# THE STEFAN PROBLEM WITH A CONVECTIVE BOUNDARY CONDITION*

By

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**Abstract.** We study the one-phase Stefan problem on a semi-infinite strip $x \geq 0$, with the convective boundary condition $-KT_x(0, t) = h[T_L - T(0, t)]$. Points of interest include: a) behavior of the surface temperature $T(0, t)$; b) asymptotic behavior as $h \to \infty$; c) uniqueness, and d) bounds on the phase change front and total system energy.

**Introduction.** In this paper we study the following problem:
*Problem* I. Find $X(t)$ and $T(x, t)$ such that

$$X(t) \text{ is Lipschitz-continuous for } t \geq 0; \tag{1.1}$$

$$X'(t) \text{ is continuous for } t > 0; \tag{1.2}$$

$$T(x, t) \text{ is continuous for } t > 0 \text{ and } 0 \leq x \leq X(t); \tag{1.3}$$

$$T_t(x, t),\ T_{xx}(x, t) \text{ are continuous for } t > 0 \text{ and } 0 < x < X(t); \tag{1.4}$$

$$-\infty < \liminf_{x, t \to 0} T(x, t),\ \limsup_{x, t \to 0} T(x, \tau) < \infty; \tag{1.5}$$

$$T_x(x, t) \text{ is continuous for } t > 0,\ 0 \leq x \leq X(t); \tag{1.6}$$

$X(t)$ and $T(x, t)$ obey the conditions

$$T_t(x, t) = \alpha T_{xx}(x, t),\ t > 0,\ 0 < x < X(t), \tag{1.7}$$

$$T(x, t) \equiv T_{cr},\ t > 0,\ x \geq X(t), \tag{1.8}$$

$$X(0) = 0, \tag{1.9}$$

$$\rho H X'(t) = -KT_x(X(t), t), \tag{1.10}$$

$$-KT_x(0, t) = h[T_L - T(0, t)],\ t > 0. \tag{1.11}$$

Here $\alpha, \rho, H, K, h$ are positive constants, $T_L$ and $T_{cr}$ are constants, and $T_L > T_{cr}$.

Eqs. (1.7–1.11) describe melting of a material due to convective heat transfer from a fluid with ambient temperature $T_L$ flowing across the face at $x = 0$. The parameters are:

$\alpha$ = material thermal diffusivity ($m^2/s$),
$K$ = material thermal conductivity (KJ/m − s − °C),

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$\rho$ = material density (Kg/m$^3$),
$H$ = latent heat of melting (KJ/Kg),
$T_{cr}$ = material melting temperature (°C),
$T_L$ = ambient fluid temperature (°C),
$h$ = fluid to material surface heat transfer coefficient (KJ/m$^2$ − s − °C).

We will also use

$$\Delta T = T_L - T_{cr} \text{ (°C)},$$

and the material specific heat

$$c = \text{specific heat (KJ/Kg − °C)}.$$

Of course $\alpha = K/c\rho$. The melting front at time $t$ is at $x = X(t)$ while $T(x, t)$ is the temperature at position $x$ and time $t$.

It is known [3] that a solution to Problem I exists. While a number of papers in the heat transfer literature are devoted to various approximations pertinent to this problem [5, 8–10], the only studies of the qualitative behavior of its solution concern existence and smoothness ([7], [12], in addition to [3]). We will study the qualitative behavior of a solution, focusing on questions pertinent to the melting (or solidification) problem from which it arises. These include

*Question* 1. How do $T(x, t)$, $X(t)$ behave at $t = 0$?
*Question* 2. How does the surface temperature $T(0, t)$ vary with $t$?
*Question* 3. What happens as $h \to \infty$?

On physical grounds it would be expected that the surface temperature $T(0, t)$ would tend to $T_{cr}$ as $t \to 0^+$, and to the fluid temperature $T_L$ as $t \to \infty$. Similarly, (1.10) and (1.11) would lead us to conjecture that $X'(0^+)$ exists and is given by

$$X'(0^+) = h(T_L - T_{cr})/\rho H.$$

The situation whereby $h \to \infty$ could arise from a greater flow rate for the transfer fluid at $x = 0$ [6], in which case we would expect that $T(0, t) \to T_L$; in this case we would also anticipate that the solution to problem I should tend to that of the problem with (1.11) replaced by

$$T(0, t) = T_L, \ t > 0. \tag{1.12}$$

This latter problem (1.1)–(1.10), (1.12), will be referred to as Problem II, and its exact solution is given by

$$X_\infty(t) = 2\lambda\sqrt{\alpha t}, \qquad t \geq 0, \tag{1.13}$$

$$T_\infty(x, t) = T_L - \Delta T \operatorname{erf}(x/2\sqrt{\alpha t})/\operatorname{erf} \lambda \tag{1.14}$$

with $\lambda$ the root of

$$\lambda e^{\lambda^2} \operatorname{erf} \lambda = \mathrm{St}/\sqrt{\pi}. \tag{1.15}$$

Here St is the so-called "Stefan number", indicating the ratio of sensible to latent heat [11], and given by

$$\mathrm{St} = c\Delta T/H.$$

Our aim is to establish these claims. To do this we use a number of moment-type relations as well as the maximum principle. These are discussed in Sec. 2. In Sec. 3 we address questions 1 and 2; what happens as $h \to \infty$ is examined in Sec. 4. We close in Sec. 5, with upper and lower bounds on the total heat in the material. In the Appendix we prove a form of the maximum principle which we use.

