CONSERVATION LAWS WITH SHARP INHOMOGENEITIES*

By

WILLIAM K. LYONS

Brown University

1. Introduction. In this paper we will discuss the solutions to the initial-value problem for a single hyperbolic conservation law in a non-homogeneous medium which has sharp discontinuities in its flux function; that is,

$$u_t(x, t) + f(u(x, t), x)_x = 0 \tag{1.1a}$$

where $f(u, x)$ is a function which is discontinuous in x. This problem has properties of nonuniqueness which have not been previously encountered. The classical theory of conservation laws has been developed, in part, to handle the question of non-uniqueness for weak solutions. Unfortunately this theory is insufficient for our study. More specifically, when waves collide with a sharp inhomogeneity, which we will call an *interface*, they may be transmitted, reflected, or partially transmitted and partially reflected. Unfortunately, the classical theory cannot be used to decide how an interface will affect a wave when such a collision occurs, leaving our solution undetermined.

We will consider the initial value problem for Eq. (1.1a) under the initial condition

$$u(x, 0) = u_0(x). \tag{1.1b}$$

We assume that for each x, $f(u, x)$ is a uniformly convex function of u which attains its minimum and for each u, $f(u, x)$ is a piecewise constant function of x. We note that if $f(u, x)$ is independent of x, (1.1) reduces to the classical conservation law studied by Hopf [9, 10], Lax [13, 15], Oleinik [19, 20] and others.

The main focus of our study deals with the situation where

$$\begin{aligned} f(u, x) &= f_1(u), \qquad x < 0, \\ &= f_2(u), \qquad x > 0. \end{aligned} \tag{1.2}$$

Eq. (1.1a) now takes the form

$$u_t(x, t) + \begin{Bmatrix} f_1(u)_x, & x < 0 \\ f_2(u)_x, & x > 0 \end{Bmatrix} = 0. \tag{1.3a}$$

This makes the line $x = 0$ an interface. An interesting feature of the above choice is that the Riemann problem for (1.3a), i.e., the problem with initial data

$$\begin{aligned} u(x, 0) &= u_l, \qquad x < 0 \\ &= u_r, \qquad x > 0 \end{aligned} \tag{1.3b}$$

attains solutions which are constant along rays $x/t =$ constant.

* Received July 10, 1981; revised version received September 22, 1981.

As can be easily seen, a function $u(x, t)$ which is smooth except along a family of smooth curves, will be a weak solution to Eq. (1.1) if and only if it satisfies (1.1) at points of smoothness, while along a curve of discontinuity $\chi(t)$ it satisfies the Rankine-Hugoniot condition

$$\dot{\chi}(t) = \frac{f(u(\chi(t)+, t), \chi(t)+) - f(u(\chi(t)-, t), \chi(t)-)}{u(\chi(t)+, t) - u(\chi(t)-, t)}.$$

In particular, along the interface $\chi(t) = 0$, for Eq. (1.3),

$$f_1(u(0-, t)) = f_2(u(0+, t)). \tag{1.4}$$

Eq. (1.4) will be referred to as the *interface condition.*

It is necessary at this point to select a unique weak solution for the Riemann problem. However, the classical admissibility criteria seem to be inadequate to single out such a solution. It is possible to construct examples for which no possible weak solution satisfies the Lax E-condition at the interface. There is no clear way to devise an entropy for our problem. If we let $\eta(u, x)$ be an entropy, there is no a priori information to decide the jump of η across the interface. Hence there is no clear way to apply the pointwise entropy criterion or the global entropy rate criterion. If we now consider the viscosity criterion, in general

$$u_t + f(u, x)_x = \varepsilon u_{xx}.$$

does not possess smooth solutions and it is not clear that it possesses a unique weak solution. In the light of what is to follow, no further investigation of the above criteria seems necessary here.

