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DIAKOPTICS OR TEARING—A MATHEMATICAL APPROACH*

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Abstract. The method of diakoptics or tearing was introduced by G. Kron in order to reduce computations in the solution of certain problems arising from large inter-connected power distribution networks. Here the method is given a purely mathematical form which can be used to solve large systems of linear equations by first solving some smaller sub-problems and then combining these solutions into a complete solution. The sub-problems are formed from sets of equations and variables which are strongly connected, within the sub-problem, but only weakly connected to those of another sub-problem.

1. Introduction. The late Gabriel Kron [7, 8] introduced the method of diakoptics in the 1950s, and since then it seems to have remained closely associated with electrical power system problems (Happ [4, 5]), and other specialised applications such as substructuring problems in civil engineering (Przemieniecki [9]). A number of authors have given alternative viewpoints, such as Brameller [2], Branin [3], Kesavan [6]. A rather complicated mathematical generalisation was given by Roth [11], while computer algorithms and mathematical analysis was given by Steward [12].

On the whole, with the exception of Steward's paper, diakoptics has remained as a complicated mixture of electrical and mathematical concepts which are difficult to follow and to use for both the engineer and mathematician. The present paper gives a purely mathematical analysis of the solution of a sparse system of a linear equation, which parallels the diakoptics approach of Kron. The mathematical method has the advantages of being a numerical technique applicable to any linear system, and does not rely on a knowledge of electrical networks. It is quite different from the approach given in the appendix to Steward's paper [12], but it results in the same formulae as those given by Brameller [2].

Section 2 of this paper gives the basic algorithms and formulae, while Sec. 3 gives a numerical illustration and Sec. 4 contains the background theory for the method.

2. Mathematical formulae and algorithms. Suppose $Az = c$ is a matrix form of a system of linear equations, where A is the order $m \times m$ coefficient matrix, z is the order $m \times 1$ column vector of variables, and c is the order $m \times 1$ column vector of constants. The

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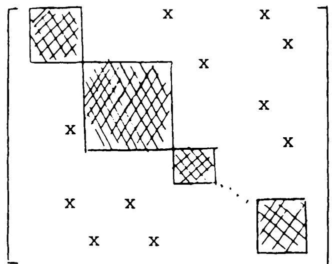
solution procedure is given in parts labelled (a) to (e).

(a) If A is sparse, then it is possible to rearrange the equations and variables so that the new coefficient matrix, M, has most non-zero entries in square diagonal blocks, as in:

A =

(1)

The algorithms for accomplishing this are not included here, but are discussed by Steward [12]. In the usual network-based diakoptics approach this diagonal block form is derived implicitly during the analysis of the network and the resulting formulation of the problem. From now on it will be assumed that A has the form (1) and that the matrix is very sparse outside of the diagonal blocks. In electrical power distribution network each diagonal block can correspond to the equations derived from an almost self-contained distribution network, while the few other non-zero entries correspond to the interconnections between those networks. Steward [12] gives a more general version of this involving a block-triangular form.

(b) Convert the system of equations into a new larger system with coefficient matrix, B, which is solely of square diagonal block form. This can be done arbitrarily, as long as the following rules are observed.

Each non-zero entry in A is also an entry in B, or is a sum of entries in B, and all entries in B are of this type. (2)

If two entries in A are in different rows (columns), then the corresponding entries, or parts of entries, in B must be in different rows (columns). [So each row (column) of B contains entries, or parts of entries, from just one row (column) of A.] (3)

The best way of satisfying these rules is usually to leave the original diagonal blocks unchanged, and to move all of the other non-zero entries into one or more new square diagonal blocks (this is illustrated in Sec. 3). Hence if there are r rows and r columns of A which have non-zero entries not in the diagonal blocks, then the new diagonal blocks must contain a total of r rows and r columns. If there are r rows and s columns with, for example, $r > s$, then in order to make the new diagonal blocks square, $r - s$ of these entries can be split as a sum of two parts, both in the same row, with the extra parts filling the empty columns of the block in B. A similar stratagem can be used if $r < s$, and is illustrated in Sec. 3.

The matrices A and B, with orders $m \times m$ and $n \times n$, are related by an equation

$$A = P'BQ', \tag{4}$$

where P' and Q' have orders $m \times n$ and $n \times m$, and are constructed as follows. The ith row of P' contains a 1 in column j if row j in B contains an entry or parts of entries, from row i in A. All other entries are zero. Notice that each column of P' contains a single non-zero entry, 1, and so it is of full row rank. Similarly the jth column of Q' contains a 1 in row i if column i in B contains entries, or parts of entries, from column j in A. All other entries are zero. Notice that each row of Q' contains a single non-zero entry, 1, and so it is of full column rank.

