QUARTERLY OF APPLIED MATHEMATICS
JULY 1982

ON STEADY STRATIFIED FLOWS IN POROUS MEDIA*

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Summary. Steady flows of an incompressible and nondiffusive fluid stratified in viscosity and in density that take place in porous media are considered, and conditions necessary for similarity between any two such flows are given with proof. Results on the vorticity and circulation in such flows are also given.

Then the problem of axisymmetric steady flow of a stratified fluid into a point sink is solved by the use of Whittaker functions, and the solution for two-dimensional flow of a stratified and diffusive fluid into a line sink is presented. The solution of these two problems illustrate many of the general results mentioned above.

1. Introduction. Flows of a fluid stratified in salinity (and therefore in density) in porous media bear on agriculture and domestic water supply, notably in Holland, and flows of a fluid stratified in viscosity in porous media occur in oil-recovery operations. Thus the modeling of stratified flows in porous media is of environmental, agricultural, and industrial interest. And yet the laws of modeling such flows have never been clearly stated. In this short note steady stratified flows in porous media are considered, and the conditions of similarity between two such flows stated, with a proof of the similarity (or at least the possibility of similarity) when these conditions are satisfied. Results on the vorticity and the circulation in steady stratified flows in porous media will also be given.

Then the problem of axisymmetric steady flow of a stratified fluid into a point sink is considered and solved in terms of Whittaker functions. Finally, the problem studied by List (1969), two-dimensional steady flow of a stratified diffusive fluid into a line sink, is considered, and it is found that List's solution does not satisfy the symmetry condition for the density at the plane of symmetry. An approximation to the correct solution for low Péclet numbers is presented here. The formulation and solution of the two special problems considered here illustrate many of the general results given in Secs. 2 and 3 of this paper, thus providing a sense of unity to the various subjects considered.

2. Governing equations. With $x_i (i = 1, 2, 3)$ denoting the ith Cartesian coordinate, u_i denoting the velocity component in the direction of increasing x_i, μ denoting the viscosity, and k, p, ρ, and g denoting the permeability, the pressure, the density, and the gravitational

* Received September 1, 1981. This work has been supported by the Office of Naval Research. It is presented to Ralph D. Cooper at the occasion of his retirement from government service, as a token of respect and friendship.

acceleration, respectively, the equations of motion are given by (Yih 1961)

$$\frac{\mu}{k} u_i = -\frac{\partial p}{\partial x_i} - \rho g \delta_{i3}, \tag{1}$$

where δ_{i3} is the Kronecker delta, if we take the direction of increasing x_3 to be opposite to that of the gravitational acceleration. In this study, we assume k to be constant.

The equation of continuity is

$$\partial(\rho u_\alpha)/\partial x_\alpha = 0, \tag{2}$$

where the summation convention is used. We shall neglect diffusive effects, so that

$$u_\alpha \frac{\partial \mu}{\partial x_\alpha} = 0, \tag{3}$$

$$u_\alpha \frac{\partial \rho}{\partial x_\alpha} = 0. \tag{4}$$

From (2) and (4) follows

$$\partial u_\alpha/\partial x_\alpha = 0. \tag{5}$$

which can be used in lieu of (2).

Now let (u_i^*, u^*, ρ^*, p^*) represent a solution of Eqs. (1)—(4), or Eqs. (1), (3), (4), and (5). To fix ideas, let us consider this solution to represent a flow E^* (E is the first letter of *écoulement*; we wish to avoid the letter F, which has been used for the Froude number) in nature, and let

$$\rho^* = \rho_0^* + \rho_1^*, \tag{6}$$

where ρ_0^* is a constant. We wish to produce a flow E in the laboratory which is similar to E^*. The first question that arises is "what constitutes similarity?" and the second question is "what must one do to achieve it?"