In Sec. 2, we will demonstrate that if the discontinuity in the flux function is smoothed, then many different solutions may be attained as the limit of different smoothings. Without a good physical reason to choose a particular smoothing, this indicates that some additional criteria must be added to our problem to single out a unique solution. In Sec. 3, a characterization is given for admissibility criteria which will select solutions in an acceptable fashion, and an existence theorem is presented for arbitrary bounded measurable initial data.

2. Smoothing of $f(u, x)$. Since it is possible to visualize our differential equation (1.1) as a model of some physical problem, we begin our study by drawing on some physical intuition. Usually, sharp discontinuities in physical problems attempt to model very rapid continuous transitions. For this reason, it looks natural to study here solutions to a sequence of smooth approximations which approach our discontinuous problem.

We would therefore like to approximate the solution to the Riemann problem

$$u_t + \begin{Bmatrix} f_1(u)_x, & x < 0 \\ f_2(u)_x, & x > 0 \end{Bmatrix} = 0, \tag{2.1a}$$

$$f_1(u(0-, t)) = f_2(u(0+, t)), \tag{2.1b}$$

$$\begin{aligned} u(x, 0) &= u_l, \qquad x < 0, \\ &= u_r, \qquad x > 0 \end{aligned} \tag{2.1c}$$

by smoothing the discontinuity in $f(u, x)$ at $x = 0$. We assume $f_1(u)$ and $f_2(u)$ are uniformly convex functions that attain their minima. We will assume that $g(u, x)$ approximates $f(u,$

x) in the following sense: $g(u, x) = f_1(u)$ for $x \leq -1$, $g(u, x) = f_2(u)$ for $x \geq 1$, and $g(u, x)$ is a C^3 function of u and x which, for each fixed x, is uniformly convex and attains its minimum. Next, we will let a_n be a sequence such that $a_n > 0$ and $a_n \to 0$ and we will investigate the solutions to

$$\begin{aligned} u_t + g(u, x/a_n)_x &= 0, \\ u(x, 0) = u_l, &\qquad x < 0, \\ = u_r, &\qquad x > 0 \end{aligned}$$

as $n \to \infty$.

The main tool in commencing our investigation is the use of generalized characteristics. Classical characteristics are defined for smooth solutions u as solutions to

$$\dot{\xi} = g_u(v, \xi), \qquad \dot{v} = -g_x(v, \xi) \tag{2.3a,b}$$

where $\xi(t)$ represents the characteristic trajectory and $v(t)$ is the restriction of the solution to this characteristic. A generalized characteristic is a solution of (2.3a) in the sense of Filippov [6]. It turns out that generalized characteristics are either classical characteristics or shocks. It has been shown [2, 3] that from each point (x, t) of the upper half plane there emanates a unique forward characteristic. On the other hand, the set of backward characteristics through (x, t) either consists of a single classical characteristic or an infinite number of characteristics spanning the sector between two classical characteristics. These two characteristics are the minimal and maximal backward characteristics, the minimal having the larger speed at (x, t).

Notice that the curves $\xi(t)$ and $v(t)$ which satisfy

$$g(v(t), \xi(t)) = k \tag{2.4}$$

are integrals of the characteristic equations (2.3) for each fixed k. Now, if we look for the point $(\tilde{u}, \tilde{x})$ which satisfies $g(\tilde{u}, \tilde{x}) = \max_x (\min_u g(u, x))$ and assume that $(\tilde{u}, \tilde{x})$ is uniquely determined, then $(\tilde{u}, \tilde{x})$ will be a saddle-point. Consider the minimizer $\mu(x)$ of $g(u, x)$ for each fixed x. By the implicit function theorem, $\mu(x) \in C^2$. Let $\bar{x}$ be the minimizer of $g(\mu(x), x)$ in $[-1, 1]$; then we have $(\tilde{u}, \tilde{x}) = (\mu(\bar{x}), \bar{x})$. From this point on, we assume that $\bar{x} \in (-1, 1)$ is unique, and $(\partial^2/\partial x^2) g(\mu(x), x)$ is strictly negative at $\bar{x}$. However, similar results may be proven for $\bar{x} \in [-1, 1]$ and $(\partial^2/\partial x^2) g(\mu(x), x)$ non-positive. It is clear from the above definitions that we have the following.