**2. Preliminaries.** The maximum principle for the heat equation is normally used in two forms [4]. The first asserts that a solution to the heat equation cannot attain its greatest or least value at an interior point $P_0$ of a domain unless it equals that value at all points influencing $P_0$. The second, due to Friedman, concerns the behavior of a nontangent temperature derivative at a boundary point. As stated in [4] it presents some difficulty due to the assumed "strong-sphere" property of the boundary. For this reason we use the following version of the maximum principle suggested by a result of Vyborny [13].

THEOREM 1. *Corner Point Maximum Principle.* Let $D$ be a simply connected domain in the $x, t$ plane and $P_0 = (x_0, t_0)$ a point of its boundary. Let $N$ be the disk

$$N = \{(x, t) \mid (x - x_0)^2 + (t - t_0)^2 < \delta^2\}.$$

Set

$$G^\circ = D \cap N \cap \{(x, t) \mid t < t_0\}, \qquad \hat{G}^\circ = \bar{G}^\circ - \partial D.$$

Suppose that $u \in C(\bar{D})$, $u_x, u_t, u_{xx} \in C(D)$, and

$$Lu = u_t - \alpha u_{xx} \leq 0 \tag{2.1}$$

in $D$. Furthermore, let

$$u(P) < u(P_0) \qquad \text{for} \qquad P \in \hat{G}^\circ, \tag{2.2}$$

$$u(P) \leq u(P_0) \qquad \text{for} \qquad P \in \partial D \cap N, \tag{2.3}$$

and suppose that $\partial D \cap N$ is a $C^1$ curve representable as $x = X(t)$. Then

$$\varlimsup_{\substack{P \to P_0 \\ P \in \hat{G}^\circ}} \frac{u(P) - u(P_0)}{|P - P_0|} < 0$$

where $P$ tends to be $P_0$ in any nontangential direction.

The proof of this theorem is given in the Appendix.

COROLLARY 1. If all of the conditions of Theorem 1 hold except for (2.2), (2.3), and if

$$\varlimsup_{\substack{P \to P_0 \\ P \in \hat{G}^\circ}} (u(P) - u(P_0)/|P - P_0|) \geq 0,$$

then either

a) there exist points $P = (x, t)$ in $G^\circ$ arbitrarily close to $(x_0, t_0)$, with $t \leq t_0$, for which $u(P) \geq u(P_0)$

or

b) there exist points $P$ on $\partial D \cap N$ arbitrarily close to $P_0$ for which $u(P) > u(P_0)$.

Reversing the inequalities in Theorem 1 and Corollary 1 yields the corresponding corner-point minimum principle.

We will use a number of integral relations satisfied by a solution $X(t)$, $T(x, t)$ to Problem I. From the continuity of $T_x(x, t)$ in any region $t \geq \tau > 0$ we find

$$\int_\tau^t h[T_L - T(0, t')]dt' = \rho H[X(t) - X(\tau)]$$

$$+ \int_0^{X(t)} c\rho[T(x, t) - T_{cr}]dx - \int_0^{X(\tau)} c\rho[T(x, \tau) - T_{cr}]dx, \tag{2.4}$$

$$\int_0^{X(t)} c\rho x[T(x, t) - T_{cr}]dx - \int_0^{X(\tau)} c\rho x[T(x, \tau) - T_{cr}]dx$$

$$+ (\rho H/2)\,[X^2(t) - X^2(\tau)] = K \int_\tau^t [T(0, t') - T_{cr}]dt', \tag{2.5}$$

$$\int_0^{X(t)} (c\rho/2)\,[T(x, t) - T_{cr}]^2 dx - \int_0^{X(\tau)} (c\rho/2)\,[T(x, \tau) - T_{cr}]^2 dx$$

$$+ \int_\tau^t \int_0^{X(t')} K\ T_x(x, t')^2 dx dt' = \int_\tau^t h[(T_L - T(0, t')][T(0, t') - T_{cr}]]dt'.$$

For example, (2.4) is derived as follows. Let $\theta$ be any value between 0 and 1/2. Consider the closed domain

$$D_0 = \{(x', t') \mid \tau \leq t' \leq t,\ \theta X(t') \leq x' \leq (1 - \theta)X(t')\} \tag{2.6}$$

By the conditions (1.1)–(1.11) we find

$$(d/dt') \int_{\theta X(t')}^{(1-\theta)X(t')} c\rho[T(x', t') - T_{cr}]dx' = (1 - \theta)X'(t')c\rho[T[(1 - \theta)X(t'), t'] - T_{cr}]$$

$$- \theta X'(t')c\rho[T[\theta X(t'), t] - T_{cr}] + KT_x[(1 - \theta)X(t'), t'] - KT_x[\theta X(t'), t'].$$

Integrating this equation with respect to $t'$ on $[\tau, t]$ and letting $\theta \to 0$ yields (2.4). Relations (2.5), (2.6) are derived similarly. From the boundedness of $T(x, t')$ and the fact that $X(t) \to 0$ as $t \to 0^+$, we conclude that in (2.4)

$$\int_0^{X(\tau)} c\rho[T(x, \tau) - T_{cr}]dx \to 0$$

as $\tau \to 0$, whence

$$\int_0^t h[T_L - T(0, t')]dt' = \rho HX(t) + \int_0^{X(t)} c\rho[T(x, t) - T_{cr}]dx, \tag{2.7}$$

which is the overall heat balance relation on [0, t]. In the same way (2.5) implies

$$\int_0^{X(t)} c\rho x[T(x, t) - T_{cr}]dx + (\rho H/2)\ X(t)^2 = K \int_0^t [T(0, t') - T_{cr}]dt'. \tag{2.8}$$