(c) From the order $m \times n$ matrix P' construct the order $n \times m$ matrix P as the matrix of zeros except for a single entry, 1, in each column. Specifically, 1 occurs in column j, row i, where in P' the *first* 1 in row j occurs in column i. In other words P is the transpose of the matrix P' with only the first non-zero entry retained in each row of P'. Readers familiar with generalized inverse matrices (see Ben-Israel [1]) will recognise P' and P as generalized inverses of each other, satisfying

$$P'P = I. \tag{5}$$

Similarly, define the $m \times n$ matrix Q as the transpose of the matrix formed from Q' by retaining only the first non-zero entry in each column of Q'. Q' and Q satisfy

$$QQ' = I. \tag{6}$$

(d) From P, P', Q, and Q' define the order $n \times (n - m)$ and $(n - m) \times n$ matrices K and L by,

$$K = \overline{I - PP'}, \qquad L = \underline{(I - Q'Q)}, \tag{7}$$

where $\overline{I - PP'}$ denotes the matrix of the $n - m$ non-zero columns in $I - PP'$, and $\underline{I - Q'Q}$the matrix of $n - m$ non-zero rows of $I - Q'Q$. The matrices K and L both have rank $n - m$. They may be constructed directly by the following algorithm.

Eliminate the first 1 in each row of P', and change all of the remaining non-zero entries from 1 to -1. Insert $n - m$ extra rows so that if column j has one of the remaining non-zero entries from P', then row j is a new row containing the single non-zero entry, 1, in column j. Eliminating the m zero columns, leaves the matrix K.

Eliminate the first 1 in each column of Q' and change all the remaining non-zero entries from 1 to -1. Insert $n - m$ extra columns so that if row i has one of the remaining non-zero entries from Q', then column i is a new column containing the single non-zero entry, 1, in row i. Eliminating the non-zero rows leaves the matrix L.

(e) The solution for the system $Az = c$ can now be formed as follows (the proof that $(LB^{-1}K)^{-1}$ exists and that z is a solution is given in Sec. 4),

$$z = Q\left(I - B^{-1}K(LB^{-1}K)^{-1}L\right)B^{-1}Pc. \tag{8}$$

The advantages of this formula over the direct solution of $Az = c$, arise from the block diagonal structure of B. The matrices $B^{-1}K$ and $B^{-1}P$ can be calculated using algorithms which utilize the individual diagonal blocks of B, thus reducing the complexity of the system which must be solved. The portion $(LB^{-1}K)^{-1}LB^{-1}Pc$ may be calculated by any of the usual algorithms as the solution, y, of the matrix system

$$(LB^{-1}K)y = LB^{-1}Pc,$$

where the coefficient matrix, $LB^{-1}K$, has order $(n - m) \times (n - m)$. Hence the size of this coefficient matrix is governed by the increase in size of B over A, which in turn is governed by the number of columns, or rows, of A, with non-zero entries outside of the diagonal blocks (see Eq. (1)).

The complexity of calculations in solving a sparse m variable system with the best direct algorithms has upper bound mt, where t is the number of non-zeros (Reid [10]) though this upper bound is rarely attained in practice. Applying this result to the modified problem analysed above gives an approximate upper bound to the complexity of

$$\sum_{i=1}^{k} s_i t_i + rt',$$

where k, s_i and t_i are, respectively, the number of blocks in B, the size of the ith block and the number of non-zero elements in it, while r and t' are the size and number of non-zero entries in $LB^{-1}K$. If the size of the largest block in B is N, then this upper bound satisfies

$$\leqslant N \sum_{i=1}^{k} t_i + rt' = Nt + rt'.$$

The first term, Nt, will usually be much smaller than the upper bound, mt, for the original problem, $Az = c$, provided the block size, N, is considerably less than the size, m, of the original matrix A. The second term, rt', depends very much on the original problem—the number r will be small compared with m since it measures the largest number of rows or columns containing nonzero entries not in the diagonal blocks, while t' is the number of non-zero entries in $LB^{-1}K$. B^{-1} itself may have considerable fill-in among the diagonal blocks when compared with B, and the product $LB^{-1}K$ combines only certain rows and columns of B^{-1}, normally resulting in a decrease in the non-zero entries.