LEMMA 2.1. $(\mu(\bar{x}), \bar{x})$ is a saddle-point of Eq. (2.3) and the stable $(\sigma_+(x))$ and unstable $(\sigma_-(x))$ manifolds are given by $g(\sigma_\pm(x), x) = g(\mu(\bar{x}), \bar{x})$, $g_u(\sigma_+(x), x) > 0$ for $x < \bar{x}$, $g_u(\sigma_-(x), x) < 0$ for $x < \bar{x}$, $g_u(\sigma_+(x), x) < 0$ for $x > \bar{x}$, and $g_u(\sigma_-(x), x) > 0$ for $x > \bar{x}$.

We shall now use the phase portrait to identify the limiting solution for any given Riemann problem. In doing so, however, we must watch out for possible collisions of characteristics which can cause shocks. We will be able to understand the limit of our solution as $a_n \to 0$ in spite of these interactions. A typical phase portrait for the above approximation is illustrated in Fig. 2.1.

We first distinguish the case $u_l > \sigma_+(-1)$ or $u_r < \sigma_+(1)$. This leads to transmitting characteristics at the interface. When $u_l > \sigma_+(-1)$ and $u_r < \sigma_+(1)$ then both waves could potentially transmit. The one that does transmit will form a shock which travels away from the t-axis. When $u_l < \sigma_+(-1)$ and $u_r > \sigma_+(1)$, we are in the critical region of the

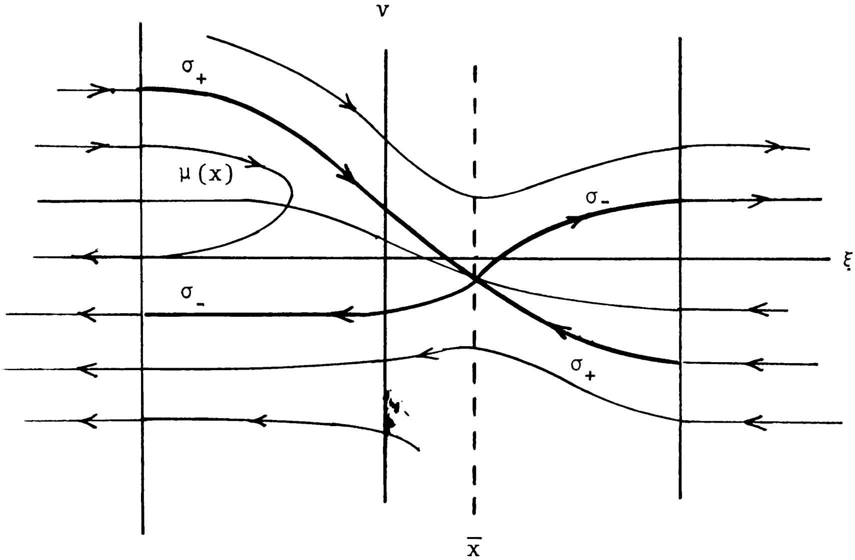

FIG. 2.1

approximation where there are characteristics that are traveling towards the critical point via the stable manifold. It is our claim that

LEMMA 2.2. Assume $u_l < \sigma_+(-1)$ and $u_r > \sigma_+(1)$; then there exists a characteristic $(\xi^*(t), v^*(t))$ such that $(\xi^*(t), v^*(t)) \rightarrow (\bar{x}, \mu(\bar{x}))$ as $t \rightarrow \infty$.