Consider (2.6). By elementary calculus

$$[T_L - T(0, t')][T(0, t') - T_{cr}] \leq (1/4)\,(T_L - T_{cr})^2.$$

Hence

$$\int_0^{X(t)} (c\rho/2)\,[T(x,t) - T_{cr}]^2 dx - \int_0^{X(\tau)} (c\rho/2)\,[T(x,\tau) - T_{cr}]^2 dx + \int_\tau^t \int_0^{X(t')} K T_x(x,t')^2 dx\,dt' \le [h(T_L - T_{cr})^2/4](t - \tau). \tag{2.9}$$

Letting $\tau \to 0$ we conclude that

$$\int_0^{X(t)} (c\rho/2)\,[T(x,t) - T_{cr}]^2 dx + \int_0^t \int_0^{X(t')} K T_x(x,t')^2 dx\,dt' \le [h(T_L - T_{cr})^2/4]\,t,$$

and, in particular, that

$$\int_0^t \int_0^{X(t')} K T_x(x,t')^2 dx\,dt' \le t[h(T_L - T_{cr})^2/4]. \tag{2.11}$$

**3. The qualitative behavior of a solution.** We now address the qualitative behavior of a solution to Problem I for a fixed $h > 0$.

THEOREM 2. The phase boundary $X(t)$ solving Problem I is always positive: $X(t) > 0$ for $t > 0$.

*Proof.* Since $X(t) \ge 0$ for all $t \ge 0$, a point $t_0 > 0$ for which $X(t_0) = 0$ must be a zero of the (continuous) derivative $X'(t)$. However, we would then have $T(0, t_0) = T_{cr}$, whence

$$\begin{aligned} 0 = \rho H X'(t_0) &= -K T_x[X(t_0), t_0] \\ &= -K T_x(0, t_0) = h(T_L - T_{cr}) \ne 0 \end{aligned}$$

and the theorem is proved.

THEOREM 3. $T_{cr} \le T(x,t) < T_L$ for $t > 0$ and $0 \le x \le X(t)$.

Our proof rests upon the following lemma, which asserts that $T(x,t)$ cannot be bounded away from $T_{cr}$ in a neighborhood of the origin.

LEMMA 1. Let $t_0 > 0$ be given. There is no function $x = x^*(t)$ satisfying the following conditions on $(0, t_0]$: a) $0 \le x^*(t) < X(t)$; b) $0 < \omega \le |\,T[x^*(t), t] - T_{cr}|$ for some $\omega$.

*Proof of Lemma* 1. Roughly speaking, we will see that if $T(x,t)$ is bounded away from $T_{cr}$, then $T_x(x,t)$ must grow in a manner inconsistent with the bound (2.11).

For suppose that $x^*(t)$ satisfies (a) and (b), and let $t \in (0, t_0]$. Since $T_x(x,t)$ is continuous on $[0, X(t)]$,

$$|\,T_{cr} - T[x^*(t), t]\,| \le \int_{x^*(t)}^{X(t)} |\,T_x(x,t)\,|\,dx \le \{X(t) \int_0^{X(t)} T_x(x,t)^2 dx\}^{1/2}$$

or by (b),

$$\omega^2 \le X(t) \int_0^{X(t)} T_x(x,t)^2 dx.$$

Integration over $[t/2, t]$ for any $t \le t_0$ yields

$$(\omega^2 t/2) \le \int_{t/2}^t X(t') \int_0^{X(t')} T_x(x,t')^2 dx\,dt'.$$

By the generalized mean value theorem,

$$(\omega^2 t/2) \le X(t^*) \int_{t/2}^{t} \int_0^{X(t')} T_x(x, t')^2 dx\, dt'$$

where $t^* \in [t/2, t]$. Now by (2.11) we obtain

$$(\omega^2 t/2) \le X(t^*)\ (ht\Delta T^2/4)$$

or

$$2\omega^2 \le X(t^*)\ h(T_L - T_{cr})^2.$$

This contradicts $X(t^*) \to 0$ as $t \to 0$ and thus proves the lemma.

*Proof of Theorem* 3. We begin by showing that $T(x, t)$ must be less than $T_L$ for all points $(x, t)$ with $t \ge 0$, $0 \le x \le X(t)$. Suppose that $T(x_0, t_0) \ge T_L$ for some $t_0 > 0$, and $x_0 \in [0, X(t_0)]$.

*Claim*: For each $t_1 \in (0, t_0)$ there is some $x^* = x^*(t_1) \in [0, X(t)]$ such that $T[x^*(t_1), t_1] \ge T_L$.

Since this directly contradicts Lemma 1 we need only establish this claim. Fix $t_1 \in (0, t_0)$ and let

$$S = \{t\colon t \in [t_1, t_0], \qquad T(0, t) \ge T_L\}.$$

If $x_0 = 0$ then $t_0 \in S$ and $S$ is not empty. If $x_0 > 0$ then, by the strong maximum principle [4] applied to $D_1 = \{(x, t)\colon 0 \le x \le X(t), t_1 \le t \le t_0\}$, $T(x, t)$ must exceed $T(x_0, t_0) \ge T_L$ somewhere on its parabolic boundary. If this occurs at some point $(x, t_1)$, $x \in [0, X(t_1)]$ the claim is proved. If it occurs on $x = 0$, i.e., for some $(0, t^*)$ with $t^* \in [t_1, t_0]$, then $t^* \in S$, so again $S$ is not empty. Let

