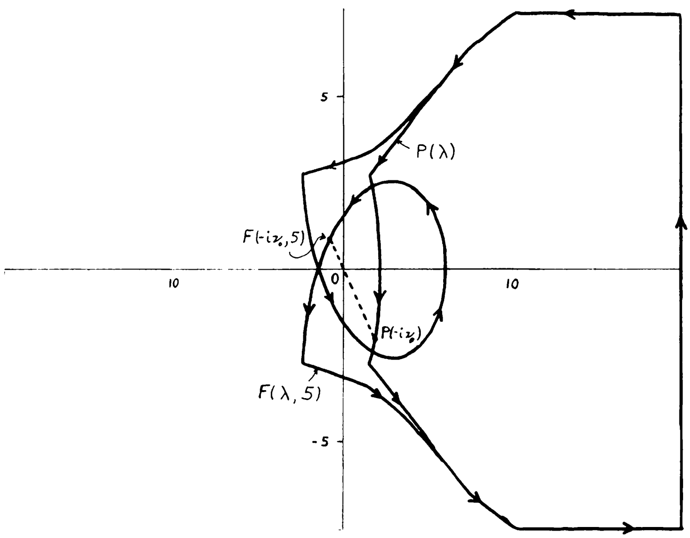

FIG. 4.2(e). $r = 5$.

hence two roots of $F(\lambda, r) = 0$ lie inside C for these values of r. Fig. 4.2d illustrates one of the three Hopf bifurcations in this example. $F(\lambda, r)$ passes through the origin at the eigenvalues $\lambda \doteq \pm 1.09i$ where $r \doteq 1.52$.

5. Conclusion. Determining stability of linear delay differential equations is more difficult than for ordinary differential equations. The delay system's associated eigenvalues are found by evaluating the characteristic equation, an exponential polynomial. The Routh-Huriwitz criterion gives an easy test for determining if an ordinary differential equation has any associated eigenvalues in the right half of the complex plane, implying an instability of the system. However, for differential delay systems there are no such easy tests. The theorems above provide simple hypotheses to determine whether the linear delay system can become unstable as the delay increases. Moreover, the example of Sec. 4 and the proofs of the theorems demonstrate how to use the contour C of Fig. 1.1 and the argument principle to determine stability of the system.

When the system is unstable the technique involving the contour C also gives bounds on the imaginary parts of the eigenvalues. These bounds are important in the study of nonlinear problems. One technique of proving the existence of periodic solutions to nonlinear differential delay equations involves the use of a fixed-point theorem of Nussbaum [6]. To apply this theorem Hale [3, p. 267] and Mahaffy [5] consider the linearization about an equilibrium solution of the nonlinear problem. The bounds on the imaginary part of the eigenvalues are used to demonstrate the property of ejectivity in Nussbaum's theorem, thus proving the existence of a non-constant periodic orbit.

The geometric nature of the arguments given in the proofs of the theorems provides a technique for future research on related problems. The contour C of Fig. 1.1 would be useful for specific problems not covered by the theorems in Sec. 2. Also, modifications of the contour C would give easy tests for stability without the stringent hypotheses of the theorems of Sec. 2.

The arguments are geometric and also computationally easy, as shown by the example in Sec. 4. The characteristic equation is analyzed by being split into its polynomial part and exponential part. These two components of the characteristic equation are compared by relating their magnitudes and arguments in the complex plane. With the parameter r fixed, the position of the exponential part of the characteristic equation is compared to that of the polynomial to give their relative orientations to the origin. By an application of the argument principle this comparison provides a simple test for stability. The computational aspect of this problem is tractable, as the purely imaginary part of C is bounded; hence stability questions for a specific problem can be answered.

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