Proof. We will first show that there is a bounded interval of $\bar{x}$, $(\bar{x} - \alpha, \bar{x} + \beta)$, so that no characteristic with $\xi(0) \in (-\infty, -1]$ or $\xi(0) \in [1, \infty)$ can enter this interval. Let α be such that $\bar{x} - \alpha$ is the largest value of $\xi(t)$ which is attained by the orbit which starts from $(\xi(0), v(0)) = (-1, u_l)$. Since $u_l < \sigma_+(-1)$ then we must have $\alpha > 0$. Also, let β be such that $\bar{x} + \beta$ is the smallest value of $\xi(t)$ which is attained by the orbit which starts from $(\xi(0), v(0)) = (1, u_r)$; then $\beta > 0$. Let $(\tilde{x}, t)$ be such that $\tilde{x} \in (\bar{x} - \alpha, \bar{x} + \beta)$; then we claim that the minimal and maximal backward characteristics through $(\tilde{x}, t)$ must have started at $\tilde{\xi}_\pm(0) \in (-1, 1)$ ($\tilde{\xi}_-(t)$ and $\tilde{\xi}_+(t)$ represent the minimal and maximal backward characteristic respectively). Indeed, if not, $\tilde{\xi}_-(0) \in (-\infty, -1]$ or $\tilde{\xi}_+(0) \in [1, \infty)$. If $\tilde{\xi}_-(0) \in (-\infty, -1]$ then $\tilde{v}_-(0) = u_l$. Now, $(\tilde{\xi}_-(t), \tilde{v}_-(t))$ is classical up until $\tilde{\xi}_-(t) = \tilde{x}$; however, by the phase portrait, the largest value which can be attained by $\tilde{\xi}_-(t)$ is $\bar{x} - \alpha < \tilde{x}$, and this is a contradiction. If $\tilde{\xi}_+(0) \in [1, \infty)$, then $\tilde{v}_+(0) = u_r$. Since the smallest value attained by $\tilde{\xi}_+(t)$ will be $\bar{x} + \beta > \tilde{x}$, we again get a contradiction. Hence, $\tilde{\xi}_\pm(0) \in (-1, 1)$.

Now, consider the sequence of points $(\bar{x}, n)$, where n is an interger, and let $(\xi_n(t), v_n(t))$ be the minimal backward characteristic through $(\bar{x}, n)$. The sequence $\xi_n(0)$ is contained in $(-1, 1)$ and so it must have a convergent subsequence $\xi_{n_i}(0)$. By continuous dependence $(\xi_{n_i}(t), v_{n_i}(t))$ converges to a solution of the characteristic equations (2.3). Let

$$(\xi^*(t), v^*(t)) = \lim(\xi_{n_i}(t), v_{n_i}(t)).$$

It is clear that $\xi^*(t)$ cannot leave $(-1, 1)$, so we must have $(\xi^*(t), v^*(t)) \rightarrow (\bar{x}, \mu(\bar{x}))$, since this is the only characteristic that remains classical for all time in $(-1, 1)$.

COROLLARY 2.3. In the limit, $\sigma_-(-1)$ and $\sigma_-(1)$ will be the values at the interface when $u_l < \sigma_+(-1)$ and $u_r > \sigma_+(1)$.

We note that, in general, solutions to the Riemann problem will consist of four constant states, two classical waves and a discontinuity at the interface. We denote these solutions by (u_l, u_-, u_+, u_r) where $u_- = \lim_{x\to 0_-} u(x, t)$ and $u_+ = \lim_{x\to 0_+} u(x, t)$ for $t > 0$ (note that u_- and u_+ are constant because of the radial nature of the solutions; also note that u_- may equal u_l or u_+ may equal u_r).

We can now use Corollary 2.3 to construct approximations which give any solution of the type (u_l, u_-, u_+, u_r), where $f'_1(u_-) < 0$ and $f'_2(u_+) > 0$, the only limitation being that if $f'_1(u_l) \geq 0$ then we must have $f_1(u_-) > f_1(u_l)$ or if $f'_2(u_r) \leq 0$ then we must have $f_2(u_+) > f_2(u_r)$. To see this, we assume (u_l, u_r) are given and choose (u_-, u_+) with the properties stated above. It is easy to select an approximation so that the maximizer of $g(\mu(x), x)$ is uniquely determined and so that $g(\mu(\bar{x}), \bar{x}) = f_1(u_-)$ and $g(\mu(\bar{x}), \bar{x}) = f_2(u_+)$. Also, $\bar{x} \in (-1, 1)$, since $f'_1(u_-) < 0$ and $f'_2(u_+) > 0$. The solutions to (2.2) will now converge, in the sense of distributions, to the desired solution. We have shown the following