$$t^{**} = \inf S.$$

Suppose $t^{**} > t_1$. Then $T(0, t_1) < T_L$ while $T(0, t^{**}) = T_L$, whence $-KT_x(0, t^{**}) = 0$ and by Corollary 1 to the corner point maximum principle either there exist points $(x, t)$ arbitrarily close to $(0, t^{**})$ with $t \le t^{**}$ for which $T(x, t) \ge T_L$, or there exists some $t < t^{**}$ for which $T(0, t) > T_L$. Either possibility violates the definition of $t^{**}$ and thus $t^{**} = t_1$. Hence $T(0, t_1) \ge T_L$, and $x^*(t_1) = 0$. Thus our claim is proved and $T(x, t) < T_L$. The proof that $T(x, t) \ge T_{cr}$ is carried out in a similar way, as we see by assuming that

$$T(x_0, t_0) \le T_{cr} - \omega$$

for $\omega > 0$ and some point $(x_0, t_0)$, $x_0 \in [0, X(t_0))$.

By the strong maximum principle we now have:

COROLLARY 2. $T(x, t) > T_{cr}$ for $t > 0$, $x \in (0, X(t))$.
This result implies that at any point $(X(t), t)$, $T(x, t)$ assumes a strictly minimum value relative to points to its left. Hence by the corner point minimum principle $\rho H X'(t) = -KT_x(X(t), t) > 0$ and we have

COROLLARY 3. $X'(t) > 0$ for $t > 0$.

We will now use the moment-type relations of Sec. 2 to derive upper bounds on $X(t)$.

THEOREM 4. For any $t > 0$ the phase boundary of Problem I obeys the conditions:

$$X(t) \le f_1(t) = ht\Delta T/\rho H, \tag{3.1}$$

$$X(t) \le f_2(t) = \{2Kt\Delta T/\rho H\}^{1/2}, \tag{3.2}$$

$$X(t) \le \begin{cases} f_1(t), & t \le t^* \\ f_2(t), & t \ge t^* \end{cases} \tag{3.3}$$

with

$$t^* = 2K\rho H/h^2\Delta T. \tag{3.4}$$

*Proof*: From Theorem 3, $T(x, t) \ge T_{cr}$, whence (2.7) implies (3.1). Similarly $T(0, t) < T_L$ whence from (2.8), $\frac{1}{2}\rho H X(t)^2 \le Kt\Delta T$, or (3.2) is proved.

By a straightforward calculation we see that $f_1(t) \le f_2(t)$ for $t \le t^*$, and $f_1(t) \ge f_2(t)$ for $t \ge t^*$, whence (3.3) holds.

Note that the bound (3.3) indicates an initial linear growth in the phase front, followed by growth as $t^{1/2}$.

THEOREM 5. $T(x, t)$ is nondecreasing in $t$; that is, $T(x, t + \Delta t) \ge T(x, t)$ for all $x \in [0, X(t)]$, $\Delta t > 0$.

*Proof.* The concept of the proof is to show that the first forward difference of $T(x, t)$ in $t$, namely

$$v(x, t, \Delta t) = T(x, t + \Delta t) - T(x, t)$$

is never negative for any choice of $\Delta t > 0$. To do this we note first that $v(x, t, \Delta t)$ is defined and satisfies the heat equation for $t > 0$, $x \in [0, X(t)]$. Moreover, by Corollary 2 and 3,

$$v(X(t), t, \Delta t) > 0.$$

At $x = 0$

$$K\, v_x(0, t, \Delta t) = hv(0, t, \Delta t).$$

Suppose now that $v(x, t, \Delta t)$ is negative at some point $(x_0, t_0)$:

$$v(x_0, t_0, \Delta t) \le -\omega < 0.$$

By an identical argument to that used in proving Theorem 3 we conclude that for each $0 < t < t_0$ there is a point $x^* = x^*(t)$ for which

$$v(x^*(t), t, \Delta t) \le -\omega.$$

Hence

$$|v(x^*(t), t, \Delta t) - v(X(t), t, \Delta t)| > \omega$$

for all $t \in (0, t_0)$. However, we may now apply the argument used in proving Lemma 1 to show that this violates (2.11) and the theorem is proved.

COROLLARY 4. $T(x, t) \to T_{cr}$ as $x, t \to 0^+$.

*Proof.* By the Theorem, $T(x, t)$ is nonincreasing for $t \to 0^+$. But then by Lemma 1 it cannot be bounded away from $T_{cr}$, whence it must tend to $T_{cr}$ as $x, t \to 0^+$. Thus, we can now extend $T(x, t)$ continuously to $(0, 0^+)$ and define it for $t \geq 0$, $x \in [0, X(t)]$ as a continuous function.

An immediate implication of Theorem 5 is that for $t > 0$, $x \in (0, X(t))$,

$$\alpha T_{xx}(x, t) = T_t(x, t) \geq 0. \tag{3.5}$$

This in turn implies the following theorem.

THEOREM 6. Let $q(x, t) = -KT_x(x, t)$ for $t > 0$, $x \in [0, X(t)]$. Then

$$\rho H X'(t) \leq q(x, t) \leq h[T_L - T(0, t)] \leq h\Delta T. \tag{3.6}$$

*Proof.* For any $\theta \in (0, 1/2)$

$$T_x(x, t) - T_x(\theta X(t), t) = \int_{\theta X(t)}^{x} T_{xx}(x', t)\, dx' \geq 0$$

whence

$$-KT_x(x, t) \leq -KT_x(\theta X(t), t);$$

letting $\theta \to 0$ and using the continuity of $T_x(x, t)$ on the closed $x$-interval, we have

$$q(x, t) = -KT_x(x, t) \leq -KT_x(0, t) = h[T_L - T(0, t)].$$

The second inequality of (3.6) is proved in the same manner.