It should also be noted that the solution z, as given in equation (8), is closely related to the solution of the system, $By = I - K(LB^{-1}K)LB^{-1}Pc$, which involves the enlarged matrix, B. In fact the solutions y and z are related by $z = Qy$. The solution y gives the solution of the torn system, from which the solution z is reconstituted.

A simple numerical example. Suppose the system $Az = c$ is the 12 variable system shown below with A already in the form of Eq. (1). The numerical solution is given in a convenient fashion without necessarily following the most efficient solution algorithm which would be used in a computer solution of a large system. Note that a blank indicates a zero entry.

A												z		c
2						2				4		z_1		1
	-1	4										z_2		3
1			2									z_3		0
1		1										z_4		-1
				1	2	1						z_5		2
				-1		-1						z_6	=	0
				4		2						z_7		0
							1	1				z_8		1
				2			1	3		1		z_9		2
									1		-1	z_{10}		0
									1	-2		z_{11}		1
-1											2	z_{12}		-3

B is chosen, using the rules (2) and (3) in part (b), as

$$B = \left[\begin{array}{cccccccccccccccc}
2 &&&&&&&&&&&&&&& \\
& -1 & 4 &&&&&&&&&&&&& \\
1 &&& 2 &&&&&&&&&&&& \\
1 && 1 &&&&&&&&&&&&& \\
&&&& 1 & 2 & 1 &&&&&&&&& \\
&&&& -1 && -1 &&&&&&&&& \\
&&&&& 4 & 2 &&&&&&&&& \\
&&&&&&& 1 & 1 &&&&&&& \\
&&&&&&& 1 & 3 &&&&&&& \\
&&&&&&&&& 1 && -1 &&&& \\
&&&&&&&&& 1 & -2 &&&&& \\
&&&&&&&&&&& 2 &&&& \\
&&&&&&&&&&&& -1 &&& \\
&&&&&&&&&&&&& 2 && \\
&&&&&&&&&&&&&& 2 & 4 \\
&&&&&&&&&&&&&&& 1
\end{array}\right]$$

Hence P' and Q' are developed, as described in part (b), as

$$P' = \left[\begin{array}{cccccccccccccccc}
1 &&&&&&&&&&&&&& 1 & \\
& 1 &&&&&&&&&&&&&& \\
&& 1 &&&&&&&&&&&&& \\
&&& 1 &&&&&&&&&&&& \\
&&&& 1 &&&&&&&&&&& \\
&&&&& 1 &&&&&&&&&& \\
&&&&&& 1 &&&&&&&&& \\
&&&&&&& 1 &&&&&&&& \\
&&&&&&&& 1 &&&&& 1 && 1 \\
&&&&&&&&& 1 &&&&&& \\
&&&&&&&&&& 1 &&&&& \\
&&&&&&&&&&& 1 & 1 &&&
\end{array}\right], \quad Q' = \left[\begin{array}{cccccccccccc}
1 &&&&&&&&&&& \\
& 1 &&&&&&&&&& \\
&& 1 &&&&&&&&& \\
&&& 1 &&&&&&&& \\
&&&& 1 &&&&&&& \\
&&&&& 1 &&&&&& \\
&&&&&& 1 &&&&& \\
&&&&&&& 1 &&&& \\
&&&&&&&& 1 &&& \\
&&&&&&&&& 1 && \\
&&&&&&&&&& 1 & \\
&&&&&&&&&&& 1 \\
1 &&&&&&&&&&& \\
&&&& 1 &&&&&&& \\
&&&&&&& 1 &&&& \\
&&&&&&&&&& 1 &
\end{array}\right]$$

Consequently P, Q, K and L, as described in parts (c) and (d), are

$$K = \left[\begin{array}{cccc}
&& 1 & \\
&&& \\
&&& \\
&&& \\
&&& \\
&&& \\
&&& \\
&&& \\
& -1 && -1 \\
&&& \\
&&& \\
-1 &&& \\
1 &&& \\
& 1 && \\
&& 1 & \\
&&& 1
\end{array}\right], \quad P = \left[\begin{array}{cccccccccccc}
1 &&&&&&&&&&& \\
& 1 &&&&&&&&&& \\
&& 1 &&&&&&&&& \\
&&& 1 &&&&&&&& \\
&&&& 1 &&&&&&& \\
&&&&& 1 &&&&&& \\
&&&&&& 1 &&&&& \\
&&&&&&& 1 &&&& \\
&&&&&&&& 1 &&& \\
&&&&&&&&& 1 && \\
&&&&&&&&&& 1 & \\
&&&&&&&&&&& 1 \\
&&&&&&&&&&& \\
&&&&&&&&&&& \\
&&&&&&&&&&& \\
&&&&&&&&&&&
\end{array}\right],$$