To answer the first question we note first of all that the boundary geometry of E must be similar to that of E^*. That is to say, the boundary shape must be the same for both flows, although the sizes do differ. Let the length scale of E^* be L^* and that of E be L, and let

$$\lambda = L^*/L, \tag{7}$$

$$y_i^* = x_i/L^*, \qquad y_i = x_i/L. \tag{8}$$

A point in E^* with (dimensionless) coordinates y_i^* and a point in E with coordinates y_i are called corresponding points if $y_i^* = y_i$. The flows E^* and E are said to be similar if at corresponding points

(a) The direction of the velocities $\mathbf{u}^*$ and $\mathbf{u}$ are the same, i.e., the flow patterns are similar, and

(b) ρ can be found from ρ^* by a linear operation, specifically

$$\rho = \rho_0 + \rho_1, \tag{9}$$

where ρ_0 is a constant, and

$$r = \rho_1^*/\rho_1 = (\rho^* - \rho_0^*)/\rho_1, \tag{10}$$

r being a constant.

We have thus defined similarity. It is an indication of the power of the transformation (Yih 1961), which is contained in the transformation to be given below, that we do not mention the μ-distribution at all in our definition of similarity, apart from the consequence of (a) and (3) that μ or μ^* is constant along a streamline in E or E^*, and the streamline patterns are similar for E and E^*. Given the upstream distribution of μ^* in E^*, any upstream distribution of μ in E will do.

The answer to the second question posed above is that, to achieve dynamical similarity, we must have, apart from the geometrical similarity stated above, the satisfaction of (9) and (10) at some upstream section, and the u_i at that section related to the u_i^* at the corresponding section in E^* in a definite way to be described below. To demonstrate this, we first write (1) in the form

$$\frac{\mu}{k} L u_i = -\frac{\partial \pi}{\partial y_i} - L \rho_1 g \delta_{i3} \tag{11}$$

where π is defined by

$$\pi = p + \rho_0 g x_3, \tag{12}$$

and ρ_0 is the constant part of ρ. Correspondingly, π^* is defined as

$$\pi^* = p^* + \rho_0^* g x_3, \tag{13}$$

and if E^* is a possible flow the equation

$$\frac{\mu^*}{k^*} L^* u_i^* = -\frac{\partial \pi^*}{\partial y_i^*} - L^* \rho_1^* g \delta_{i3} \tag{14}$$

is satisfied for $i = 1, 2, 3$.

Then we consider ρ_1 given by (10) and the u_i and π given by[1] (remember $y_i = y_i^*$ in similarity considerations)

$$\mu_i = \frac{\mu^*}{r\kappa\mu} \mu_i^*, \quad \pi = \frac{1}{r\lambda} \pi^* + \text{any constant}, \tag{15a, b}$$

with

$$\kappa = k^*/k, \tag{16}$$

and see whether (10) and (15) satisfy (11). In demonstrating this, we are of course borrowing (10), but (10) stands everywhere if it stands at some upstream section in E (and the corresponding section in E^*), which we demand, and if the velocity relationship (15a) is satisfied. Upon substitution of (10) and (15) into (14), we obtain (11) after division by λr throughout. Thus, if we choose a ρ_0 and arrange to have (10) and (15a) satisfied at some upstream section in E, the quantities (u_i, μ, ρ, p) represent a possible flow, since they satisfy the equation of motion and (3), (4), and (5) as well. That they satisfy (3) follows from (15a). For (15a) gives the flow pattern in E, and along the streamlines we simply assign to μ its upstream value, whatever it is, so that the satisfaction of (3) is trivial. Similarly, we demand the satisfaction of (4), and since at an upstream section (9) and (10) are satisfied, they are

[1] (1), (15a), and subsequent arguments imply that for similarity between two flows the dimensionless number $g\,|\,\mathrm{grad}\,\rho\,|\,Lk/\mu u$ must be the same at corresponding points of the two flows.

satisfied at any two corresponding points downstream from that section, justifying the use of (9) and (10) to go from the satisfaction of (14) to that of (11). Thus it remains only to show that (5) is satisfied if

$$\partial u_\alpha^*/\partial x_\alpha = 0. \tag{17}$$

But from (15a) and (17), on account of (3) and (4), whose satisfaction has just been shown, and of

$$u_\alpha^* \frac{\partial \mu^*}{\partial x_\alpha} = 0 \qquad \text{and} \qquad u_\alpha^* \frac{\partial \rho^*}{\partial x_\alpha} = 0,$$

whose satisfaction has been presumed, it is an easy though far from trivial exercise to show that (5) is indeed satisfied by the u_i given by (15a).