The key difficulty in understanding the convective boundary condition lies in the variability of the surface temperature $T(0, t)$. We will now obtain a bound on it describing its long-term behavior.

THEOREM 7. For any $t > 0$, the surface temperature $T(0, t)$ of Problem I obeys

$$0 < T_L - T(0, t) \leq (1 + \mathrm{St})(2K\rho H\Delta T)^{1/2}/ht^{1/2}. \tag{3.7}$$

*Proof.* From the heat balance relation (2.7) and the fact that $T(x, t) \in [T_{\mathrm{cr}}, T_L]$,

$$\int_0^t h[T_L - T(0, t')]dt' \leq H\rho X(t)[1 + \mathrm{St}],$$

and using the upper bound (3.2)

$$X(t) \leq \{2Kt\Delta T/\rho H\}^{1/2}$$

we find

$$\int_0^t h[T_L - T(0, t')]dt' \leq \{2K\rho Ht\Delta T\}^{1/2}[1 + \mathrm{St}].$$

However, $T(0, t)$ is nondecreasing for increasing $t$, whence

$$ht(T_L - T(0, t)) \leq \int_0^t h[T_L - T(0, t')]dt'$$

and (3.7) holds true.

COROLLARY 5. $T(0, t) \to T_L$ as $t \to \infty$.

We will now further examine the behavior of $X(t)$, $T(x, t)$ at the origin.

THEOREM 8. $X(t)$ has a right-hand derivative at $t = 0$, given by

$$X'(0^+) = (h\Delta T/\rho H). \tag{3.8}$$

*Proof*: From (2.7)

$$(X(t)/t) = (h/t\rho H) \int_0^t (T_L - T(0, t'))\, dt' - (c/tH) \int_0^{X(t)} (T(x, t) - T_{cr})\, dx.$$

Since $T(0, t)$ is continuous for $t \geq 0$,

$$(1/t) \int_0^t (T_L - T(0, t'))\, dt' = T_L - T(0, \theta t),\ \theta \in [0, 1].$$

Moreover,

$$(c/tH) \int_0^{X(t)} (T(x, t) - T_{cr})\, dx = [cX(t)/Ht]\left\{(1/X(t)) \int_0^{X(t)} (T(x, t) - T_{cr})\, dx\right\}$$

$$= (c/H)\,(X(t)/t)\,(T(x^*(t), t) - T_{cr})$$

for $x^*(t) \in [0, X(t)]$. Hence

$$(X(t)/t)\{1 + (c/H)[T[x^*(t), t] - T_{cr}]\} = (h/\rho H)\,(T_L - T(0, \theta t)).$$

Letting $t \to 0$ yields the asserted result.

COROLLARY 6. $T_t(0, 0^+)$ exists and equals $(h\Delta T)^2/\rho HK$.

*Proof*: For any $t > 0$,

$$\begin{aligned}(T(0, t) - T_{cr}/t) &= (T(0, t) - T(X(t), t)/t \\ &= -(X(t)/t)\ T_x(x^*(t), t), \qquad 0 \leq x^*(t) \leq X(t).\end{aligned}$$

Moreover, from (3.6)

$$(\rho HX'(t)/K) \leq -T_x(x^*(t), t) \leq (h/K)\,(T_L - T(0, t)),$$

and as $t \to 0$ this implies

$$-T_x(0^+, 0^+) = (h\Delta T/K),$$

whence

$$T_t(0, 0^+) = ((h\Delta T)^2/\rho HK).$$

The bound (3.6) on $|T_x|$ is the principal tool needed for proving uniqueness of the solution, using the approach of Douglas [2]. Because of the direct nature of the proof we will merely state

THEOREM 9. The solution to problem I is unique.

**4. Dependence on the heat transfer coefficient.** We now address the question of how the solution to Problem I depends upon $h$. Indeed, from (3.7) of Theorem 7 we can state

THEOREM 10. As $h \to \infty$, $T(0, t) \to T_L$ in a pointwise manner for all $t \to 0$.

Similarly we may assert:

THEOREM 11. The solution to Problem I depends monotonically on $h$. In particular, if $(T^1(x, t), X_1(t))$ and $(T^2(x, t), X_2(t))$ are the solutions to Problem I for $h = h_1, h_2$, respectively and if $h_1 < h_2$, then $X_2(t) > X_1(t)$ for $t > 0$ and $T^2(x, t) > T^1(x, t)$ wherever they are both defined.

*Proof*: From Theorem 8, Corollary 6 and the maximum principle, there is some $t_0 > 0$ such that our assertion is true when $t < t_0$. This is seen by considering the difference

$$v(x, t) = T^2(x, t) - T^1(x, t).$$

at points where they are both defined. Let

$$t^* = \sup\{t \mid T^2(0, t) > T^1(0, t)\}, \qquad t^{**} = \sup(t \mid X_2(t) > X_1(t)).$$

By definition

$$v_x(0, t) = (h_1 - h_2)(T_L - T^2(0, t)) + h_1 V(0, t).$$

Suppose that $t^*, t^{**} < \infty$.

*Claim 1.* $t^* \neq t^{**}$.

Suppose that $t^* = t^{**}$. Then

a) $X_1(t^*) = X_2(t^*)$,

b) $X_1'(t^*) \geq X_2'(t^*)$,

c) $v(X_1(t^*), t^*) = 0$,

while for $t < t^*$, $0 \leq x < X_1(t)$, $v(x, t) > 0$. But by (b), $v_x(X_1(t^*), t^*) \geq 0$, which would contradict the corner minimum principle since

$$v(x, t) > v(X_1(t^*), t^*) \qquad \text{for} \qquad t < t^*, 0 \leq x \leq X_1(t).$$

*Claim 2.* $t^* < t^{**}$ is impossible.