$$L = \left[\begin{array}{cccccccccccccccc} -1 & & & & & & & & & & & & 1 & & & \\ & & & & -1 & & & & & & & & & 1 & & \\ & & & & & & -1 & & & & & & & & 1 & \\ & & & & & & & & & & -1 & & & & & 1 \end{array}\right], \quad Q = \left[\begin{array}{cccccccccccc} 1 & & & & & & & & & & & \\ & 1 & & & & & & & & & & \\ & & 1 & & & & & & & & & \\ & & & 1 & & & & & & & & \\ & & & & 1 & & & & & & & \\ & & & & & 1 & & & & & & \\ & & & & & & 1 & & & & & \\ & & & & & & & 1 & & & & \\ & & & & & & & & 1 & & & \\ & & & & & & & & & 1 & & \\ & & & & & & & & & & 1 & \\ & & & & & & & & & & & 1 \end{array}\right]$$

Only at this point are calculations carried out for the solution z as given in equation (8) of part (e).

$$B^{-1} = \left[\begin{array}{cccccccccccccccc} .5 & & & & & & & & & & & & & & & \\ -2 & -1 & & 4 & & & & & & & & & & & & \\ -.5 & & & 1 & & & & & & & & & & & & \\ -.25 & & .5 & & & & & & & & & & & & & \\ & & & & & 1 & .5 & & & & & & & & & \\ & & & & .5 & .5 & & & & & & & & & & \\ & & & & & -2 & -.5 & & & & & & & & & \\ & & & & & & & 1.5 & -.5 & & & & & & & \\ & & & & & & & -.5 & .5 & & & & & & & \\ & & & & & & & & & 1 & & .5 & & & & \\ & & & & & & & & & .5 & -.5 & .25 & & & & \\ & & & & & & & & & & & .5 & & & & \\ & & & & & & & & & & & & -1 & & & \\ & & & & & & & & & & & & & .5 & & \\ & & & & & & & & & & & & & & .5 & -2 \\ & & & & & & & & & & & & & & & 1 \end{array}\right], \quad B^{-1}K = \left[\begin{array}{cccc} & & -5 & \\ & & 2 & \\ & & .5 & \\ & & .25 & \\ & & & \\ & & & \\ & & & \\ & .5 & & .5 \\ & -.5 & & -.5 \\ -.5 & & & \\ -.25 & & & \\ -.5 & & & \\ -1 & & & \\ & .5 & & \\ & & .5 & -2 \\ & & & 1 \end{array}\right]$$

$$LB^{-1}K = \left[\begin{array}{cccc} -1 & & .5 & \\ & .5 & & \\ & & .5 & -2 \\ .25 & & & 1 \end{array}\right], \quad (LB^{-1}K)^{-1} = \frac{1}{3}\left[\begin{array}{cccc} -2 & & 2 & 4 \\ & 6 & & \\ 2 & & 4 & 8 \\ .5 & & -.5 & 2 \end{array}\right].$$

Hence,

$$Pc = \begin{bmatrix} 1 \\ 3 \\ 0 \\ -1 \\ 2 \\ 0 \\ 0 \\ 1 \\ 2 \\ 0 \\ 1 \\ 3 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad B^{-1}Pc = \begin{bmatrix} .5 \\ -9 \\ -1.5 \\ -.25 \\ 0 \\ 1 \\ 0 \\ .5 \\ .5 \\ 1.5 \\ .25 \\ 1.5 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad LB^{-1}Pc = \begin{bmatrix} -.5 \\ 0 \\ 0 \\ -.25 \end{bmatrix}, \quad (LB^{-1}K)^{-1}LB^{-1}Pc = \begin{bmatrix} 0 \\ 0 \\ -1 \\ -.25 \end{bmatrix}$$

$$B^{-1}K(LB^{-1}K)^{-1}LB^{-1}Pc = \begin{bmatrix} .5 \\ -2 \\ -.5 \\ -.25 \\ 0 \\ 0 \\ 0 \\ -.125 \\ .125 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -.25 \end{bmatrix}$$