Finally, we note that the boundaries may either be impermeable boundaries or free surfaces. The former kind needs no comment. We note here that if ρ_0 and ρ_0^* are taken to be zero, π is p and π^* is p^*. If π^* is constant on a free surface, (15b) shows that π is also constant on the corresponding surface in the laboratory flow E. Thus free surfaces are not excluded.

We note also that although we assume the macro-geometry to be similar, we do not include in this geometry the micro-geometry of the grains and interstices constituting the porous media. Indeed, we allow κ to take any value, larger or less than 1. This gives some freedom in the choice of the porous medium used in the laboratory.

3. Generation of vorticity and circulation. Let the components of a pseudo-velocity, denoted by u_i', be defined by (Yih 1961)

$$u_i' = \frac{\mu}{\mu_0} u_i, \tag{18}$$

where μ_0 is a constant viscosity. Then, because of (3), it can be shown trivially that

$$\frac{\mu_0}{k} u_i' = -\frac{\partial p}{\partial x_i} - \rho g \delta_{i3}, \tag{19}$$

and not so trivially that, on account of (3),

$$\partial(\rho u_\alpha')/\partial x_\alpha = 0. \tag{20}$$

Eqs. (3) and (4) can be written as

$$u_\alpha' \frac{\partial \mu}{\partial x_\alpha} = 0, \tag{21}$$

$$u_\alpha' \frac{\partial \rho}{\partial x_\alpha} = 0. \tag{22}$$

On account of (22), (20) can be replaced by

$$\partial u_\alpha'/\partial x_\alpha = 0. \tag{23}$$

Thus we have reduced the flow of a fluid stratified in viscosity to that of a fluid of constant density. This is known (Yih 1961). We have reproduced this result here only because we need to use (19), and its use without a mention of the significance or rather the admissibility of the velocity field u_i' as a solution for seepage of a fluid of constant viscosity (though stratified in density) would seem incomplete.

The vorticity of the flow is defined by

$$\boldsymbol{\xi} = \mathbf{curl}\ \mathbf{u}, \tag{24}$$

which has the components (ξ_1, ξ_2, ξ_3). Similarly, we define a pseudo-vorticity by

$$\boldsymbol{\xi}' = \mathbf{curl}\ \mathbf{u}', \tag{25}$$

which has components (ξ_1', ξ_2', ξ_3').

From the first two equations in (19), i.e., for $i = 1$ and 2, we obtain

$$\xi_3' = 0 \tag{26}$$

by cross-differentiation. Thus the vertical component of the pseudo-vorticity is always zero. That, however, does not mean that ξ_3 is zero. Next, from (19) we easily obtain

$$\frac{\mu_0}{k}\,\xi_1' = -g\,\frac{\partial \rho}{\partial x_2}, \qquad \frac{\mu_0}{k}\,\xi_2' = g\,\frac{\partial \rho}{\partial x_1}, \tag{27}$$

which can be combined with (26) into the vector form

$$\frac{\mu_0}{k}\,\boldsymbol{\xi}' = -g\,\mathbf{curl}(\rho\mathbf{k}), \tag{28}$$

where $\mathbf{k}$ is a unit vertical vector. The significance of (28) or (27) is that it is the horizontal variation of ρ that produces the pseudo-vorticity.

Thus true vorticity is produced in two ways: (1) by the viscosity variation which would produce true vorticity even in the absence of density variation (or of gravity), even though it would not produce any pseudo-vorticity, and (2) through the creation of pseudo-vorticity by the density variation in the presence of a gravitational field.

The circulation along any circuit is defined by

$$\Gamma = \oint u_i\,dx_i. \tag{29}$$

We shall define a pseudo-circulation by

$$\Gamma' = \oint u_i'\,dx_i. \tag{30}$$

From (19) we have

$$\frac{\mu_0}{k}\,\Gamma' = -g\oint \rho\ dx_3. \tag{31}$$

If ρ is constant, or if on the circuit it is constant, Γ' is zero. Also, of course if the circuit is horizontal Γ' is zero. This is to be expected since $\xi_3' = 0$, and an application of Stokes theorem will give a zero Γ'.

4. Steady flows into a point sink. To illustrate the general results obtained in the preceding sections, we consider axisymmetric flows of a stratified fluid confined between two horizontal boundaries into a point sink.