On $[0, t^*]$, $X_2(t) > X_1(t)$ whence $v(X_1(t), t) > 0$. Hence we must have $v(0, t^*) = 0$ with $v(0, t) > 0$ for $t < t^*$. But then $v(0, t^*)$ is a minimum value up to time $t^*$ whence $v_x(0, t^*) > 0$, which contradicts

$$v_x(0, t^*) = (h_1 - h_2)(T_L - T^2(0, t^*)) < 0.$$

*Claim 3.* $t^{**} < t^*$ is impossible, since

$$T^2(X_2(t^{**}), t^{**}) = T^1(X_1(t^{**}), t^{**}) = T_{cr}.$$

Thus Theorem 11 is proved.

The solution to Problem II (see Sec. 1) is given explicitly by [1]

$$X_\infty(t) = 2\lambda\sqrt{(\alpha t)} \tag{4.1}$$

$$T^\infty(x, t) = T_L - (\Delta T/\operatorname{erf}\lambda)\operatorname{erf}(x/2\sqrt{(\alpha t)}) \tag{4.2}$$

where $\lambda$ is the root of

$$\lambda e^{\lambda^2}\operatorname{erf}\lambda = \mathrm{St}/\sqrt{(\pi)}. \tag{4.3}$$

We claim that this solution constitutes an upper bound for that of Problem I, namely

THEOREM 12. Let $h > 0$, and let $X_h(t)$, $T^h(x, t)$ denote the solution to Problem I for this value of the heat transfer coefficient. Then $X_\infty(t) > X_h(t)$ for all $t > 0$, and $T^\infty(x, t) > T^h(x, t)$ for all $(x, t)$ for which both functions are defined.

*Proof*: We note first that since $X_h(t) \le (ht\Delta T/\rho H)$, we find $X_\infty(t) > X_h(t)$ for

$$0 < t < t_0 = (4\lambda^2 K\rho H^2/ch^2\Delta T^2).$$

Moreover, for $t < t_0$, $T^\infty(0, t) = T_L > T^h(0, t)$ and

$$T^\infty(X_h(t), t) > T^h(X_h(t), t) = T_{cr}.$$

Let $t < t_0$ and $x \in [0, X_h(t)]$. Then by the mean value theorem

$$\begin{aligned} T^\infty(x, t) - T^h(x, t) &= (T^\infty(x, t) - T_{cr}) - (T^h(x, t) - T_{cr}) \\ &= (x - X_\infty(t))\, T_x^\infty(x', t) - (x - X_h(t))\, T_x^h(x'', t) \end{aligned}$$

for $x', x'' \in (x, X_h(t))$. But then from (4.2)

$$\begin{aligned} T^\infty(x, t) - T^h(x, t) &= [(X_\infty(t) - x)\Delta T/\sqrt{(\pi\alpha t)}\, \mathrm{erf}\lambda]\, e^{-x'^2/4\alpha t} + \big(X_h(t) - x\big)\, T_x^h(x'', t) \\ &\ge \big(X_\infty(t) - x\big)\, \Delta T e^{-\lambda^2}/\sqrt{(\pi\alpha t)}\mathrm{erf}\lambda - \big(X_h(t) - x\big)(h\Delta T/K) \\ &\ge \big(X_h(t) - x\big)\, \Delta T e^{-\lambda^2}/\sqrt{(\pi\alpha t)}\mathrm{erf}\lambda - (h\Delta T/K) \\ &> 0 \end{aligned}$$

for $t < t_1 = \big(e^{-\lambda^2}K/h\sqrt{(\pi\alpha)}\mathrm{erf}\lambda\big)^2$. It is easily seen that $t_1 < t_0$. Thus for $t < t_1$ the solution to problem II bounds that of problem I.

Let

$$\begin{aligned} t^* &= \sup\{t\colon X_\infty(t) > X_h(t)\}, \\ t^{**} &= \sup\{t\colon T^\infty(x, t) > T^h(x, t)\}, \text{ for } 0 \le x \le \min\big(X_\infty(t), X_h(t)\big). \end{aligned}$$

Let

$$v(x, t) = T^\infty(x, t) - T^h(x, t)$$

where both functions are defined. Suppose that $t^*, t^{**} < \infty$.

*Claim 1*: It is not possible to have $t^{**} < t^*$.

Indeed, suppose that $t^{**} < t^*$. Then $v(x, t)$ would vanish at some point $(x, t^{**})$ for $x \in (0, X_h(t^{**}))$ while it is positive on the line $t = t_1/2$ and at $x = 0$ and $x = X_h(t)$ for $t < t^{**}$, violating the maximum principle.

*Claim 2*: It is not possible to have $t^* < \infty$.

For at $t^*$,

$$\begin{aligned} X_\infty(t^*) &= X_h(t^*), & v\big(X_h(t^*), t^*\big) &= 0, \\ v(x, t) &> 0 \text{ for } t < t^*, & X_h'(t^*) &\ge X_\infty'(t^*) \end{aligned}$$

whence

$$v_x\big(X_h(t^*), t^*\big) \ge 0,$$

and by the corner point maximum principle $v(x, t)$ could not have a minimum at $\big(X_h(t^*), t^*\big)$. Thus the claim is proved, and $t^*, t^{**}$ must be infinite, proving the theorem.