THEOREM 2.4. If $f'_1(u_l) < 0$ and $f'_2(u_r) > 0$, then there exists an approximation $g(u, x)$ such that the solutions to (2.2) converge to (u_l, u_-, u_+, u_r) for any pair (u_-, u_+) where $f_1(u_-) = f_2(u_+)$, $f'_1(u_-) < 0$ and $f'_2(u_+) > 0$. Also, if $f'_1(u_l) \geq 0$ or $f'_2(u_r) \leq 0$ then for $f_1(u_-) > f_1(u_l)$ or $f_2(u_+) > f_2(u_r)$, respectively, there is an approximation so that the solutions to (2.2) converge to (u_l, u_-, u_+, u_r) where $f_1(u_-) = f_2(u_+)$, $f'_1(u_-) < 0$ and $f'_2(u_+) > 0$.

3. Tractable admissibility criteria and existence. At this point it is clear that if a physical problem is modeled in a manner which leads to solving the discontinuous conservation law (1.5), the model is incomplete. It is the opinion of the author that if such a case arises, one should study the disturbance at the interface more closely. If this disturbance can be shown to be the limit of some continuous approximation, one can use the method of smoothing to identify good solutions.

If some physical motivation is presented, criteria may be proposed to resolve the indeterminacy at the interface. An admissibility criterion is said to be tractable if there is one smooth approximation which, in the limit, yields the same solution at the interface. It is clear that this insures that our equation remains evolutionary since backward characteristics can be defined which will extend all the way back to the x-axis. Along these characteristics $f(u, x)$ will remain constant. Therefore, if we assume that a solution exists for

$$u_t + \begin{Bmatrix} f_1(u)_x & x < 0 \\ f_2(u)_x & x > 0 \end{Bmatrix} = 0,$$

$$u(x, 0) = u_0(x),$$

and assume $u_0(x)$ is bounded, then it must follow that $u(x, t)$ is bounded.

Even if we are using a tractable admissibility criterion, it is possible to transmit a wave with bounded variation and produce a wave with unbounded variation along the interface.

Assume $u(0-, t)$ is of bounded variation and there exists an $a > 0$ so that $|f'_2(u(0+, t))| \geq a$. Then the total variation of $u(0+, t)$ is bounded. In view of the interface condition, we have

$$|u(0+, t) - u(0+, t')| \leq \frac{K}{a} |u(0-, t) - u(0-, t')|$$

where $K = \max f'_1(u(0-, t))$. The difficulty occurs when $f'_2(u(0+, t)) = 0$.

Consider the following example: $f_1(u) = (1/2)u^2$, $f_2(u) = (1/2)(u + 1)^2$,

$$u_0(x) = \begin{cases} 2, & x \le -2, \\ 1 + \dfrac{1}{i^2}, & x = -\left(1 + \dfrac{1}{i^2}\right)^-, \\ 1 - \dfrac{(i+1)^2}{2i+1}\left(x + 1 + \dfrac{1}{(i+1)^2}\right), & -\left(1 + \dfrac{1}{i^2}\right) < x < -\left(1 + \dfrac{1}{(i+1)^2}\right), \\ 1, & x = -\left(1 + \dfrac{1}{(i+1)^2}\right)^+, \\ 1, & x \ge -1. \end{cases}$$

We will use any tractable admissibility criterion which requires that a wave which can transmit will transmit to decide the interactions at the interface. There are no shocks formed to the left of the interface, so

$$\text{TV } u(0-, t) = \text{TV } u_0(t) = 2\sum_{i=1}^{\infty} \frac{1}{i^2} < \infty.$$

However, by the interface condition

$$\text{TV } u(0+, t) = \sum_{i=1}^{\infty} \left(\frac{2}{i^2} + \frac{1}{i^4}\right)^{1/2},$$

which diverges.