In cylindrical coordinates (r, ϕ, z), with z increasing vertically upward, and with the pseudo-velocity (Yih 1961) defined by

$$(u', w') = \frac{\mu}{\mu_0}(u, w), \tag{32}$$

the equations of motion are

$$\frac{\mu_0}{k}u' = -\frac{\partial p}{\partial r}\frac{\mu_0}{k}, \qquad w' = -\frac{\partial p}{\partial z} - g\rho. \tag{33}$$

Note that the r is now not the r defined in (10); we use the same symbol since there is little danger of confusion. In (32) and (33), μ_0 is a (constant) reference viscosity, u and w are the velocity components in the directions of increasing r and z, respectively, and the other symbols have the same meanings as they have in (1). As shown by Yih (1961), because μ does not change along a streamline in steady flows, u' and w' satisfy the equation of continuity and we can use a new stream function ψ' and write

$$u' = -\frac{1}{r}\frac{\partial\psi'}{\partial z}, \qquad w' = \frac{1}{r}\frac{\partial\psi'}{\partial r}. \tag{34}$$

The final equation governing the flow is then, upon elimination of p in (33), if the fluid is incompressible and therefore ρ is a function only of ψ' in steady flows (Yih 1961),

$$\left(\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)\psi' = -\frac{kgr}{\mu_0}\frac{d\rho}{d\psi'}\frac{\partial\psi'}{\partial r}. \tag{35}$$

The horizontal boundaries are at $z = 0$ and $z = d$, and the sink is at $r = 0$, $z = d$. First, we note that the flow at infinite r must be horizontal, since there the velocity is zero and from (33) we see that the pressure is hydrostatic and the isopycnic lines horizontal. Then at infinite r the streamlines are also horizontal and ψ' must be a function only of z, and (35) shows that ψ' must be linear in z at infinite r. Hence the flow at infinity is given by

$$u' = -\frac{Q}{2\pi rd}, \tag{36}$$

where Q is a pseudo-discharge which will be used to measure the strength of the sink. At infinite r, then,

$$\psi' = \frac{Q}{2\pi d}z. \tag{37}$$

We emphasize that whatever the sink distribution at the axis may be, (37) always holds. Let the density at infinite r be given by

$$\rho = \rho_0\left(1 - \frac{\beta z}{d}\right). \tag{38}$$

Then

$$\frac{d\rho}{d\psi'} = -\frac{2\pi\beta\rho_0}{Q}. \tag{39}$$

We now use the dimensionless quantities

$$(\xi, \zeta) = \left(\frac{r}{d}, \frac{z}{d}\right), \qquad \Psi = \frac{2\pi\psi'}{Q}. \tag{40}$$

Then (35) becomes

$$\left(\frac{\partial^2}{\partial\xi^2} - \frac{1}{\xi}\frac{\partial}{\partial\xi} + \frac{\partial^2}{\partial\zeta^2}\right)\Psi = \lambda^2\xi\,\frac{\partial\Psi}{\partial\xi}, \tag{41}$$

where

$$\lambda^2 = 2\pi\beta\rho_0\,kgd^2/\mu_0\,Q. \tag{42}$$

The boundary conditions are

$$\Psi = 0 \quad \text{at} \quad \zeta = 0, \tag{43}$$

$$\Psi = 1 \quad \text{at} \quad \zeta = 1, \tag{44}$$

$$\Psi = 0 \quad \text{at} \quad \xi = 0, \tag{45}$$

$$\Psi = \zeta \quad \text{at} \quad \xi = \infty. \tag{46}$$

The discontinuity of Ψ at $\xi = 0$ and $\zeta = 1$ represents the sink.

The solution by the separation of variables is of the form

$$\Psi = \zeta + \sum_{n=1}^{\infty} A_n\, f_n(\xi)\,\sin n\pi\zeta, \tag{47}$$

which satisfies (43) and (44). The coefficients A_n are determined by (45) to be

$$A_n = \frac{2(-1)^n}{n\pi f_n(0)}, \tag{48}$$

and the condition (46) is satisfied if

$$f_n(\infty) = 0. \tag{49}$$

It is evident that the solution $f_n(\xi)$ must satisfy not only (49), but also

$$f_n(0) \neq 0, \tag{50}$$

in order for A_n to be finite.