We now assert that as $h \to \infty$ the solution to Problem I converges to that of Problem II.

THEOREM 13. Let $t > 0$. Then as $h \to \infty$,

$$X_h(t) \to X_\infty(t), \qquad T^h(x, t) \to T^\infty(x, t).$$

The proof rests upon the following observation:

LEMMA 2. The relation (2.8)

$$\int_0^{X(t)} c\rho x[T(x, t) - T_{cr}]dx + (1/2)\ \rho H\ X(t)^2 = K \int_0^t [T(0, t') - T_{cr}]dt' \tag{2.8}$$

holds for $X_h$, $T^h$ as well as for $X_\infty$, $T^\infty$.

Indeed, the factor $x$ in the spatial integral prevents the flux at $x = 0$ from entering into the equation. Of course

$$T^\infty(0, t') \equiv T_L, \quad t' > 0.$$

*Proof of Theorem 13.* For any $h > 0$, by Lemma 2,

$$\int_0^{X_\infty(t)} c\rho x[T^\infty(x, t) - T_{cr}]dx + (1/2)\ \rho H\ X_\infty(t)^2 = K \int_0^t [T_L - T_{cr}]dt', \tag{4.4}$$

$$\int_0^{X_h(t)} c\rho x[T^h(x, t) - T_{cr})dx + (1/2)\ \rho H\ X_h(t)^2 = K \int_0^t [T^h(0, t') - T_{cr}]dt'. \tag{4.5}$$

Recalling that $X_\infty(t) > X_h(t)$ and $T^\infty(x, t) > T^h(x, t)$ and subtracting (4.5) from (4.4), we find, using the estimate (3.7) on $(T_L - T^h(0, t))$, that

$$\begin{aligned}\int_0^{X_h(t)} c\rho x[T^\infty(x, t) - T^h(x, t)]dx + \int_{X_h(t)}^{X_\infty(t)} c\rho x[T^\infty(x, t) - T_{cr}]dx \\ + (1/2)\ \rho H[X_\infty(t)^2 - X_h(t)^2] = K \int_0^t (T_L - T^h(0, t'))\ dt' \\ \leq (2K\sqrt{(2K\rho H\Delta T)}(1 + \mathrm{St})/h)\sqrt{(t)},\end{aligned} \tag{4.6}$$

which immediately implies that as $h \to \infty$

$$X_h(t) \to X_\infty(t).$$

Consider the family of functions $\{T^h(x, t)\}$ for $h \to \infty$. By (3.6) and (3.7), for any $x \in [0, X_h(t)]$

$$-T_x(x, t) \leq h[T_L - T^h(0, t)] \leq (1 + \mathrm{St})(2K\rho H\Delta T)^{1/2}/t^{1/2}.$$

Hence for any $t > 0$ the functions $T^h(x, t)$ are equicontinuous; since they are all bounded by $T^\infty(x, t)$ and monotonically increasing in $h$ the Arzela-Ascoli lemma implies their uniform convergence on $[0, X_\infty(t)]$ (assuming them extended beyond $X_h(t)$ as $T_{cr}$) to a limiting function. By (4.6) this limit must coincide with $T^\infty(x, t)$ and the theorem is proved.

Relation (4.6) yields the following interesting observation.

THEOREM 14. As $t \to \infty$, $(X_h(t)/X_\infty(t)) \to 1$; that is, the fronts for finite and infinite $h$ agree asymptotically.

*Proof*: From (4.6),

$$(1/2)\rho H\ [X_\infty(t)^2 - X_h(t)^2] \leq (2K\sqrt{(2K\rho H\Delta T)}\ (1 + \mathrm{St})\sqrt{(t)}/h.$$

Division by $X_\infty(t) = 2\lambda\sqrt{(\alpha t)}$ yields

$$0 \leq 1 - (X_h(t)/X_\infty(t))^2 \leq \theta/\sqrt{(t)}$$

for $\theta = (c(1 + \mathrm{St})/\lambda^2 h\sqrt{(2K\rho\Delta T/H)}$, or

$$(1 - (\theta/\sqrt{(t)}))^{1/2} \leq (X_h(t)/X_\infty(t) < 1 \tag{4.7}$$

which, letting $t \to \infty$, implies our result. We note that (4.7) provides a potentially useful bound on $X_h(t)$,

$$X_\infty(t)\sqrt{(1 - (\theta/\sqrt{(t)}))} \leq X_h(t) < X_\infty(t). \tag{4.8}$$

**5. A bound on the total system energy.** If the motivation for studying problem I lies in the goal of storing heat in a phase-changing material, then the total heat stored by time $t$,

$$Q(t) = h\int_0^t [T_L - T(0, t')]dt'$$

assumes a special importance. Using the relations (2.7, 2.8) we can obtain useful upper and lower bounds on $0(t)$. Thus we assert:

THEOREM 15. At time $t > 0$,

$$Q(t) \geq F_0(t) = (K\rho H/h)\sqrt{[1 + (2th^2\Delta T/K\rho H)]} - 1, \tag{5.1}$$

$$Q(t) \leq F_1(t) = K\rho H(1 + \mathrm{St}/2)^2\{[1 + (2t\Delta Th^2/K\rho H(1 + \mathrm{St}/2)^2)]^{1/2} - 1\}/h.$$

Moreover,

$$0 \leq F_1 - F_0 \leq (\alpha^2 t^2 h^4 c^2 \Delta T^2/K^4 H^2) = (t^2 h^4 \Delta T^2/K^2\rho^2 H^2). \tag{5.3}$$

*Proof of (5.1).* We note that

$$\int_0^{X(t)} xc\rho[T(x, t) - T_{\mathrm{cr}}]dx \leq X(t)\int_0^{X(t)} c\rho[T(x, t) - T_{\mathrm{cr}}]dx$$

while

$$\int_0^t [T(0, t') - T_{\mathrm{cr}}]dt' = -(1/h)Q(t) + t\Delta T,$$

whence, after some manipulation, (2.7) implies

$$[X(t) - (Q(t)/\rho H)]^2 + ((2K/\rho Hh)[ht\Delta T - Q(t)] - [Q(t)^2/(\rho H)^2]) \leq 0$$

and since $X(t)$ is real, we find

$$(2K/\rho Hh)[ht\Delta T - Q(t)] \leq [Q(t)^2/(\rho H)^2].$$

Further manipulation yields the bound (5.1).