We now turn to the question of global solutions to

$$u_t + \begin{Bmatrix} f_1(u)_x & x < 0 \\ f_2(u)_x & x > 0 \end{Bmatrix} = 0, \tag{3.1a}$$

$$u(x, 0) = u_0(x). \tag{3.1b}$$

We assume that $f_1(u)$ and $f_2(u)$ are uniformly convex and that $u_0(x)$ is an arbitrary bounded measurable function. We also postulate any admissibility criterion, tractable in the above sense, which yields uniqueness for the Riemann problem at the interface. Let $g(u, x)$ be the approximation that induces the criterion.

Pick $a_n > 0$ such that $a_n \to 0$ as $n \to \infty$ and let $u_n(x, t)$ be the unique weak solution to

$$u_t + g(u, x/a_n)_x = 0, \qquad u(x, 0) = u_0(x)$$

which must exist for all time, by Kružkov [12].

For $u_n(x, t)$, minimal and maximal backward characteristics through any point (x, t) are well-defined, in the sense of Dafermos [2]. The backward characteristics $\zeta_\pm(\tau; x, t)$ for $x \notin (-a_n, a_n)$ either enter the strip $(-a_n, a_n) \times [0, \infty)$ or they emanate from the x-axis without intersecting $x = \pm a_n$. If they enter the strip let $T_\pm(x, t)$ be the first time of entry; otherwise let $\xi_\pm(x, t)$ be their starting points from the x-axis. We note the preservation of ordering for these points. Assume $a_n < x < y < \infty$; then if $\zeta_+(\tau; x, t)$ does not enter the strip, we must have that $\zeta_\pm(\tau; y, t)$ do not enter the strip and $\xi_+(x, t) \le \xi_\pm(y, t)$. Also, if $\zeta_-(\tau; y, t)$ enters the strip then we have that $\zeta_\pm(\tau; x, t)$ enter the strip and $T_-(y, t) \le T_\pm(x, t)$. The same type of ordering is preserved in the negative quadrant. Let $\varepsilon > 0$; then there is an $N(\varepsilon)$ so that $n \ge N(\varepsilon)$ implies that $a_n \le \varepsilon/2$.

LEMMA 3.1. For $x < y < -\varepsilon$ or $\varepsilon < x < y, t > 0$ and $n \geq N(\varepsilon)$ there is a K such that

$$\frac{u_n(y\pm, t) - u_n(x\pm, t)}{y - x} \leq \frac{K}{\min(t, \varepsilon)}. \tag{3.2}$$

Proof. Let $(I(t), t)$ be the set of points whose backward characteristics enter the strip $(-a_n, a_n) \times [0, \infty)$ for each fixed t. Also assume that if $x \notin I(t)$, then the backward characteristics through (x, t) do not enter the strip. The boundary points of $I(t)$ may have one characteristic which enters the strip and one which does not. Clearly $I(t)$ is a bounded interval for each fixed t.

If $x > 0$, $x \notin I(t)$, then the starting points of $\zeta_\pm(\tau;\ x, t)$ and $\zeta_\pm(\tau;\ y, t)$ are $\xi_\pm(x, t)$ and $\xi_\pm(y, t)$, respectively. We have $\xi_\pm(x, t) \leq \xi_\pm(y, t)$. By Dafermos [2],

$$\xi_\pm(x, t) = x - tf'_2(u_n(x\pm, t)), \qquad \xi_\pm(y, t) = y - tf'_2(u_n(y\pm, t)).$$

So we must have

$$\frac{f'_2(u_n(y\pm, t)) - f'_2(u_n(x\pm, t))}{y - x} \leq \frac{1}{t}. \tag{3.3}$$

Similarly, if $y < 0$ and $y \notin I(t)$, then

$$\frac{f'_1(u_n(y\pm, t)) - f'_1(u_n(x\pm, t))}{y - x} \leq \frac{1}{t}. \tag{3.4}$$