The differential equation satisfied by $f_n(\xi)$ is

$$f_n'' - \left(\lambda^2\xi + \frac{1}{\xi}\right)f_n' - n^2\pi^2 f_n = 0. \tag{51}$$

Power-series expansion by the Frobenius method gives two independent solutions for f_n : f_{n2} which is a power series in ξ^2 starting with ξ^2, and f_{n1} which is of the form

$$f_{n1}(\xi) = 1 + a_1\xi^2 + \cdots + f_{n2}(\xi)\ln\xi. \tag{52}$$

It is evident that $f_{n2}(\xi)$ does not satisfy (50). So the solution has to be f_{n1} or a linear combination of f_{n1} and f_{n2}. The crititerion is that it has to satisfy (49). The determination of the right combination is far from trivial. We choose to give an explicit solution of (51) that satisfies both (49) and (50).

Let

$$\eta = \lambda^2 \xi^2 / 2. \tag{53}$$

Then (51) becomes

$$\frac{d^2 f}{d\eta^2} - \frac{df}{d\eta} + \frac{k_n}{\eta} f = 0, \tag{54}$$

where

$$k_n = -n^2 \pi^2 / 2\lambda^2. \tag{55}$$

Now let

$$f = e^{\eta/2} h(\eta), \tag{56}$$

which transforms (54) to

$$\frac{d^2 h}{d\eta^2} + \left(-\frac{1}{4} + \frac{k_n}{\eta} \right) h = 0. \tag{57}$$

The Whittaker equation in its cannonical form (Whittaker and Watson 1945, p. 337), is

$$\frac{d^2 W}{d\eta^2} + \left(-\frac{1}{4} + \frac{k}{\eta} + \frac{1 - 4m^2}{4\eta^2} \right) W = 0. \tag{58}$$

Taking $2m = 1$, and identifying our k_n with the k in (58), we obtain

$$h = e^{-\eta/2} \eta^{k_n} \int_0^\infty t^{-k_n} \left(1 + \frac{t}{\eta} \right)^{k_n} e^{-t}\, dt, \tag{59}$$

which, except for a multiplicative constant which is inconsequential here, is the same as that given for $W_{k,\,0.5}(\eta)$ by Whittaker and Watson (1945, p. 340). Thus

$$f_n(\eta) = \eta^{k_n} \int_0^\infty t^{-k_n} \left(1 + \frac{t}{\eta} \right)^{k_n} e^{-t}\, dt. \tag{60}$$

It is a simple calculation by the use of (60) to show that

$$f_n(0) = 1. \tag{61}$$

Hence (48) gives

$$A_n = \frac{2}{n\pi} (-1)^n. \tag{62}$$

Furthermore, it is evident that (49) is satisfied. Thus (47), (60), and (62) gives the solution. We note in passing that w' on the axis of symmetry cannot be obtained by differentiation of (47) with respect to ξ term by term. This is a peculiarity arising from eigenfunction expansions, and has been noted before in similar problems (Yih, O'Dell, Debler 1962).

The flow patterns for various values of λ^2 are similar to those given for the two-dimensional case by Yih (1961). The larger λ^2 is, the more gradually the streamlines rise to the sink. We therefore refrain from presenting the flow patterns. Instead, we make the following observations on the connection of the results for this problem with the general results given in Secs. 2 and 3.

First we note that two flows into a point sink are similar if the λ^2 defined by (42) is the same for both flows. And for linear upstream density distributions, such as given by (38), this statement is quite equivalent to requiring the constancy of r in (10) and the requirement of (15a) far upstream.

Then we note that the left-hand side of (35), divided by $-r$, is the pseudo-vorticity, and this pseudo-vorticity is zero at infinite r, but is created by the combined action of gravity and stratification as the streamlines deviate from horizontality, in agreement with (28).

The fact that (37) holds for *any* sink distribution on the axis is of practical importance. For if one wishes to draw a lighter fluid (say oil) and does not wish to have water underlying the oil come out with the oil, one merely has to use two sinks of the appropriate strengths calculated from the depths of the oil and the water far upstream and from their viscosities. The upper sink will then draw only oil and the lower one only water. So while it is futile to try to separate oil and water by using only one sink, however small the discharge is (since (37) always holds), it is possible to separate oil from water by using two sinks of the appropriate strengths—and it does not seem very inconvenient to do so.