To obtain the lower bound we note that [by (3.5)]

$$T(x, t) \leq T_L - [x\Delta T/X(t)],$$

whence

$$\int_0^{X(t)} c\rho[T(x, t) - T_{\mathrm{cr}}]dx \leq [c\rho X(t)\Delta T/2]$$

and so

$$Q(t) \leq \rho HX(t)\{1 + (1/2)\mathrm{St}).$$

But from (2.8),

$$X(t) \leq \sqrt{[(2K/\rho H)]\{-[Q(t)/H] + t\Delta T\}},$$

whence (5.3) yields

$$Q(t) \leq [1 + (1/2)\mathrm{St}]\sqrt{(2K\rho H\{t\Delta T - [Q(t)/H]\}}.$$

By straightforward manipulations we are then led to (5.2).

It is interesting to note that $F_1 - F_0 = O(\mathrm{St}^2)$. Indeed, if we introduce the nondimensional parameters

$$F_0 = (\alpha t/L^2),\ \mathrm{Bi} = (hL/K),$$

for $L$ a representative length, then

$$F_1 - F_0 \leq (F_0\,\mathrm{Bi}^2\mathrm{St})^2.$$

Thus the bounds (5.1, 5.2) are effective for small values of the parameters St and/or Bi; they may be used to augment previously derived approximations for the surface temperature and moving boundary history [8].

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**Appendix: Proof of the corner point maximum principle.** Let $h(x,t) = |X(t) - x|^{3/2} + \beta|X(t) - x|$ in $\bar{G}^\circ$, with $\beta = \text{const} > 0$ to be chosen. Note that

$$h > 0 \text{ in } \hat{G}^\circ, \qquad h|_{\partial D} \equiv 0$$

Then

$$Lh = \pm\{\tfrac{3}{2}|x - X(t)|^{1/2} + \beta\}\ X'(t) - \tfrac{3}{4}|x - X(t)|^{-1/2}$$

where $\pm$ correspond to the cases where $X$ lies to the right or left of $x$, respectively. We can thus choose $N$ and $\beta > 0$ so small that $Lh < 0$ in $\hat{G}^\circ$. Let $v(P) = u(P) + \varepsilon h(P)$, $\varepsilon > 0$. Then

$Lv \le \varepsilon Lh < 0$ in $\hat{G}^\circ$ and $v \in C(\bar{G}^\circ)$. Thus $v(P)$ attains its maximum value in $\bar{G}^\circ$ on the boundary of $G^\circ$. Now

$$\partial G^\circ = \partial_0 G^\circ \cup \partial_1 G^\circ \cup \partial_2 G^\circ$$

where

$$\partial_0 G^\circ = \bar{G}^\circ \cap \{t = t_0\}, \qquad \partial_1 G^\circ = \partial G^\circ \cap \partial D, \qquad \partial_2 G^\circ = \bar{G}^\circ \cap \partial N,$$

Suppose $M = \max_{P \in \bar{G}^\circ} v(P)$ is attained at a point $P^* \neq P_0$. There are three possibilities:

a) $P^* \in \partial_0 G^\circ$. Then at $P^*$, $v_t \ge 0$, $v_{xx} \le 0$, whence $Lv \ge 0$, which is not possible since $Lv < 0$ in $\hat{G}^\circ$. Thus $P^* \notin \partial_0 G^\circ - P_0$.

b) $P^* \in \partial_1 G^\circ$. Then $M = v(P^*) = u(P^*) \le u(P_0) = v(P_0)$, whence $v(P_0)$ would equal $M$ (which is claimed) or exceed $M$ (which is not possible).

c) $P^* \in \partial_2 G^\circ$. Now $M = v(P^*) = u(P^*) + \varepsilon h(P^*)$. But $u(P^*) < u(P_0)$ and we may choose $\varepsilon$ so small that

$$v(P^*) = u(P^*) + \varepsilon h(P^*) < u(P_0) = v(P_0).$$

Hence in all cases

$$u(P_0) \ge v(P),\ P \in \bar{G}^\circ.$$

Thus

$$\begin{aligned} 0 > [v(P) - v(P_0)/|P - P_0|] &= [v(P) - u(P_0)/|P - P_0|] \\ &= [u(P) - u(P_0)/|P - P_0|] + \varepsilon[h(P) - h(P_0)/|P - P_0|] \end{aligned}$$

or

$$[u(P) - u(P_0)/|P - P_0|] \le -\varepsilon[h(P) - h(P_0)/|P - P_0|]$$

whence, by the form of $h$,

$$\overline{\lim_{\substack{P \to P_0 \\ P \in \hat{G}^\circ}}} \frac{u(P) - u(P_0)}{|P - P_0|} < 0$$

for $P \to P_0$ in a nontangential direction. Thus the principle is proved.