Now, if $\varepsilon < x$, $y \in I(t)$, then the starting points from $x = a_n$, $\zeta_\pm(\tau;\ x, t)$ and $\zeta_\pm(\tau;\ x, t)$ are $T_\pm(x, t)$ and $T_\pm(y, t)$, respectively. We must have $T_\pm(x, t) \geq T_\pm(y, t)$. It is clear that

$$x - a_n = f'_2(u_n(x\pm, t))(t - T_\pm(x, t)), \qquad y - a_n = f'_2(u_n(y\pm, t))(t - T_\pm(y, t)).$$

Hence we must have

$$\frac{f'_2(u_n(y\pm, t)) - f'_2(x\pm, t))}{y - x} \leq \frac{f'_2(u_n(x\pm, t))}{x - a_n}.$$

Now, $u_0(x)$ is uniformly bounded, so $u_n(x, t)$ must be uniformly bounded. Hence there must be a $C_1 < \infty$ such that $f'_2(u_n(x\pm, t)) < C_1$. Since $x \geq \varepsilon > 0$ and $a_n \leq \varepsilon/2$, then

$$\frac{f'_2(u_n(y\pm, t)) - f'_2(u_n(x\pm, t))}{y - x} \leq \frac{2C_1}{\varepsilon}. \tag{3.5}$$

Similarly, when $y < -\varepsilon$ and $x \in I(t)$, there is a C_2 such that

$$\frac{f'_1(u_n(y\pm, t)) - f'_1(u_n(x\pm, t))}{y - x} \leq \frac{2C_2}{\varepsilon}. \tag{3.6}$$

Combining Eqs. (3.3)–(3.6), we obtain

$$\frac{g_u(u_n(y\pm, t), y) - g_u(u_n(x\pm, t), x)}{y - x} \leq \frac{\max(1, 2C_1, 2C_2)}{\min(t, \varepsilon)}. \tag{3.7}$$

Since $g(u, x)$ is uniformly convex in u for each fixed x, there is an $a > 0$ such that $g_{uu}(u, x) \geq a$, so by (3.7) and the mean value theorem we arrive at (3.2) where $K = (1/a) \max(1, 2C_1, 2C_2)$.

COROLLARY 3.2. For $M < \infty$,

$$\operatorname*{Var}_{\substack{|x|\geq \varepsilon \\ |x|\leq M}} |u_n(x, t)| \leq K_1(\varepsilon, t, M), \qquad n \geq N(\varepsilon),$$

for each $\varepsilon > 0$ and $t > 0$.

Now, let $t' > t > 0$; then

$$u_n(x, t') - u_n(x, t) = \int_t^{t'} \{g(u_n(x, \tau), x)\}_x \, d\tau.$$

So we must have

$$\int_{\substack{|x|\geq \varepsilon \\ |x|\leq M}} |u_n(x, t') - u_n(x, t)| \, dx \leq K_2 \int_t^{t'} \operatorname*{Var}_{\substack{|x|\geq \varepsilon \\ |x|\leq M}} u_n(x, \tau) \, d\tau$$

where $|f_u(u_n(x, t), x)| \leq K_2$. Hence

$$\int_{\substack{|x|\geq \varepsilon \\ |x|\leq M}} |u_n(x, t') - u_n(x, t)| \, dx \leq K_1 K_2 |t' - t|.$$

THEOREM 3.3. Assuming an admissibility criterion which is compatible with the approximation $g(u, x)$, there exists a global solution to

$$u_t + \begin{Bmatrix} f_1(u)_x & x < 0 \\ f_2(u)_x & x > 0 \end{Bmatrix} = 0,$$

$$u(x, 0) = u_0(x)$$

where $f_1(u)$ and $f_2(u)$ are uniformly convex.

Proof. By Corollary 3.2, Helly's theorem and the diagonalization process there must be a subsequence $u_{n'}(x, t)$ of $u_n(x, t)$ which converges to a weak solution to (3.1).

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