5. Sink flows when diffusion is taken into account. List (1969) gave an exact solution for a two-dimensional diffusive flow of a stratified fluid into a line sink, and showed that my solution (1961) for nondiffusive flow remains intact when diffusion is considered. The simplicity of his solution has an elegance that is very appealing. However, upon closer examination one notes that the boundary condition for the temperature (or density) at the plane of symmetry has not been considered by List. The most natural form of this condition is

$$\partial\rho/\partial\xi = 0 \qquad \text{at} \qquad \xi = 0, \tag{63}$$

if we write now

$$(\xi, \eta) = \left(\frac{x}{d}, \frac{y}{d}\right). \tag{64}$$

where x is measured horizontally and y increases vertically upward, d being the spacing between two impermeable planes. The sink is situated at

$$\xi = 0, \qquad \eta = 1.$$

List's solution does not satisfy (63). Therefore his solution requires a heat-sink distribution on the plane of symmetry $\xi = 0$. Nor is the ρ given by his solution constant on $\xi = 0$. It varies linearly with η. Thus his solution is for an unrealistic temperature distribution at the plane of symmetry, where this distribution is artificially maintained. It does not solve the problem of symmetric flow into the sink from left and right. The elegance of his solution is therefore illusory, and this is most unfortunate since the solution has such appealing simplicity.

The problem admits only an approximate solution if it is to be solved properly. We shall consider the effect of temperature variation on the viscosity μ to be small, and take μ to be constant. Furthermore, we shall use ρ instead of the temperature, implicitly assuming the variation of ρ with the temperature to be linear. Then the equation of motion is (Yih 1961)

$$\mu\nabla^2\psi = gk\rho_x, \tag{65}$$

where ψ is the usual stream function and k again denotes permeability, as in Sec. 2.

The velocity given by Darcy's law at any "point" is the mean velocity over not just the pores but over a mean, sufficiently small area. We therefore do not use the porosity as List (1968, 1969) did in his diffusion equation. To be consistent we also consider the diffusivity α (thermal or saline, as the case may be) to be determined in a similar average way, and not to be the molecular diffusivity in the absence of the grain matrix. Then the diffusion equation is

$$\psi_y \rho_x - \psi_x \rho_y = \alpha \nabla^2 \rho. \tag{66}$$

We shall deal with the region $x \leq 0$ first.

To render the equations dimensionless, and to make things convenient, let

$$\frac{\psi}{Ud} = \eta + F(\xi, \eta), \tag{67}$$

$$\frac{\rho}{\rho_0} = 1 - \beta\eta + \theta(\xi, \eta), \tag{68}$$

where U is the velocity at $x = -\infty$, ρ_0 is the density at $y = 0$, and $1 - \beta\eta$ is the ρ/ρ_0 at $x = -\infty$. Then (65) becomes

$$\nabla^2 F = A\theta_\xi, \tag{69}$$

where

$$A = gk\rho_0/\mu U, \tag{70}$$

and (66) becomes

$$\nabla^2\theta - \text{Pé}(\theta_\xi + \beta F_\xi) = \text{Pé}(F_\eta \theta_\xi - F_\xi \theta_\eta), \tag{71}$$

where Pé is the Péclet number Ud/α. In (69) and (71), ∇^2 now is

$$\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}.$$

The boundary conditions are

$$F(\xi, 0) = 0 = F(\xi, 1), \tag{72}$$

$$F(-\infty, \eta) = 0, \tag{73}$$

$$F(0, \eta) = -\eta, \tag{74}$$

$$\theta(\xi, 0) = 0 = \theta(\xi, 1), \tag{75}$$

$$\theta(-\infty, \eta) = 0, \tag{76}$$

$$\theta_\xi(0, \eta) = 0. \tag{77}$$

The solution by Yih (1961) is for infinite Pé. Now we shall consider the case of low Pé only. For this case we can expand F and θ in powers of Pé. But this is not an efficient way of finding the solution. The most efficient way is to multiply the right-hand side of (71) by ε, solve the equations by a power expansion in ε, and then make ε equal to 1. In this way the terms containing Pé on the left-hand side of (71) appear in the first approximation already. For the first approximation, then, we have (69) and

$$\nabla^2\theta = \text{Pé}(\theta_\xi + \beta F_\xi). \tag{78}$$

Elimination of θ between (69) and (78) gives

$$\nabla^2\nabla^2 F - \text{Pé}(\nabla^2 F_\xi + A\beta F_{\xi\xi}) = 0. \tag{79}$$