Hence

$$z = Q(B^{-1}Pc - B^{-1}K(LB^{-1}K)^{-1}LB^{-1}Pc) = Q\begin{bmatrix} 0 \\ -7 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ .625 \\ .375 \\ 1.5 \\ .25 \\ 1.5 \\ 0 \\ 0 \\ 0 \\ .25 \end{bmatrix} = \begin{bmatrix} 0 \\ -7 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ .625 \\ .375 \\ 1.5 \\ .25 \\ 1.5 \end{bmatrix}$$

4. Proofs. The following facts need to be proved in order to justify the algorithm given in Sec. 2:

(I) The inverse of $LB^{-1}K$ exists;

(II) $z = Q(I - B^{-1}K(LB^{-1}K)^{-1}L)B^{-1}Pc$, as given in Eq. (8), is a solution of $Az = c$.

These results are established in the following two lemmas.

LEMMA 1. The matrix $LB^{-1}K$ is non-singular if and only if the matrix A is non-singular.

Proof. First it is shown that the columns of Q' are a basis for the null space of L (the set of x with $Lx = 0$). $L = \overline{I - Q'Q}$, by equation (7), where $\overline{I - Q'Q}$, consists of the non-zero rows of $I - Q'Q$. Also, $(I - Q'Q)Q' = 0$, by Eq. (6), L has rank $n - m$, and Q' has rank m, and so the columns of Q' are a basis of the null space of L. Similarly the columns of K span the null space of P', and so $P'K = 0$.

Suppose there is a vector w with $LB^{-1}Kw = 0$, then $B^{-1}Kw$ is in the null space of L. If $w \neq 0$, then $B^{-1}Kw \neq 0$, since K has full column rank and B is non-singular. It follows that for some order $m \times 1$ vector q

$$B^{-1}Kw = Q'q,$$

and if $w \neq 0$, then $q \neq 0$. By equation (4),

$$Aq = P'BQ'q = P'BB^{-1}Kw = P'Kw = 0.$$

Hence, if $LB^{-1}Kw = 0$ for some $w \neq 0$, then $Aq = 0$ for some $q \neq 0$, and so if A is non-singular, then so is $LB^{-1}K$.

Conversely, suppose q is such that $Aq = 0$, then by Eq. (4),

$$Aq = P'BQ'q = 0,$$

and if $q \neq 0$, then $BQ'q \neq 0$ since Q' has full column rank and B is non-singular. Since the null space of P' is spanned by the columns of K,

$$BQ'q = Kw$$

for some w, and if $q \neq 0$, then $w \neq 0$. Therefore

$$LB^{-1}Kw = LB^{-1}BQ'q = LQ'q = 0,$$

since the columns of Q' are in the null space of L.

Hence, if A is singular, then so is $LB^{-1}K$, and the result follows.

LEMMA 2. If A and B are non-singular, then the matrix system, $Az = c$, has solution given by

$$z = Q\left(I - B^{-1}K(LB^{-1}K)^{-1}L\right)B^{-1}Pc.$$

Proof. Let $D = (I - B^{-1}K(LB^{-1}K)^{-1}L)B^{-1}Pc$, for convenience.

$$\begin{aligned} Az &= P'BQ'z, \quad \text{by Eq. (4)} \\ &= P'BQ'QD = P'BD - P'B(I - Q'Q)D \\ &= P'BB^{-1}Pc - P'BB^{-1}K(LB^{-1}K)^{-1}LB^{-1}Pc - P'B(I - Q'Q)D \\ &= c - P'B(I - Q'Q)D, \end{aligned}$$

since $BB^{-1} = I$, $P'P = I$, by Eq. (5), and $P'K = P'\overline{(I - P'P)} = \overline{P' - P'PP} = 0$, by Eqs. (5) and (7) (where $\overline{I - P'P}$ represents the non-zero columns of $I - P'P$). It remains to show that $P'B(I - Q'Q)D = 0$. However,

$$\begin{aligned} LD &= LB^{-1}Pc - (LB^{-1}K)(LB^{-1}K)^{-1}LB^{-1}Pc \\ &= LB^{-1}Pc - LB^{-1}Pc = 0. \end{aligned}$$

Since, by definition, $I - Q'Q$ differs from L only by having some extra zero rows, it follows that if

$$LD = 0, \quad \text{then } P'B(I - Q'Q)D = 0.$$

Hence, the results, $Az = c$, follows.

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