Using the method of separation of variables, we have

$$F = \sum_{n=1}^{\infty} A_n F_n(\xi)\sin n\pi\eta, \tag{80}$$

$$\theta = \sum_{n=1}^{\infty} A_n \theta_n(\xi)\sin n\pi\eta, \tag{81}$$

which satisfies (72) and (75) exactly. The function F_n satisfies

$$[(D^2 - n^2\pi^2)^2 - \text{Pé}(D^2 - n^2\pi^2 + A\beta D)D]F_n = 0. \tag{82}$$

Of the four independent solutions of (82), all exponential functions, only two satisfy (73), and these are $\exp(-a_n\xi)$ and $\exp(-b_n\xi)$, where, when terms of $O(\text{Pé})$ are neglected (since the right-hand side of (71) has been dropped in this approximation),

$$a_n = n\pi(1 + Q), \tag{83}$$

$$b_n = n\pi(1 - Q), \tag{84}$$

with

$$Q = (\text{Pé}A\beta)^{1/2}. \tag{85}$$

The solution for $F_n(\xi)$ is

$$F_n(\xi) = e^{-a_n\xi} + \gamma e^{-b_n\xi}, \tag{86}$$

where γ will be determined by (77). Substituting (86) into (69) and applying (77), we find, by using (83) and (84), that $\gamma = 1$. Then integrating (78), using (86), we obtain

$$\theta_n = \frac{1}{1 - Q^2}\left(\frac{\text{Pé}\beta}{A}\right)^{1/2}(-b_n e^{-a_n\xi} + a_n e^{-b_n\xi}), \tag{87}$$

which satisfies (76) and (77).

Only (74) needs to be satisfied by determining the Fourier coefficients A_n. These are found to be

$$A_n = \frac{2}{\pi n}(-1)^n.$$

Thus

$$F = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}F_n(\xi)\sin n\pi\eta, \tag{88}$$

$$\theta = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\theta_n(\xi)\sin n\pi\eta, \tag{89}$$

and the solution is obtained for the first approximation. Higher approximations can be found by successive substitution of the results of the previous approximation into the right-hand side of (71) and integration. But it should be remembered that the indicial equation for (82) has to be solved to higher orders in Pé, since terms of $O(\text{Pé})$ have been neglected in (83) and (84). The higher approximations present no real difficulty, and we content ourselves with the solution given by (67), (68), (88), and (89) for low Péclet numbers.

The solution for $x \geq 0$ is given by

$$\psi/Ud = -\eta - F(-\xi, \eta),$$

$$\rho/\rho_0 = 1 - \beta\eta + \theta(-\xi, \eta),$$

with F and θ given by (88) and (89). The reflection of the solution across the axis of symmetry is possible because

$$\psi = 0 \qquad \text{and} \qquad \left(\frac{\rho}{\rho_0}\right)_\xi = 0 \qquad \text{at} \qquad \xi = 0.$$

A similar solution for the axisymmetric case is possible. But to save space we shall not present it, and note here only that it involves Hankel functions that vanish at large distances from the axis of symmetry.

The solution given by (88) and (89) is in a sense already the second approximation. A truly first approximation is obtained if we drop all terms containing Pé in (71). Then we obtain

$$\theta = 0$$

exactly, and (69) shows that the flow will be irrotational. We mention this fact to show that when ρ_ξ (or ρ_x) is zero there is no vorticity, in agreement with (28) in Sec. 3, where $\boldsymbol{\xi}'$ is the true vorticity $\boldsymbol{\xi}$ in the present case of constant μ. The results in Secs. 4 and 5 illustrate the general results given in Secs. 1 and 2, thus providing a sense of unity to the various subjects treated in this paper.

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